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The Space W_2 of Isometric Minimal Immersions of the Three-Dimensional Sphere into Spheres

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Introduction

An immersion f of an m-dimensional sphere S^m into an M-dimensional sphere $S^M(r)$ is called an isometric minimal immersion $f: S^m(1) \to S^M(r)$ if $f: S^m \to S^M(r)$ is a minimal immersion, and, at the same time, $f: S^m(1) \to S^M(r)$ is an isometric immersion. Some special cases of such immersions were studied by E. Calabi [1] and by M. do Carmo and N. Wallach [2] and the general cases by M. do Carmo and N. Wallach [3]. In the present introduction we quote, with a little change of style, those results in [3] which have intimate relation with the present paper.

When m is given. essentially important cases of isometric minimal immersions f of a standard m-sphere $S^m(1)$ into a sphere $S^M(r)$ are the following ones. For each positive integer $s>1$ there exists a class of isometric minimal immersions

$$f_s: S^m(1) \to S^{n-1}(r)$$

such that

$$n=(2s+m-1)(s+m-2)!/(s!(m-1)!)\ ,$$
$$r^2=m/(s(s+m-1))\ .$$

We consider the cases $m\geqq 3$ and $s\geqq 4$. Then in each of the classes mentioned above there exist three kinds of isometric minimal immersions, namely standard minimal immersions, nonstandard full isometric minimal immersions and non full isometric minimal immersions.

Suppose we have fixed a rectangular coordinate system in $\boldsymbol{R}^n$ and consider $S^{n-1}(r)$ as the hypersphere of radius r whose center is the origin 0. Then the image $f_s(S^m(1))$ is expressed by n coordinates $f^A (A=1, \cdots, n)$ which are eigenfunctions of the Laplacian Δ_m on $S^m(1)$ with eigenvalue

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$\lambda_s = s(s+m-1)$, satisfying

$$\sum_A (f^A)^2 = r^2$$

and the isometry condition.

We also see in [3] that the set of equivalence classes of isometric minimal immersions is parametrized by a compact convex body L in a certain vector space W_2. The interior points of L correspond to the equivalence classes of full isometric minimal immersions and the boundary points of L correspond to those of non full isometric minimal immersions.

Such minimal immersions were studied or are being studied by several mathematicians (see [4] and [7]). The present author also studied the same object in his own way [5] and it became clear that the vector space W_2 corresponds to a space of some bi-symmetric tensors in R^{m+1} of degree $2s$ which we call $D^m_{s,s}$.

The purpose of the present paper is to investigate in more detail the space $D^3_{4,4}$, namely, the space W_2 of do Carmo and Wallach for the case $m=3$, $s=4$. Thus it is proved that $\dim D^3_{4,4}=18$ and that there exist mappings φ_J and φ_I from the space H^3_4 of harmonic polynomials of R^3 of degree 4 into $D^3_{4,4}$ such that, if $\{a_1, \cdots, a_9\}$ is an orthonormal basis of H^3_4, then $\{\varphi_J a_1, \cdots, \varphi_J a_9, \varphi_I a_1, \cdots, \varphi_I a_9\}$ is an orthonormal basis of $D^3_{4,4}$.

REMARK 1. The use of letters m and n in [5] and in the present paper differs from that in [3]. Also the curvature of the sphere to be immersed differs from that in [3] being fixed to be 1. This is preferred only with the purpose of simplicity.

REMARK 2. The inner product in H^3_4 is defined in §5.

In §1 we reproduce some of the results of the previous paper [5] and explain the linear space $D^m_{s,s}$. As R^4, where the standard sphere $S^3(1)$ is embedded as the unit hypersphere, admits the group $SO(4)$ which has the well-known special property, attention is attracted to this fact in §2. There we introduce six transformations $J_1, J_2, J_3, I_1, I_2, I_3$ which play important role in our study. When a harmonic polynomial of degree 4 and with three variables is given, we get from it some elements of $D^3_{4,4}$. This fact is explained in §3. From this result we can deduce mappings φ_J and φ_I of the space H^3_4 into $D^3_{4,4}$. These mappings are studied in §4 and their images D_J, D_I in §5. The fact that the dimension of W_2 is not less than 18 for $m\geqq 3$, $s\geqq 4$ was established by do Carmo and Wallach [3]. But we wanted to verify $\dim D^3_{4,4}=18$. As we could not find a short cut proof of this fact, without which we cannot state the main results in §7, it is proved through a lengthy calculation in §6. In §8 parameters given in §6 are evaluated for $\varphi_J a$ and $\varphi_I a$ in a typical case. In §9 the action of some subgroups of $SO(4)$ on $D^3_{4,4}$ is studied.

