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An Environment of Quasi-Valuation Domains

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Introduction

Any domain W has an ordered group $G(W)$. This group, the set of non-zero principal fractional ideals of W with $xW \leqq yW$ if and only if xW contains yW, is called the group of divisibility of W. Let $K^{\times} = K \backslash \{0\}$ be the multiplicative group of quotient field of W and $U(W)$ the group of units of W, then $G(W)$ is order isomorphic to $K^{\times}/U(W)$, where $xU(W) \leqq yU(W)$ if and only if $y/x \in W$. It is wellknown that $G(W)$ is linearly ordered if and only if W is a valuation domain.

In section 1, to define a good preordered group (2.1), we study an additive abelian group admitting two co-linear preorder relations compatible with the group operation.

In section 2, using the basic results of section 1, we discuss some facts related to a domain W under the assumption that $G(W)$ is a good preordered group. Then W is dominated by a valuation domain V. We call this domain W a quasi-valuation domain; in particular, in case V is integral over W we call W a prevaluation domain. Furthermore, there are many similarities between quasi-valuation domains and valuation domains. In fact $V \backslash U(V) = W \backslash U(W)$. Then it is only natural that a quasi-valuation domain has some normalities. A quasi-valuation domain W is really seminormal, i.e., $\mathrm{Pic}(W) \to \mathrm{Pic}(W[X])$ is an isomorphism, where $\mathrm{Pic}(W)$ is the Picard group of W and X is an indeterminate. Therefore, for a domain R, it stands to reason that we should think about $\cap W_{\lambda}$, the intersection ranging over all quasivaluation domains containing R. This domain $R^{\#} = \cap W_{\lambda}$ is seminormal; $R^{\#}$ is not always the seminormalization R^{+} of R, however.

In section 3, we show that $R^{\#}$ is the largest subdomain R' of $\tilde{R}$ containing R such that, for all $p' \in \mathrm{Spec}(R')$, the canonical homomorphism $k(p' \cap R) \to k(p')$ is an isomorphism, where $\tilde{R}$ is the derived normal ring

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of R and $k(p')$ is the residue field of $R'_{p'}$, and give some properties of prenormal domains. By the way, a domain R is quasinormal if and only if $\mathrm{Pic}(R) \to \mathrm{Pic}(R[X, X^{-1}])$ is an isomorphism. To the writer's knowledge, the quasinormalization hadn't been given even in one-dimensional case. Our prenormalization provides the quasinormalization in the case.

In section 4, we define an M-prenormalization of a domain and show that an M-prenormal domain is quasinormal under the noetherian assumption. This is proved in the same way as the proof of ([3], 3.6).

All rings considered in this paper will be commutative with unit.

§ 1. Preordered groups.

In this section we turn to an information of additive abelian groups (for short, groups) admitting a preorder relation and an order relation compatible with the group operation and we give some definitions. H denotes a group.

If $\leqq$ is a relation defined on H, we say that $\leqq$ is a preorder on H and that H is preordered under $\leqq$ if $\leqq$ is reflexive and transitive. If a preorder $\leqq$ is asymmetric, we say that $\leqq$ is an order on H and that H is ordered under $\leqq$. If, for any $x, y \in H$, $x \leqq y$ or $y \leqq x$, then $\leqq$ is a linear preorder on H, and H is said to be linearly preordered under $\leqq$.

DEFINITION 1.1. If $\leqq_1$ and $\leqq_2$ are preorders on H, then we say that $\leqq_1$ and $\leqq_2$ are co-linear on H to each other, and that $\leqq_1$ is co-linear with respect to $H=(H, \leqq_2)$ if $x \leqq_1 y$ or $y \leqq_2 x$ for all $x, y \in H$.

DEFINITION 1.2. If $\leqq$ is a preorder on H compatible with the group operation on H, i.e., for all $x, y, z \in H$, $x \leqq y$ implies that $x+z \leqq y+z$, we say that H is a preordered group.

DEFINITION 1.3. Let $H=(H, \leqq)$ be a preordered group and $x \in H$. We say that x is prepositive if $x \in P=\{h \in H; 0 \leqq h\}$, and x is prenegative if $x \in (-P)$; x is strictly prepositive if $x \in P^+=P \backslash Z$ where $Z=P \cap (-P)$, and x is strictly prenegative if $x \in (-P^+)$.

Hereafter the set of prepositive element of a preordered group $H=(H, \leqq)$, the set of strictly prepositive elements of H and the set of prepositive and prenegative elements of H are denoted by P, P^+ and Z respectively.

PROPOSITION 1.4. *Let $H=(H, \leqq)$ be a preordered group. Then, the following statements hold.*

(1.4.1) *P is a subsemigroup.*

(1.4.2) *$P \ni y-x$ if and only if $x \leqq y$.*

(1.4.3) *Z is a subgroup of H.*

(1.4.4) *H/Z is an ordered group under the preordering on H/Z induced by $\leqq$.*

DEFINITION 1.5. Let $\leqq_1$ and $\leqq_2$ be preorders on H. Then we say that the preorder $\leqq_2$ is finer than the preorder $\leqq_1$ if the following two conditions are satisfied:

(1.5.1) $x \leqq_1 y$ implies that $x \leqq_2 y$ for all $x, y \in H$.

(1.5.2) If $x \in P_1^+$, then $x \in P_2^+$ where $P_j^+ = \{h \in H; 0 \leqq_j h, h \nleqq_j 0\}$.

PROPOSITION 1.6. *Let $\leqq_1$ and $\leqq_2$ be preorders on H. If $\leqq_1$ and $\leqq_2$ are co-linear and $\leqq_2$ is finer than $\leqq_1$, then H is linearly preodered under $\leqq_2$.*

PROOF. Take any $h, h' \in H$. Since $\leqq_1$ and $\leqq_2$ are co-linear, either $h \leqq_1 h'$ or $h' \leqq_2 h$ must hold. If $h' \nleqq_2 h$ then $h \leqq_1 h'$. Thus, by (1.5.1), $h \leqq_2 h'$.

COROLLARY 1.7. *Under the assumptions as in* (1.6), *$H = P_2 \cup (-P_2)$.*

PROPOSITION 1.8. *Suppose, in addition to the circumstances that $\leqq_1$ is an order. If $h \in P_2^+$ then $0 <_1 h$.*

PROOF. If $h \in P_2^+$ then $h \notin (-P_2)$, i.e., $h \nleqq_2 0$. Then $0 \leqq_1 h$, since $\leqq_1$ and $\leqq_2$ are co-linear. Since $h \in P_2^+$, $h \neq 0$. Then, since $\leqq_1$ is an order, $0 <_1 h$.

PROPOSITION 1.9. *Let P_3 be a subsemigroup of H between P_1 and P_2. Then,*

(1.9.1) *The relation $\leqq_3$ on H defined by "$x \leqq_3 y$ if and only if $y - x \in P_3$" is a preorder on H.*

(1.9.2) *There are the order $\leqq_3'$ and the preorder $\leqq_2'$ induced by P_3 on $H' = H/Z_3$ where $Z_3 = P_3 \cap (-P_3)$.*

(1.9.3) *$\leqq_2'$ is finer than $\leqq_3'$.*

(1.9.4) *$\leqq_2'$ and $\leqq_3'$ are co-linear on H.*

PROOF. (1.9.1) and (1.9.2) are clear by (1.4). By (1.8), we have $P_3^+ = P_2^+$, thus $\leqq_2'$ is finer than $\leqq_3'$. (1.9.4): Since $\leqq_3$ is finer than $\leqq_1$ and $\leqq_1$ and $\leqq_2$ are co-linear, $\leqq_2$ and $\leqq_3$ are co-linear, hence $\leqq_2'$ and $\leqq_3'$ are co-linear on H.

PROPOSITION 1.10. *Under the assumptions as in* (1.9), *there is a one to one correspondence between all P_j's and all Z_j's.*

PROOF. Since $P_j = P_j^+ \cup Z_j$ and $P_j^+ = P_i^+$ for all i, j, the statement above holds.

§ 2. Quasi-valuations and prevaluations.

In this section $H=(H, \leqq_1, \leqq)$ always denotes a good preordered group defined as follows:

DEFINITION 2.1. Let $H=(H, \leqq)$ be a preordered group. If H is an ordered group under the order $\leqq_1$ compatible with the same group operation, $\leqq$ and $\leqq_1$ are co-linear and $\leqq$ is finer than $\leqq_1$, then we say that H is a good preordered group and we write $H=(H, \leqq, \leqq_1)$.

EXAMPLE 2.2. Let G be an additive abelian group and $F=(F, \leqq)$ be a linearly ordered additive abelian group. We put $H=G\times F$. Let $\leqq_1$ be an order on H defined by "$(g, f)\leqq_1(g', f')$ if and only if ($g=g'$ and $f=f'$) or ($f<f'$)" and $\leqq_2$ be a preorder on H defined by "$(g, f)\leqq_2(g', f')$ if and only if $f\leqq f'$". Then $H=(H, \leqq_1, \leqq_2)$ is a good preordered group.

DEFINITION 2.3. Let K be a field. A mapping w of $K^\times = K\backslash\{0\}$ into a suitable good preordered group $H=(H, \leqq, \leqq_1)$ is called a quasi-valuation on K if the following conditions are satisfied for all $x, y\in K^\times$;

(2.3.1) $w(xy)=w(x)+w(y)$.

(2.3.2) (1) If $w(x)-w(y)\in P^+$, $w(y)\leqq_1 w(x+y)$.

(2) If $w(x)-w(y)\in(-P^+)$, $w(x)\leqq_1 w(x+y)$.

(3) If $w(x)-w(y)\in Z\backslash\{0\}$, then for some $z\in H$ such that $w(x)-z\in Z$, $z\leqq_1 w(x+y)$.

(4) If $w(x)-w(y)=0$, $w(x)\leqq_1 w(x+y)$.

(2.3.3) $w(-1)=0$.

PROPOSITION 2.4. *Under the circumstances, the followings hold.*

(2.4.1) $w(x^{-1})=-w(x)$, $w(1)=0$ *and* $w(x)=w(-x)$.

(2.4.2) *If* $w(y)-w(x)\in P^+$, *then* $w(x+y)=w(x)$.

PROOF. We are to prove (2.4.2). By (1.8), we note that $h\in P^+$ if and only if $0<_1 h$.

(1) If $w(x+y)-w(-y)\in P^+$, then $w(x)<_1 w(y)=w(-y)\leqq_1 w(-y+(x+y))=w(x)$, a contradiction.

(2) If $w(x+y)-w(-y)\in(-P^+)$, then $w(x)\leqq_1 w(x+y)\leqq_1 w(-y+(x+y))=w(x)$ so that $w(x)=w(x+y)$.

(3) If $w(x+y)-w(-y)\in Z\backslash\{0\}$, for some $z\in H$ such that $w(-y)-z=z'\in Z$, then $z+z'=w(y)>_1 w(x)=w(-y+(x+y))\geqq_1 z$. Thus $z+z'>_1 z$. Hence $z'>_1 0$, i.e., $z'\in P^+$, a contradiction.

(4) If $w(x+y)-w(-y)=0$, $w(x)\leqq_1 w(x+y)\leqq_1 w(-y+(x+y))=w(x)$, so that $w(x+y)=w(x)$.

Proposition 2.5. *Under the circumstances, we set $V=\{x\in K^\times; 0\leqq w(x)\}\cup\{0\}$ and $W=\{x\in K^\times; 0\leqq_1 w(x)\}\cup\{0\}$. Then V is a valuation domain of K and W is a local domain dominated by V with the quotient field K. Moreover the maximal ideal of V is equal to the maximal ideal of W.*

Proof. We note that $\tilde{H}=H/Z$ is a linearly ordered group. Considering $\tilde{w}$; $K^\times\to H$,

$$\begin{tikzcd} K^\times \arrow[r, "w"] \arrow[dr, "\tilde{w}"'] & H \arrow[d] \\ & \tilde{H} \end{tikzcd}$$

$V=\{x\in K^\times; 0\leqq'\tilde{w}(x)\}\cup\{0\}$ where $\leqq'$ is the order on $\tilde{H}$ induced by P. Then the conditions of (2.3) induce the condition of a valuation of K. Hence V is a valuation domain of K.

Condition (2.3.1) implies that W is closed under multiplication. Take $x, y\in W$, so that $0\leqq_1 w(x)$ and $0\leqq_1 w(y)$.

(1) If $w(x)-w(-y)\in P^+$, then $0\leqq_1 w(y)\leqq_1 w(x-y)$.

(2) If $w(x)-w(-y)\in(-P^+)$, then $0\leqq_1 w(y)\leqq_1 w(x-y)$.

(3) If $w(x)-w(-y)\in Z$, then we may assume that $w(x)=w(y)=0$ and that $0<_1 w(x)$, $0<_1 w(y)$.

(a) If $w(x)=w(y)=0$, $0=w(x)\leqq_1 w(x-y)$.

(b) If $0<_1 w(x)$, $0<_1 w(y)$ and $w(x)-w(-y)\in Z\backslash\{0\}$, then $z\leqq_1 w(x-y)$ for some $z\in H$ such that $w(x)-z\in Z$. Then $z=w(x)+z'$ where $z'\in Z$. Hence $z\in P^+$. By (1.8), we have $0<_1 z\leqq_1 w(x-y)$.

(c) If $0<_1 w(x)$, $0<_1 w(y)$ and $w(x)-w(-y)=0$, then $0<_1 w(x)\leqq_1 w(x-y)$. Thus in all cases $0\leqq_1 w(x-y)$. We have proved that W is a domain with identity. Moreover the maximal ideal of $V=\{x\in K^\times; 0<\tilde{w}(x)\}\cup\{0\}=\{x\in K^\times; w(x)\in P^+\}\cup\{0\}=\{x\in K^\times; 0<_1 w(x)\}\cup\{0\}\subset W$. This shows that W is a local domain (W, n) dominated by $V=(V, m)$ with the quotient field K and that $m=n$.

Definition 2.6. Under the circumstances, we say that (V, n) is the valuation domain of $\tilde{w}$ and that (W, n) is the quasi-valuation domain of w dominated by V. Two quasi-valuations w, w' of a field K are equivalent to each other if the quasi-valuation domain of w coincides with the quasi-valuation domain of w'. We call $w(K^\times)$ the quasi-value group of w.

Then we see the following result immediately.

THEOREM 2.7. *Let* (V, m) *be a valuation domain of a quotient field* K *and* W *a subdomain of* V *such that* $W \leqq V$ *and* $Q(W)=K$. *Then the following statements are equivalent.*

(2.7.1) W *is a quasi-valuation domain of* K.