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§ 1. Preliminaries.

Let us consider $S^m(1)$ as the unit hypersphere of $\boldsymbol{R}^{m+1}$ where we have fixed an orthonormal basis $\{e_1, \cdots, e_{m+1}\}$. On the other hand let us take in R^n an orthonormal basis $\{\tilde{e}_1, \cdots, \tilde{e}_n\}$ and a hypersphere $S^{n-1}(r)$ where the center is the origin and the radius is r. We use indices as follows:

$$A, B, C, \cdots = 1, \cdots, n ,$$
$$a, b, c, \cdots, h, i, j, \cdots = 1, \cdots, m+1 ,$$
$$\alpha, \beta, \gamma, \cdots, \kappa, \lambda, \mu, \cdots = 1, \cdots, m ,$$

and adopt the usual summation convention if possible.

Let $f\colon S^m(1) \to S^{n-1}(r)$ be an immersion such that

$$f(u) = f^A(u)\tilde{e}_A , \quad u \in S^m(1) .$$

Then $\sum_A (f^A)^2 = r^2$. If f^A satisfy

$$\Delta_m f^A = \lambda_s f^A , \quad \lambda_s = s(s+m-1) \tag{1.1}$$

where Δ_m is the Laplacian on the standard sphere $S^m(1)$, then f is called an immersion of order s. A theorem of Takahashi [6] states that, if f is an isometric minimal immersion, then f is necessarily an immersion of order s.

As $S^m(1)$ is the unit hypersphere of $\boldsymbol{R}^{m+1}$, we can put $u = u^i e_i$, $\sum_i (u^i)^2 = 1$, and the functions u^h are eigenfunctions of Δ_m satisfying $\Delta_m u^h = m u^h$. It is well-known that, for each eigenfunction ψ of Δ_m satisfying $\Delta_m \psi = \lambda_s \psi$, there exists just one harmonic polynomial

$$F = F_{i_1 \cdots i_s} X^{i_1} \cdots X^{i_s}$$

of degree s such that

$$\psi(u) = F_{i_1 \cdots i_s} u^{i_1}(u) \cdots u^{i_s}(u) .$$

The number $n = (2s+m-1)(s+m-2)!/(s!(m-1)!)$ gives the dimension of the space H^{m+1}_s of harmonic polynomials of degree s in $\boldsymbol{R}^{m+1}$. It is also clear that a harmonic polynomial F of degree s determines a symmetric tensor t such that $t(v, \cdots, v)=F(v, \cdots, v)$, hence we can consider F as a symmetric tensor satisfying

$$\sum_i F(e_i, e_i, v_3, \cdots, v_s)=0 ,$$

where $v_3, \cdots, v_s$ are arbitrary vectors of R^{m+1}.

This fact implies that, when an immersion f_s: $S^m(1) \to S^{n-1}(r)$ of order s is given, we have n tensors F^A such that

$$f_s(u)=F^A(u, \cdots, u)\tilde{e}_A , \quad \sum_i F^A(e_i, e_i, v_3, \cdots, v_s)=0 .$$

In terms of the components with respect to the frame $\{e_1, \cdots, e_{m+1}\}$ F^A satisfy

$$\sum_i F^A_{iij_3\cdots j_s}=0 . \tag{1.2}$$

F^A are called the tensors of degree s associated with the immersion f_s.

REMARK. In the present paper we do not use the letter p for a point of $S^m(1)$.

DEFINITION OF $B^m_{s,s}$. Now we define a linear space $B^m_{s,s}$ by saying that $C \in B^m_{s,s}$ if and only if the tensor C of degree $2s$ satisfies the following conditions where $v_1, \cdots, v_{2s}$ are arbitrary vectors of R^{m+1}:

(i) $C(v_1, \cdots, v_s; v_{s+1}, \cdots, v_{2s})$ is symmetric both in $v_1, \cdots, v_s$ and in $v_{s+1}, \cdots, v_{2s}$,

(ii) $C(v_1, \cdots, v_s; v_{s+1}, \cdots, v_{2s})=C(v_{s+1}, \cdots, v_{2s}; v_1, \cdots, v_s)$

(iii) $\sum_i C(e_i, e_i, v_3, \cdots, v_s; v_{s+1}, \cdots, v_{2s})=0$.