(2.7.2) *For any* $x, y \in W$, *it holds that either* $x \in yW$ *or* $y \in xV$.

(2.7.3) *The maximal ideal* m *of* V *is set-theoretically equal to the maximal ideal of* W.

(2.7.4) *For any prime ideal* p *of* W, *the maximal ideal of* V_p *is set-theoretically equal to* p.

(2.7.5) *If* $x \in K$, *then either* $x \in W$ *or* $x^{-1} \in V$.

(2.7.6) *There exists a subfield* k *of* V/m *such that* $W=\{x \in V; x \bmod m \in k\}$.

PROOF. (2.7.1)$\rightarrow$(2.7.5): If $x \notin V$, then $x^{-1} \in m$. By (2.4), we have $x^{-1} \in m \subseteqq W$.

(2.7.5)$\rightarrow$(2.7.1): We set $H=\{xW; x \in K\backslash\{0\}\}$. We define $xW \leqq_1 yW$ if and only if $y/x \in W$ for xW, $yW \in H$, then the relation $\leqq_1$ is an order on H. Moreover we define $xW \leqq yW$ if and only if $y/x \in V$ for xW, $yW \in H$, then the relation $\leqq$ is a preorder on H. Hence we have that $\leqq$ is finer than $\leqq_1$ and that $\leqq$ and $\leqq_1$ are co-linear to each other. We write the group operation on H as addition: $xW+yW=xyW$. Since, for all $z \in K\backslash\{0\}$, $xW \subseteqq yW$ implies $xzW \subseteqq yzW$, the order $\leqq_1$ and $\leqq$ are compatible with the group operation on H. Then the mapping w such that $w(x)=xW$ $(x \in K\backslash\{0\})$ is a homomorphism from $K^{\times}$ onto H. Hence the mapping w satisfies the conditions of (2.3). Then w is a quasi-valuation of K. Hence $W=\{x \in K; 0 \leqq_1 w(x)\} \cup \{0\}$, i.e., W is a quasi-valuation domain of K.

(2.7.1)$\rightarrow$(2.7.3): This is the statement of (2.5).

(2.7.3)$\rightarrow$(2.7.1): We have only to show that (2.7.3)$\rightarrow$(2.7.5). Take $x \in K$. If $x \notin V$, then $x^{-1} \in m \subset W$.

(2.7.2)$\leftrightarrow$(2.7.5): This is nothing but a restatement.

(2.73)$\rightarrow$(2.7.4): We may assume that (2.7.3)$\leftrightarrow$(2.7.5). Let m' be the maximal ideal of a valuation domain V_p. Take $x=t/s \in m' \subseteqq V_p$ $(s \in W\backslash p$, $t \in V)$. If $t \notin sW$, then $s \in tV$, i.e., $s/t \in V \subseteqq V_p$, hence s/t is a unit in V_p, a contradiction. Thus $t \in sW$, i.e., $x \in W \subseteqq W_p$ and x is not a unit in $W_p \subseteqq V_p$. Hence $x \in pW_p$, i.e., $x \in p=pW_p \cap W$.

Of course (2.7.4) implies (2.7.3).

(2.7.3)$\leftrightarrow$(2.7.6): This is nothing but a restatement.

PROPOSITION 2.8. *Let* W *be a quasi-valuation domain of* K, W' *any* *domain between W and K and $p \in \operatorname{Spec}(W)$. Then the following statement hold.*

(2.8.1) *W' is a quasi-valuation domain of K.*

(2.8.2) *If x is an element of W which is not in p, then p is contained in xW.*

(2.8.3) *p is set-theoretically equal to pW_p.*

(2.8.4) *W/p is a quasi-valuation domain.*

(2.8.5) *If L is a subfield of K, $L \cap W$ is a quasi-valuation domain of L.*

PROOF. (2.8.1): Let V' be a valuation domain between W' and K and V a valuation domain which dominates W. Take $x \notin W'$.

(1) If $V' \subseteqq V$, then $W \subseteqq W' \subseteqq V' \subseteqq V$. Hence $V' = V$. Thus $x^{-1} \in V = V'$.

(2) If $V \subseteqq V'$, then $x^{-1} \in V \subseteqq V'$.

(3) If V and V' are incomparable, then, by the theorem of independence of valuation, $R = V' \cap V$ is a semi-local domain which is not local. On the other hand, as $W \subseteqq R \subset V$, R must be local, which is nonsense.

(2.8.2): Let y be an arbitrary element of p. Suppose y is not in xW. By (2.7.2), x is in $yV \subseteqq pV = pV_p = pW_p = p$, a contradiction.

(2.8.3): By (2.7.4), $p \subseteqq pW_p \subseteqq pV_p = p$.

(2.8.4): Let V be a valuation domain which dominates W. Then V/p is a valuation domain which dominates W/p. The statement is therefore immediate from (2.7).

(2.7.5): Let V be a valuation domain which dominates W. Then $V \cap L$ is a valuation domain which dominates $W \cap L$. If y is an element of L which is not in $W \cap L$, then $x^{-1} \in V \cap L$.

(W, n, k) denotes a local ring (W, n) with the residue field k.

COROLLARY 2.9. *Let (W, n, k) and (W', n', k') be quasi-valuation domains dominated by the valuation domain V. Then, $k = k'$ if and only if $W = W'$.*

PROOF. It is easy and we omit it.

PROPOSITION 2.10. *Let (W, n) be a quasi-valuation domain of a field K dominated by the valuation domain (V, n) of K and W^* a quasi-valuation domain of the residue field V/n dominated by the valuation domain V^* of V/n. Then the set $W' = \{x \in V; \bmod n \in W^*\}$ is a quasi-valuation domain of K dominated by the composite of V with V^*, i.e., $V' = \{x \in V; x \bmod n \in V^*\}$.*

PROOF. Take $x \in K$, $x \notin W'$. If $x \notin V$, then $x^{-1} \in n \subset V$, $x^{-1} \bmod n \in V^*$

and $x^{-1} \in V'$. Assume that $x \in V$. Since W^* is a quasi-valuation domain, $x^{-1} \bmod n \in V^*$, which shows that $x^{-1} \in V'$. Thus W' is a quasi-valuation domain dominated by V'.

REMARK 2.11. The domain $W''=\{x \in W; x \bmod n \in W^*\}$ is not always a quasi-valuation domain dominated by V'. Let Q be the field of rationals, C the field of complexes, X, Y indeterminates and $K=C((x))$ the quotient field of $C[[X]]$. We set $W=Q+YK[[Y]]$, $W^*=Q+XC[[X]]$ and $V^*=C[[X]]$. We put $f=2^{1/2}X$. Then, $f \notin W''$ and $f^{-1} \notin V'$, which show that W'' is not a quasi-valuation domain dominated by V'.

PROPOSITION 2.12. *Let R be a subdomain of a field K and let p a prime ideal of R. Then there exists a quasi-valuation domain W of K such that W has a prime ideal n with $k(p)=k(n)$.*

PROOF. It is well-known that there exists a valuation domain V of K such that V has a prime ideal n lying over p. We set $W=\{x \in V; x \bmod n \in k(p)=k(n)\}$. Then W is a quasi-valuation domain dominated by V such that W has a prime ideal n lying over p with $k(p)=k(n)$.

To introduce the concept of a prevaluation domain, we define a w-subgroup of $(H, \leqq, \leqq_1)$.

DEFINITION 2.13. Let w be a quasi-valuation of K with the quasi-value group $(H, \leqq, \leqq_1)$, Z' a subgroup of Z and w' a mapping of $K^\times$ to H' $(=H/Z)$ induced by Z' and w. If w' satisfies the conditions of (2.3), i.e., w' is a quasi-valuation with the quasi-value group H', then we say that Z' is a w-subgroup of H.

PROPOSITION 2.14. *Let w be a quasi-valuation with the quasi-value group $(H, \leqq, \leqq_1)$. If Z_2 and Z_3 are w-subgroups of H, then $Z_2 \cap Z_3$ and Z_2+Z_3 are w-subgroups of H.*

PROOF. $Z_4=Z_2 \cap Z_3$ is a w-subgroup: Let $\leqq_j$ $(j=2, 3, 4)$ be a preorder on H induced by Z_j. We note that $\leqq_j$ $(j=2, 3, 4)$ is finer than $\leqq_1$ and that $P^+=P_1=P_2=P_3=P_4$.

(1) If $w(x)-w(y) \in P^+$, $w(y) \leqq w(x+y)$, hence $w(y) \leqq_4 w(x+y)$.

(2) If $w(x)-w(y) \in (-P^+)$, $w(x) \leqq_1 w(x+y)$, hence $w(x) \leqq_4 w(x+y)$.

(3) If $w(x)-w(y) \in Z \backslash Z_4$, for some $z \in Z$ such that $w(x)-z \in Z$, $z \leqq_1 w(x+y)$ by (2.3.2), hence $z \leqq_4 w(x+y)$.

(4) If $w(x)-w(y) \in Z_4=Z_2 \cap Z_3$, i.e., $w(x)-w(y) \in Z_j$ $(j=2, 3)$, then $w(x) \leqq_j w(x+y)$ $(j=2, 3)$, i.e., $w(x+y)-w(x) \in P_2 \cap P_3=(P_2^+ \cup Z_2) \cap (P_3^+ \cup Z_3)=(P_1^+ \cup Z_2) \cap (P_1^+ \cup Z_3)=P_1^+ \cup (Z_2 \cap Z_3)=P_4^+ \cup Z_4=P_4$, hence $w(x) \leqq_4 w(x+y)$.

Thus the mapping w_4 induced by Z_4 and w is a quasi-valuation with the quasi-value group H/Z_4, i.e., $Z_2 \cap Z_3$ is a w-subgroup of H.

$Z_5 = Z_2 + Z_3$ is a w-subgroup: Let $\leqq_5$ be a preorder on H induced by Z_5. We note that $P^+ = P_5^+$ and that $\leqq_5$ is finer than $\leqq_1$.

(1) If $w(x) - w(y) \in P^+$, $w(y) \leqq_1 w(x+y)$, hence $w(x) \leqq_5 w(x+y)$.

(2) If $w(x) - w(y) \in (-P^+)$, $w(x) \leqq_1 w(x+y)$, hence $w(x) \leqq_5 w(x+y)$.

(3) If $w(x) - w(y) \in Z \backslash Z_5$, for some $z \in Z$ such that $w(x) - z \in Z$, $z \leqq_1 w(x+y)$, hence $z \leqq_5 w(x+y)$.

(4) Let w_j ($j = 1, 2, 3, 4$) be a quasi-valuation with the quasi-value group H/Z_j induced by Z_j. We illustrate groups in the figure, where f and f' are cannonical homomorphism:

$$
\begin{tikzcd}
 & K^{\times} \arrow[dl, "w_5"'] \arrow[d] \arrow[dr, "w"] & \\
H/Z_3 & H/Z_4 \arrow[l, "f"] & H \arrow[l, "f'"] \\
Z_5/Z_3 \arrow[u, hook] & Z_2/Z_4 \arrow[l, "\sim"'] \arrow[u, hook] &
\end{tikzcd}
$$

We note that Z_5/Z_3 is isomorphic to Z_2/Z_4. Since Z_3 is a w-subgroup of H, if $w(x) - w(y) \in Z_2 + Z_3$, then $w_3(x) - w_3(y) \in Z_2 + Z_3/Z_3$. Hence $w_4(x) - w_4(y) \in Z_2/Z_4$. Since w_4 is a quasi-valuation with the quasi-value group H/Z_4, $w_4(x+y) - w_4(x) \in P_2/Z_4 \hookrightarrow H/Z_4$. Then, by the isomorphism f of Z_2/Z_4 to Z_5/Z_3, $w_3(x+y) - w_3(x) \in P_5/Z_3$. It follows that $w(x+y) - w(x) \in P_5$.

Thus a mapping w_5 induced by Z_5 and w is a quasi-valuation with the quasi-value group H/Z_5, i.e., Z_5 is a w-subgroup of H.

The next proposition is an immediate corollary.

PROPOSITION 2.15. *Under the circumstances, there is a one-to-one order-preserving correspondence between all the w-subgroups of H and all the quasi-valuation domains dominated by V containing W.*

PROOF. Let Z' be a w-subgroup of H, w' a quasi-valuation of K with the quasi-value group H/Z' and $\leqq'$ an order on H/Z induced by Z'. We set $W' = \{x \in K^{\times}; 0 \leqq' w(x)\} \cup \{0\}$. Then W' is a quasi-valuation domain dominated by V. Conversely, let W' is a quasi-valuation domain dominated by V and $U(W')$ a unit group of W. We set $Z' = \{w(x); x \in U(W')\}$. Then Z' is a w-subgroup of H.

DEFINITION 2.16. We say that a chain of distinct w-subgroups $Z'=Z_0\supset Z_1\supset\cdots\supset Z_n$ is of length n. We say that Z' has w-rank n if there exists a chain of length n descending from Z' but no longer chain. We say that Z' has w-rank ∞ if there exist arbitrarily long chains descending from Z'. Our notation for w-rank is $w\text{-rk}(Z')$.

PROPOSITION 2.17. *Under the circumstances, let Z' be a w-subgroup of H of finite w-rank and (W', n', k') a quasi-valuation domain corresponding to Z'. Then W' is integral over (W, n, k).*

PROOF. Let $\tilde{k}$ be the residue field of the valuation domain of w. By (2.9) and (2.15), there is a one-to-one order-preserving correspondence between all the w-subgroups of H and all the intermediate fields between k and $\tilde{k}$. Moreover we note that k' is algebraic over k if and only if W' is integral over W. Since $w\text{-rk}(Z')$ is finite, the number of intermediate fields between k and k' is finite, hence k' is algebraic over k, i.e., W' is integral over W.

A finiteness of w-rank of a w-subgroup motivates the next definition.

DEFINITION 2.18. Let w be a quasi-valuation of K with the quasi-value group H. We say that w is a prevaluation of K if, for all $x\in K^\times$ such that $w(x)\in Z$, the w-subgroup Z_x of H generated by $w(x)$ is of finite w-rank. Then, we say that a quasi-valuation domain W (corresponding to w) is a prevaluation domain and that $w(K^\times)$ is called the prevalue group of w. Two prevaluation w, w' of K are equivalent to each other if the prevaluation domain of w coincides with the prevaluation domain of w'.

Then we see the following results. $\tilde{W}$ denotes the derived normal ring of a domain W.