$B^m_{s,s}$ is called the space of bi-symmetric harmonic tensors of bi-degree (s, s).

DEFINITION OF $D^m_{s,s}$. We define a linear subspace $D^m_{s,s}$ in $B^m_{s,s}$ as follows: $C \in D^m_{s,s}$ if and only if $C \in B^m_{s,s}$ and satisfies

(iv) $C(w, w, v, \cdots, v; v, \cdots, v)=0$

for arbitrary vectors v and w of R^{m+1}.

DEFINITION OF $f_{s,s}$. When F^A are the tensors of degree s associated with an immersion f_s of order s, we define $f_{s,s}$ by

$$f_{s,s}=\sum_A F^A \otimes F^A . \tag{1.3}$$

$f_{s,s}$ belongs to $B^m_{s,s}$ and is called the tensor of degree $2s$ associated with the immersion f_s.

Let f_s and f'_s be isometric minimal immersions and let $f_{s,s}$ and $f'_{s,s}$ be tensors of degree $2s$ associated with f_s and f'_s respectively. Then $f_{s,s}=f'_{s,s}$ if and only if f_s and f'_s belong to the same equivalence class (see [5] Theorem 3.3). The tensor of degree $2s$ associated with standard minimal immersions h_s is denoted by $h_{s,s}$. Then, for any isometric minimal immersion f_s we have $f_{s,s}-h_{s,s}\in D^m_{s,s}$ ([5] §6). Conversely, if $\hat{d}_{s,s}\in D^m_{s,s}$ and if $t_1<t<t_2$ where (t_1, t_2) is a certain interval depending on $\hat{d}_{s,s}$, then $h_{s,s}+t\hat{d}_{s,s}$ is the tensor $f_{s,s}$ of degree $2s$ associated with some isometric minimal immersion $f_s\colon S^m(1)\to S^{n-1}(r)$. This suggests that we can find many important properties of such immersions from the properties of $D^m_{s,s}$.

Let g be any element of $SO(m+1)$. For any element C of $D^m_{s,s}$ let us define gC by

$$gC(v_1, \cdots, v_s; v_{s+1}, \cdots, v_{2s})=C(g^{-1}v_1, \cdots, g^{-1}v_s; g^{-1}v_{s+1}, \cdots, g^{-1}v_{2s}) \tag{1.4}$$

where $v_1, \cdots, v_{2s}$ are arbitrary vectors of $\boldsymbol{R}^{m+1}$. Then it is easy to verify that $gC\in D^m_{s,s}$.

For any tensors T_1, T_2 of R^{m+1} of the same degree the inner product $\langle T_1, T_2\rangle$ is defined as usual. Then the following lemma is easily verified.

LEMMA 1. *Let C_1, C_2 be elements of $D^m_{s,s}$. Then $\langle gC_1, gC_2\rangle=\langle C_1, C_2\rangle$.*

§ 2. Some orthogonal transformations of $\boldsymbol{R}^4$.

Let us fix an orthonormal basis $\{e_1, e_2, e_3, e_4\}$ in $\boldsymbol{R}^4$. On the other hand, we take a rectangular coordinate system in $\boldsymbol{R}^3$ and express a point p of R^3 by $p=(x, y, z)$. If we take linear transformations $J_p=xJ_1+yJ_2+zJ_3$ defined by

$$\begin{aligned} J_pe_1&=-xe_2+ye_3-ze_4\ ,\\ J_pe_2&=xe_1-ye_4-ze_3\\ J_pe_3&=-xe_4-ye_1+ze_2\ ,\\ J_pe_4&=xe_3+ye_2+ze_1\ ,\end{aligned}$$

then J_p is an orthogonal transformation when p is a point of the unit sphere $S^2(1)$. As it is well-known, J_1, J_2, J_3 satisfy $J_2J_3=-J_3J_2=J_1$, $J_3J_1=-J_1J_3=J_2$, $J_1J_2=-J_2J_1=J_3$.