THEOREM 2.19. *Let W be a domain with a quotient field K. Then the following statement are equivalent.*

(2.19.1) *W is a prevaluation domain K.*

(2.19.2) *For any $x, y\in W$, it holds that either $x\in yW$ or $y\in x\tilde{W}$.*

(2.19.3) *If $x\in K$, then either $x\in W$ or $x^{-1}\in\tilde{W}$.*

(2.19.4) *W is a quasi-valuation domain (W, n, k) of K dominated by the valuation domain $(V, n, \tilde{k})$ and $\tilde{k}$ is algebraic over k.*

(2.19.5) *W is a quasi-valuation domain of K dominated by the valuation domain V and V is integral over W.*

(2.19.6) *$\tilde{W}$ is a valuation domain and the maximal ideal of $\tilde{W}$ is set-theoretically equal to the maximal ideal of W.*

(2.19.7) *$\tilde{W}$ is a valuation domain and, for any prime ideal p of*

w, a maximal ideal of W_p is set-theoretically equal to p.

PROOF. First, by (2.17), we note that the valuation domain V dominating W is integral over W.

(2.19.1)$\leftrightarrow$(2.19.4)$\leftrightarrow$(2.19.5): Trivial.

(2.19.2)$\leftrightarrow$(2.19.3): This is nothing but a restatement.

(2.19.5)$\rightarrow$(2.19.3): Take any $x \in K$, $x \notin W$. Since W is a quasi-valuation domain of K and V is a integral over W, $x^{-1} \in V \subseteqq \tilde{W}$.

(2.19.3)$\rightarrow$(2.19.5): It is easy to see that $\tilde{W}$ is a valuation domain of K. Then W is a quasi-valuation domain of K dominated by $\tilde{W}$ and $\tilde{W}$ is integral over W.

(2.19.3)$\rightarrow$(2.19.7): Let p be a prime ideal of W. Since $\tilde{W}_p=(\tilde{W}_p, \tilde{m})$ is a valuation domain, W_p is a local domain (W_p, m) hence $m \subseteqq \tilde{m}$. Take an element of $\tilde{m}$, say x. Then $x^{-1} \notin \tilde{W}_p$, hence, by (2.19.3)$\leftrightarrow$(2.19.5) and (2.8.1), $x \in m=pW_p=p$ (cf. (2.8.3)), i.e., $p=\tilde{m}$.

(2.19.7)$\rightarrow$(2.19.6): Trivial.

(2.19.6)$\rightarrow$(2.19.3): Take $x \in K$. If $x^{-1} \notin W$, then $x \in \tilde{n}=n \subset W$.

PROPOSITION 2.20. *Let W be a prevaluation domain of K and p any prime ideal of W. Then, the following statements hold.*

(2.20.1) *If W' is any domain between W and K, then W' is a prevaluation domain.*

(2.20.2) *If $x \in W$ and $x \notin p$, then $p \subset xW$.*

(2.20.3) *p is set-theoretically equal to pW_p.*

(2.20.4) *W/p is prevaluation domain.*

PROOF. (2.20.1): If $x \notin W'$, then $x \notin W$, hence $x^{-1} \in \tilde{W} \subseteqq \tilde{W}'$.

(2.20.2): Take any $y \in p$. If $x \notin yW$, then $y \in x\tilde{W}$, hence $p \subseteqq xW$. Let p be a prime ideal of W lying over p. Therefore $yW \subseteqq \tilde{p}=p \subset x\tilde{W}$, i.e., $x \notin y\tilde{W}$, thus $y \in xW$, i.e., $p \subset xW$.

(2.20.3): Take $x=y/z \in pW_p$ $(y \in p,\ z \in W\backslash p)$. Since $z \notin p$, by (2.20.2), $p \subset zW$, hence $y=zy'$ $(y' \in W)$. Thus $x \in W \cap pW_p=p$.

(2.20.4): By (2.8.4), W/p is a quasi-valuation domain dominated by the valuation domain $\tilde{W}/p$ and $\tilde{W}/p$ is integral over W/p. It follows that W is a prevaluation domain.

§ 3. Prenormality and seminormality.

The definition of a seminormalization which was given by Traverso [3] is as follows.

DEFINITION 3.1. Let R be a domain, T an overdomain of R integral over R. We define

$$R_T^+(p)=\{x\in T;\ x\in R_p+J(T_p) \text{ for all } p\in \operatorname{Spec}(R)\}$$

where $J(T_p)$ is the Jacobson radical of T_p and $R_T^+=\cap R_T^+(p)$, the intersection ranging over all prime ideals of R. We say that the ring $R_T^+(p)$ is obtained by glueing T over p and that the ring R_T^+ is the seminormalization of R in T. If $T=\tilde{R}$, then we call R_T^+ the seminormalization of R and denote it by R^+; we say that R is seminormal in T if $R=R_T^+$.

PROPOSITION 3.2. *R^+ is the largest subring T of $\tilde{R}$ containing R such that*

(3.2.1) *For any $p\in \operatorname{Spec}(R)$ there is exactly one $q\in \operatorname{Spec}(T)$ lying over p, and*

(3.2.2) *The canonical homomorphism $k(p)\to k(q)$ is an isomorphism.*

We first begin with the next proposition.

PROPOSITION 3.3. *A prevaluation domain is seminormal.*

PROOF. Let R be a prevaluation domain, p any prime ideal of R. Since R_p is a prevaluation domain by (2.20.1), $J(\tilde{R}_p)=J(R_p)$ by (2.19.7). Hence $R_p\supset J(\tilde{R}_p)$. Thus $R^+=\cap (R_p+J(R_p))=\cap R_p=R$.

LEMMA 3.4. *Let $k_0\supset k$ be fields. There is a sequence of fields*

$$k_0\supseteqq k_1\supseteqq k_2\supseteqq k_3\supseteqq\cdots\supseteqq k_n\supseteqq\cdots\supseteqq k\,,\quad \bigcap_{n\geq 0} k_n=k$$

where k_n is algebraic over k_{n+1}.

COROLLARY 3.5. *A quasi-valuation domain is seminormal.*

PROOF. This is proved in the same way as (3.3); we give another proof which is useful for (3.10) and (3.11). Let (W, n, k) be a quasi-valuation domain dominated by a valuation (V, n, k_0). By Lemma 3.4, we have the fields k_n's between k and k_0 such that k_n is algebraic over k_{n+1} with $\cap_{n\geq 0} k_n=k$. We set $V_n=\{x\in V;\ x \bmod n\in k_n\}$. Then V_n is a prevaluation domain dominated by V such that V_n is integral over V_{n+1} with $\cup_{n\geq 0} V_n=W$. Since V_n is seminormal, so is W (cf. Hamman's criterion [4]).

In the normal case, the following theorem is well-known: A domain R is normal if and only if R is an intersection of valuation domains containing R. One can ask the following question: Let R be a seminormal domain with a quotient field K and the W_λ's prevaluation domains between R and K. Then $R=\cap W_\lambda$? Proposition 3.15 shows that the above question has really a negative answer. From now on, we discuss some facts related to this question. We start with some definitions.

DEFINITION 3.6. Let R be a domain, T an overdomain of R integral over R. We define

$$R_T^\sharp(p)=\{x\in T;\ x\in R_p+qT_p \text{ for all } q(\in \operatorname{Spec}(T)) \text{ lying over } p\}$$

and $R_T^\sharp=\cap R_T^\sharp(p)$, the intersection ranging over all prime ideals of R. We say that the ring $R_T^\sharp(p)$ is obtained by ($\sharp$)-glueing T over p and that the ring $R_T^\sharp$ is the prenormalization of R in T. If $T=\widetilde{R}$, then we call $R_T^\sharp$ the prenormalization of R and denote it by $R^\sharp$, we say that the ring R is prenormal in T if $R=R_T^\sharp$.

REMARK 3.7. A quasi-valuation domain and a prevaluation domain are prenormal.

We give some basic results.

PROPOSITION 3.8. *$R_T^\sharp$ is the largest subring R' of T containing R such that*

(3.8.1) *For all $p'\in\operatorname{Spec}(R')$, the canonical homomorphism $k(p'\cap R)\to k(p')$ is an isomorphism.*

PROOF. R_T satisfies (3.7.1): Let $q\in\operatorname{Spec}(R_T^\sharp)$ and $p=q\cap R\in\operatorname{Spec}(R)$. As $R_p\subseteq R_T^\sharp\subseteq R_p+q'T_p$ where $q(\in\operatorname{Spec}(T))$ is lying over q, $k(p)=R_p/q'T_p\cap R_p\subseteq R_T^\sharp/q'T_p\cap R_T^\sharp(=k(p))\subseteq R_p+q'T_p/q'T_p=R_p/q'T_p\cap R_p=k(p)$, hence $k(p)=k(q)$.

Now we shall prove that if R satisfies (3.7.1), then $R'\subseteq R_T^\sharp$. Let $x\in R'$, $x\notin R_T^\sharp$. Then there are $q'\in\operatorname{Spec}(R')$ and $p=q'\cap R\in\operatorname{Spec}(R)$ such that $x\notin R_p+q'R'_p$. Then $k(p)=R_p/pR_p\cong R_p+q'R'_p/q'R'_p\hookrightarrow R'_p/q'R'_p=k(q')$. If $k(p)\to k(q')$ is bijective, $R_p+q'R'_p=R'_p$, hence $x\in R'\subseteq R'_p=R_p+q'R'_p$, a contradiction.

COROLLARY 3.9. *$R_T^\sharp$ is seminormal in T.*

PROPOSITION 3.10. *A domain R is prenormal if and only if R is an intersection of prevaluation domains containing R.*

PROOF. Let $T=\cap W_\lambda$ where W_λ's are prevaluation domains containing R. Let $p'\in\operatorname{Spec}(T)$, $p=p'\cap R\in\operatorname{Spec}(R)$ and $(W,n)\supseteq R$ a prevaluation domain such that $n\cap T=p'$. If $k(n)=k(p)$ all is done; and we may assume that $k(n)\supsetneqq k(p)$; by (3.4), there is a sequence of fields $k(n)=k_0\supsetneqq k_1\supsetneqq\cdots\supsetneqq k(p)$, $\cup_{i\geq 0}k_i=k(p)$ where k_i is algebraic over k_{i+1}, hence there is a sequence of prevaluation domains $W=W_0\supseteqq(W_1, n)\supseteqq\cdots\supseteqq(W_i, n)\supseteqq\cdots\supseteqq(\bigcap_{i\geq 0}W_i, n)$ where $W_i/n=k_i$, $(\bigcap_{i\geq 0}W_i)/n=k(p)$. As $(\bigcap_{i\geq 0}W_i, n)$ T and $n\cap T=p'$, $(\bigcap_{i\geq 0}W_i, n)\supseteqq k(p')$, i.e., $k(p')=k(p)$, hence $T\subseteqq R^{\natural}=R\subseteqq T$, i.e., $R=T$.

COROLLARY 3.11. *A domain R is prenormal if and only if R is an intersection of quasi-valuations domains containing R.*

We give some criteria of prenormality.

PROPOSITION 3.12. *Let S be a multiplicative closed set in a domain R. If R is prenormal, so is $S^{-1}R$.*

PROOF. By definition,

$$(S^{-1}R)^{\natural}=\bigcap_{S\cap p=\emptyset}(S^{-1}R)^{\natural}(S^{-1}p)=\bigcap_{S\cap p=\emptyset}R^{\natural}(p)\subseteqq S^{-1}\tilde{R}\,.$$

Take $y=x/s\in(S^{-1}R)^{\natural}$ $(x\in\tilde{R},\ s\in S)$. Then we have $x\in\tilde{R}^{\natural}(p)$ for all $p\in$ Spec(R) where $p\cap S=\emptyset$. Moreover $x\in R+\tilde{p}$ for all $\tilde{p}(\in\operatorname{Spec}(\tilde{R}))$ lying over all $p(\in\operatorname{Spec}(R))$ that meet S. Therefore $x\in R+\tilde{q}$ for all $\tilde{q}(\in\operatorname{Spec}(\tilde{R}))$ lying over all $q(\in\operatorname{Spec}(R))$ that meet S. Hence $x\in R^{\natural}(q)$ for all $q(\in\operatorname{Spec}(R))$ that meet S. Therefore $x\in R^{\natural}(p)$ for all $p(\in\operatorname{Spec}(R))$, i.e., $x\in R^{\natural}=R$. Hence $sy\in R$, i.e., $y\in S^{-1}R$.

COROLLARY 3.13. *Under the circumstances $S^{-1}(R^{\natural})=(S^{-1}R)^{\natural}$.*

PROOF. Since $S^{-1}R\subseteqq S^{-1}(R^{\natural})$; by (3.12), $(S^{-1}R)^{\natural}\subseteqq(S^{-1}(R^{\natural}))^{\natural}=S^{-1}(R^{\natural})$.

COROLLARY 3.14. *Let R be a domain. The followings are equivalent.*
(3.14.1) *R is prenormal.*
(3.14.2) *R_p is prenormal for all $p\in\operatorname{Spec}(R)$.*
(3.14.3) *R_m is prenormal for all $m\in\operatorname{Max}(R)$.*

PROOF. $R^{\natural}\subseteqq\bigcap(R^{\natural})_p=\bigcap(R_p)^{\natural}=\bigcap R_p=R$.

We discuss a little further the properties of the prenormality.

PROPOSITION 3.15. *Let (R, m) be a local domain, $(T, M_1, \cdots, M_s)$ an overdomain of R integral over R. Then $|\operatorname{Max}(R_T^{\natural})|=s$.*

PROOF. By $\bigoplus_{j=1}^{s}(R+M_j)/M_j\hookrightarrow\bigoplus_{j=1}^{s}T/M_j$ and $T/J(T)\cong\bigoplus_{j=1}^{s}T/M_j$, we have $\bigoplus_{j=1}^{s}(R+M_j)/M_j\hookrightarrow T/J(T)$. Let f be the canonical epimorphism $f\colon T\to T/J(T)$. Then it is easy to show that $f^{-1}(\bigoplus_{j=1}^{s}(R+M_j)/M_j)=\bigcap_{j=1}^{s}(R+M_j)$. Hence, $\bigcup_{j=1}^{s}(R+M_j)\to\bigoplus_{j=1}^{s}(R+M_j)/M_j$ is surjective. Thus $M_j'=(\bigcap_{j=1}^{s}(R+M_j))\cap M_j$ is a maximal ideal of $\bigcup_{j=1}^{s}(R+M_j)$. Moreover $M_j'\neq M_i'$ $(j\neq i)$. Therefore $|\operatorname{Max}(\bigcup_{j=1}^{s}(R+M_j))|=s$.