Similarly, let $I_p=xI_1+yI_2+zI_3$ be defined by

$$\begin{aligned} I_pe_1&=-xe_2+ye_3+ze_4\ ,\\ I_pe_2&=xe_1+ye_4-ze_3\ ,\\ I_pe_3&=xe_4-ye_1+ze_2\ ,\\ I_pe_4&=-xe_3-ye_2-ze_1\ .\end{aligned}$$

Then I_p is an orthogonal transformation when p is a point of $S^2(1)$. We can easily see that $I_2I_3=-I_3I_2=I_1$, $I_3I_1=-I_1I_3=I_2$, $I_1I_2=-I_2I_1=I_3$ and moreover

$$J_\kappa I_\lambda = I_\lambda J_\kappa \quad (\kappa, \lambda=1, 2, 3)\ , \tag{2.1}$$

$$\sum_i \langle J_\kappa e_i, I_\lambda e_i\rangle = 0 \tag{2.2}$$

where $\langle\ ,\ \rangle$ denotes the inner product in $\boldsymbol{R}^4$.

Furthermore, J_κ and I_κ satisfy

$$J_\mu J_\lambda + J_\lambda J_\mu = -2\delta_{\mu\lambda} J_0\ , \quad I_\mu I_\lambda + I_\lambda I_\mu = -2\delta_{\mu\lambda} I_0 \tag{2.3}$$

where $I_0=J_0$ is the identity transformation. Then the set $\{aI_0+bI_1+cI_2+dI_3,\ a^2+b^2+c^2+d^2=1\}$ is a subgroup of $SO(4)$. Let us denote this subgroup by O_I. Similarly, $\{aJ_0+bJ_1+cJ_2+dJ_3,\ a^2+b^2+c^2+d^2=1\}$ is another subgroup of $SO(4)$. Let us denote this by O_J. O_I and O_J commute and generate $SO(4)$.

§ 3. Harmonic polynomials of $\boldsymbol{R}^3$ and elements of $D^3_{4,4}$.

Here and in the sequel we use indices as follows:

$$a, b, c, \cdots, h, i, j, \cdots =1, 2, 3, 4\ ,$$
$$\alpha, \beta, \gamma, \cdots, \kappa, \lambda, \mu, \cdots =1, 2, 3$$

and adopt the usual summation convention if possible. Using x^1, x^2, x^3, hence x^κ collectively, for the rectangular coordinates in $\boldsymbol{R}^3$, we see that a harmonic polynomial $a(x)$ in $\boldsymbol{R}^3$ of degree 4 can be written as

$$a(x) = a_{\kappa\lambda\mu\nu} x^\kappa x^\lambda x^\mu x^\nu \tag{3.1}$$

where $a_{\kappa\lambda\mu\nu}$ are symmetric in the lower indices and satisfy

$$\sum_\kappa a_{\kappa\kappa\mu\nu} = 0\ . \tag{3.2}$$

When a harmonic polynomial $a(x)$ is given, let us define a tensor $C_J^{(a)}$ of degree 8 by

$$C_J^{(a)}(v_1, v_2, v_3, v_4;\ w_1, w_2, w_3, w_4) = \mathscr{S}_v \mathscr{S}_w a^{\kappa\lambda\mu\nu} \langle J_\kappa w_1, v_1\rangle \langle J_\lambda w_2, v_2\rangle \langle J_\mu w_3, v_3\rangle \langle J_\nu w_4, v_4\rangle \tag{3.3}$$

where $\mathscr{S}_w$ (resp $\mathscr{S}_v$) denotes the symmetrizer with respect to w_1, w_2, w_3, w_4 (resp v_1, v_2, v_3, v_4) and $a^{\kappa\lambda\mu\nu}=a_{\kappa\lambda\mu\nu}$. Then we can prove that $C_J^{(a)}$ is an element of $D^3_{4,4}$ as follows.

$C_J^{(a)}$ satisfies the condition (i) because of $\mathscr{S}_w\mathscr{S}_v$. (ii) is satisfied because of $\langle J_\kappa w, v\rangle = -\langle J_\kappa v, w\rangle$. As we have

$$\begin{aligned}\sum_i \langle J_\kappa w_1, e_i\rangle\langle J_\lambda w_2, e_i\rangle &= \langle J_\kappa w_1, J_\lambda w_2\rangle \\ &= \langle w_1, w_2\rangle \qquad \text{if} \quad \kappa=\lambda\ , \\ &= -\langle J_\kappa w_2, J_\lambda w_1\rangle \quad \text{if} \quad \kappa\neq\lambda\ ,\end{aligned}$$

we get, in view of (3.2),

$$\sum_i C_J^{(a)}(e_i, e_i, v_3, v_4; w_1, w_2, w_3, w_4) = 0\ ,$$

hence (iii) is satisfied. That $C_J^{(a)}$ satisfies the condition (iv) is easy to see.