The next theorem follows directly from (3.15).

THEOREM 3.16. *Let (R, m) be a one-dimensional noetherian local domain. If R is prenormal, then R is a prevaluation domain.*

We introduce at this point some definitions.

DEFINITION 3.17 ([2], 4.1). Let X be a free abelian group. A ring R is said to be quasi-normal if and only if the canonical homomorphism $\mathrm{Pic}(R) \to \mathrm{Pic}(R[X])$ is an isomorphism.

DEFINITION 3.18 ([2], 4.3). A domain R is locally unibranche (LUB) if and only if R_m is unibranche for all $m \in \mathrm{Max}(R)$, i.e., the canonical map $\mathrm{Max}(\tilde{R}) \to \mathrm{Max}(R)$ is bijective.

REMARKS 3.19 ([2], 4.2).
(3.19.1) A normal ring is quasinormal.
(3.19.2) A quasinormal ring is seminormal.

THEOREM 3.20. *Let R be a one-dimensional noetherian domain such that R is a finitely generated R-module. Then the followings are equivalent.*
(3.20.1) *R is quasinormal.*
(3.20.2) *R_m is quasinormal for all $m \in \mathrm{Max}(R)$.*
(3.20.3) *R_m is seminormal and LUB for all $m \in \mathrm{Max}(R)$.*
(3.20.4) *R is prenormal.*
(3.20.5) *R_m is prenormal for all $m \in \mathrm{Max}(R)$.*
(3.20.6) *R_m is a prevaluation domain for all $m \in \mathrm{Max}(R)$.*

EXAMPLE 3.21. Let Q be the field of rationals, C the field of complexes, and X, Y indeterminates.
(3.21.1) $C[X, Y]/(X^2 - Y^3)$ is seminormal, but it is not prenormal.
(3.21.2) $Q[[X, Y]]/(X^2 + Y^2)$ is prenormal.

REMARK 3.22. From (3.20), in case R is a one-dimensional noetherian domain with finite normalization, we can give actually the quasinormalization of R as the prenormalization of R.

LEMMA 3.23 ([1], 5.6). *Let R be a noetherian domain with finite normalization, X a free abelian group and I an invertible ideal of $R[X]$ such that $I_0 = I \cap R$. Let $p_1, \cdots, p_t$ be the prime divisors of I and $q_j = p_j \cap R$. Then $I = I_0 R[X]$ if and only if each $I \cdot R_{q_j}[X]$ is principal.*

Now we have the corollary of (3.20).

PROPOSITION 3.24. *Let R be a noetherian S_2-domain with finite normalization. Then R is quasinormal if and only if R_p is a prevaluation domain for all prime ideals p with* $\mathrm{ht}(p)=1$.

§ 4. M-prenormality and quasinormality.

In this section we give some properties of the M-prenormal domains and show that any M-prenormal domain is quasinormal. Our notations and terminologies are much the same as those in [1] and we assume that X is a free abelian group. All rings are assumed to be noetherian with finite normalization.

DEFINITION 4.1. Let R be a domain, T an overdomain of R integral over R. We define $R_T^b = \cap R_T^{\natural}(m)$, the intersection ranging over all maximal ideals of R. We say that the ring R_T^b is the M-prenormalization of R in T. If $T=\tilde{R}$, then we call R_T^b the M-prenormalization of R and denote it by R^b; we say that R is M-prenormal in T if $R=R_T^b$.

From the definition and (3.15) we derive:

PROPOSITION 4.2. *Any M-prenormal domain is prenormal and LUB.*

PROPOSITION 4.3. *Let (R, m) be an M-prenormal local domain which is not normal and $x \in R\backslash U(R)$. Then $\tilde{R}\backslash U(\tilde{R})=m$ and m is a prime divisor of xR.*

PROOF. By (4.2), R is LUB, hence $R=R^b=R+(\tilde{R}\backslash U(\tilde{R}))$, i.e., $\tilde{R}\backslash U(\tilde{R})=m$. Take $u \in U(\tilde{R})\backslash U(R)$, Then $xR\colon ux=m$. This means that m is a prime divisor of xR.

We note next that it is possible to localize to preserve the M-prenormality.

PROPOSITION 4.4. *Let (R, m) be an M-prenormal local domain. Then R_p is M-prenormal for all $p \in \mathrm{Spec}(R)$.*

PROOF. Take $y=x/s \in \tilde{R}_p\backslash R_p$ where $x \in \tilde{R}\backslash R$ and $s \in R\backslash p$. By (4.3) $x^{-1} \in \tilde{R}\backslash R$, hence $y^{-1}=sx^{-1} \in \tilde{R}_p$, i.e., $y \in U(\tilde{R}_p)$, i.e., $\tilde{R}_p\backslash R_p \subseteq U(R_p)$. Thus there is a unique prime ideal p in R lying over p, hence $\tilde{p}\tilde{R}_p=pR_p$. Therefore $(R_p)^b=\tilde{R}_p \cap (R_p+\tilde{p}\tilde{R}_p)=\tilde{R}_p \cap R_p=R_p$.

COROLLARY 4.5. *Under the circumstances R_p is unibranche for all $p \in \mathrm{Spec}(R)$.*

The next is a basic structure theorem for seminormal rings due to Traverso.

THEOREM 4.6 ([3]). *Let R be reduced seminormal ring.*

(4.6.1) *There is a sequence of rings*

$$\tilde{R}=B_0 \supseteqq B_1 \supseteqq \cdots \supseteqq B_n=R$$

where B_{i+1} is obtained from B_i by a finite number of glueings over prime ideals of R of height $i+1$.

(4.6.2) *If $x \in R$ is not a zero-divisor, then each associated prime divisor of xR has height $\leqq n$.*

Then we have;

PROPOSITION 4.7. *Let R be an n-dimensional domain which is not normal.*

(4.7.1) *There is an element x in R such that xR has a prime divisor of height n.*

(4.7.2) *There is a sequence of domains*

$$\tilde{R}=C_0 \supseteqq C_1 \supseteqq \cdots \supseteqq C_n=R$$

where C_{i+1} is obtained from C_i by a finite number of (#)-glueing over prime ideals of R height $i+1$.

PROOF. (4.7.1): Let m be a maximal ideal of height n. Take $x \in m$. Then, by (4.3), mR_m is a prime divisor of xR_m of height n, hence m is a prime divisor of xR of height n.

(4.7.2): Since R is seminormal, then, by (4.6.1), there is a sequence of domains

$$\tilde{R}=B_0 \supseteqq B_1 \supseteqq \cdots \supseteqq B_k=R .$$

Let

$$B_i=R^{+}_{B_{i-1}}(p_{i1}) \cap \cdots \cap R^{+}_{B_{i-1}}(p_{in_i})$$

where $\mathrm{ht}\,(p_{ij})=i$. Here we define

$$C_0=B_0 , \qquad C_i=R^{\#}_{C_{i-1}}(p_{i1}) \cap \cdots \cap R^{\#}_{C_{i-1}}(p_{in_i}) .$$

By (4.5), $R^{\#}_{C_{i-1}}(p_{ij})=R^{+}_{C_{i-1}}(p_{ij})$. From $C_0=B_0$, $B_i=C_i$ for all i. Thus $C_k=R$ for some k. If $k<n$, by (4.6.2) and (4.3), a contradiction, hence $k=n$.

DEFINITION 4.8 ([1], 5.3; [3], 3.1). If S be a multiplicative closed set, we shall write inv (R, S) for the subgroup of the group of invertible fractionary R-ideals spanned by the integral invertible R-ideals that meet S.

PROPOSITION 4.9 ([1], 5.5; [3], 3.4). *Let R be a domain and S a multiplicative closed set in R. Assume that if $s \in S$ and p is a prime divisor of sR, then* $\mathrm{Pic}(R_p[X])=0$. *Then* $\mathrm{inv}(R, S) \to \mathrm{inv}(R[X], S)$ *is an isomorphism.*

PROPOSITION 4.10. *Under the assumptions as in* (4.9), *assuming that $S^{-1}R$ is semilocal, there is an exact sequence*

$$0 \longrightarrow \mathrm{Pic}(R) \longrightarrow \mathrm{Pic}(R[X]) \longrightarrow \mathrm{Pic}(S^{-1}R[X)] \,.$$

(4.9) and (4.10) are proved in the same way as [1], 5.5, 5.7 or [3], 3.3, 3.4.

REMARK 4.11. We don't know whether the domain assumption in (4.9) and (4.10) can be deleted.

The next lemma is needful to prove (4.13).

LEMMA 4.12 ([3], 3.5). *Let T be a finite overring of a ring R and I the conductor of R in T. Let f be the inclusion of R in T and $\bar{f}$ the inclusion of $\bar{R}=R/I$ in $\bar{T}=T/I$. Then* $\mathrm{Pic}(\Phi f) \to \mathrm{Pic}(\Phi \bar{f})$ *is an isomorphism.*

We can now state the following theorem.

THEOREM 4.13. *Any M-prenormal domain is quasinormal.*

PROOF. By (4.7.2) we have a sequence of domains

$$\tilde{R}=C_0 \supsetneqq C_1 \supsetneqq \cdots \supsetneqq C_n=R$$

where C_i is obtained from C_{i-1} by a finite sequence of (#)-glueings over prime ideals of R of height i and $n=\text{krull-dim}(R)$.

For $n=0$ the assertion is vacuous, so we assume $n>0$ and use induction on n. By induction we may assume that $\tilde{R}=C_{n-1}$, $R=C_n$. Let $I=C_{n-1}\colon C_n$ and $p_1, \cdots, p_t$ the prime divisors of I in R. Then, for all j, $\mathrm{ht}(p_j)=n$. Let $S=R\backslash(p_1 \cup \cdots \cup p_t)$. Pick $s \in S$. Let p be a prime divisor of sR. As $p \not\supseteq I$, $R_p=(C_{n-1})_p$. By induction $\mathrm{Pic}(R_p[X])=\mathrm{Pic}((C_{n-1})_p[X])=0$. Therefore, by (4.10), we have an exact sequence

$$0 \longrightarrow \mathrm{Pic}(R) \longrightarrow \mathrm{Pic}(R[X]) \longrightarrow \mathrm{Pic}(S^{-1}R[X]) \,.$$

We are to prove $\mathrm{Pic}(S^{-1}R[X])=0$. Let $\bar{R}=S^{-1}R$, $\bar{T}=S^{-1}(C_{n-1})$, $R'=S^{-1}R/S^{-1}I$ and $T'=S^{-1}(C_{n-1})/S^{-1}I$. f, f', f_* and f'_* denote the inclusion $\bar{R} \to \bar{T}$, $R' \to T'$,

$\bar{R}[X] \to \bar{T}[X]$ and $R'[X] \to T'[X]$ respectively. Since R' and T' are direct sum of fields, $\mathrm{Pic}\,(R'[X]) = \mathrm{Pic}\,(T'[X]) = 0$. By induction, $0 = \mathrm{Pic}\,(\bar{T}) = \mathrm{Pic}\,(\bar{T}[X])$. Here we note that, for a commutative square of rings,

$$\begin{array}{ccc} R_1 & \xrightarrow{f} & R_2 \\ \downarrow & & \downarrow \\ R_3 & \xrightarrow{g} & R_4 \end{array} \tag{4.13.1}$$

there is the map of exact sequences induced by (4.13.1)

$$\begin{array}{ccccccccc} U(R_1) & \longrightarrow & U(R_2) & \longrightarrow & \mathrm{Pic}\,(\Phi f) & \longrightarrow & \mathrm{Pic}\,(R_1) & \longrightarrow & \mathrm{Pic}\,(R_2) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ U(R_3) & \longrightarrow & U(R_4) & \longrightarrow & \mathrm{Pic}\,(\Phi g) & \longrightarrow & \mathrm{Pic}\,(R_3) & \longrightarrow & \mathrm{Pic}\,(R_4)\,. \end{array} \tag{4.13.2}$$

Therefore we have

$$U(\bar{R}) \longrightarrow U(\bar{T}) \longrightarrow \mathrm{Pic}\,(\Phi f) \longrightarrow 0 \longrightarrow 0 \tag{4.13.3}$$

$$U(R') \longrightarrow U(T') \longrightarrow \mathrm{Pic}\,(\Phi f') \longrightarrow 0 \longrightarrow 0 \tag{4.13.4}$$

$$U(\bar{R}[X]) \longrightarrow U(\bar{T}[X]) \longrightarrow \mathrm{Pic}\,(\Phi f_*) \longrightarrow \mathrm{Pic}\,(\bar{R}[X]) \longrightarrow 0 \tag{4.13.5}$$

$$U(R'[X]) \longrightarrow U(T'[X]) \longrightarrow \mathrm{Pic}\,(\Phi f'_*) \longrightarrow 0 \longrightarrow 0 \tag{4.13.6}$$

and, by [1], 5.12, we have

$$\begin{aligned} U(\bar{R}[X]) &= U(\bar{R}) + X \otimes_{Z} H_0(\bar{R}) \\ U(\bar{T}[X]) &= U(\bar{T}) + X \otimes_{Z} H_0(\bar{T}) \end{aligned} \tag{4.13.7}$$

$$H_0(\bar{R}) = H_0(\bar{T}) \tag{4.13.8}$$

$$\begin{aligned} U(R'[X]) &= U(R') + X \otimes_{Z} H_0(R') \\ U(T'[X]) &= U(T') + X \otimes_{Z} H_0(T')\,; \end{aligned} \tag{4.13.9}$$

moreover, by the M-prenormality ((4.5) and (4.7.2)),

$$H_0(R') = H_0(T')\,. \tag{4.13.10}$$

Then, by (4.13.4) and (4.13.6), $\mathrm{Pic}\,(\Phi f') \to \mathrm{Pic}\,(\Phi f'_*)$ is an isomorphism. On the other hand, by (4.12), $\mathrm{Pic}\,(\Phi f) \to \mathrm{Pic}\,(\Phi f')$ and $\mathrm{Pic}\,(\Phi f_*) \to \mathrm{Pic}\,(\Phi f'_*)$ are isomorphism, hence $\mathrm{Pic}\,(\Phi f) \to \mathrm{Pic}\,(\Phi f_*)$ is an isomorphism. Applying the five lemma to (4.13.3) and (4.13.5), we have $\mathrm{Pic}\,(\bar{R}[X]) = 0$, i.e., $\mathrm{Pic}\,(S^{-1}R[X]) = 0$. This completes the proof.

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