Similarly, we can define an element $C_I^{(a)}$ of $D^3_{4,4}$ by

$$\begin{aligned}(3.4) \qquad & C_I^{(a)}(v_1, v_2, v_3, v_4; w_1, w_2, w_3, w_4) \\ &= \mathscr{S}_v\mathscr{S}_w a^{\kappa\lambda\mu\nu}\langle I_\kappa w_1, v_1\rangle\langle I_\lambda w_2, v_2\rangle\langle I_\mu w_3, v_3\rangle\langle I_\nu w_4, v_4\rangle\ .\end{aligned}$$

We see immediately that (3.3) shows the existence of a linear map φ_J of the space H^3_4 of harmonic polynomials of $\boldsymbol{R}^8$ of degree 4 into $D^3_{4,4}$ such that $\varphi_J(a) = C_J^{(a)}$. Similarly we have a linear map φ_I. Let us define D_J, D_I by $D_J = \varphi_J(H^3_4)$, $D_I = \varphi_I(H^3_4)$. These are linear subspaces of $D^3_{4,4}$.

§ 4. Some properties of the mappings φ_J and φ_I.

The inner product $\langle A, B\rangle$ for the elements A, B of $D^3_{4,4}$ can be written

$$\begin{aligned}(4.1) \qquad \langle A, B\rangle = \sum_i^* \sum_j^* & A(e_{i_1}, e_{i_2}, e_{i_3}, e_{i_4}; e_{j_1}, e_{j_2}, e_{j_3}, e_{j_4}) \\ & \cdot B(e_{i_1}, e_{i_2}, e_{i_3}, e_{i_4}; e_{j_1}, e_{j_2}, e_{j_3}, e_{j_4})\end{aligned}$$

where

$$\sum_i^* = \sum_{i_1 i_2 i_3 i_4}\ , \quad \sum_j^* = \sum_{j_1 j_2 j_3 j_4}\ .$$

First we calculate the inner product $\langle C_J^{(a)}, C_J^{(b)}\rangle$. As $a^{\kappa\lambda\mu\nu}$ are symmetric in the upper indices, we can write (3.3) in the form

$$\begin{aligned}& C_J^{(a)}(v_1, v_2, v_3, v_4; w_1, w_2, w_3, w_4) \\ &\quad = (1/24)a^{\kappa\lambda\mu\nu}\sum_P \langle J_\kappa w_{P(1)}, v_1\rangle\langle J_\lambda w_{P(2)}, v_2\rangle\langle J_\mu w_{P(3)}, v_3\rangle\langle J_\nu w_{P(4)}, v_4\rangle\end{aligned}$$

where P is a permutation of 1, 2, 3, 4 and $\sum_P$ means summation over all permutations. As $b^{\alpha\beta\gamma\delta}$ are also symmetric in the upper indices, we have, taking (4.1) into account, the following formula:

$$\begin{aligned}&\langle C_J^{(a)}, C_J^{(b)}\rangle\\ &\quad =(1/24)a^{\kappa\lambda\mu\nu}b^{\alpha\beta\gamma\delta}\sum_i^{*}\sum_j^{*}\sum_P \langle J_\kappa e_{i_1}, e_{j_1}\rangle\langle J_\lambda e_{i_2}, e_{j_2}\rangle\langle J_\mu e_{i_3}, e_{j_3}\rangle\langle J_\nu e_{i_4}, e_{j_4}\rangle\\ &\qquad\cdot \langle J_\alpha e_{i_{P(1)}}, e_{j_1}\rangle\langle J_\beta e_{i_{P(2)}}, e_{j_2}\rangle\langle J_\gamma e_{i_{P(3)}}, e_{j_3}\rangle\\ &\qquad\cdot \langle J_\delta e_{i_{P(4)}}, e_{j_4}\rangle .\end{aligned}$$

Then we get

$$\langle C_J^{(a)}, C_J^{(b)}\rangle=(1/24)\sum_P c_P , \tag{4.2}$$

$$\begin{aligned}c_P=a^{\kappa\lambda\mu\nu}b^{\alpha\beta\gamma\delta}\sum_i^{*}&\langle J_\alpha J_\kappa e_{i_1}, e_{i_{P(1)}}\rangle\langle J_\beta J_\lambda e_{i_2}, e_{i_{P(2)}}\rangle\\ &\cdot\langle J_\gamma J_\mu e_{i_3}, e_{i_{P(3)}}\rangle\langle J_\delta J_\nu e_{i_4}, e_{i_{P(4)}}\rangle\end{aligned} \tag{4.2$_P$}$$

in view of

$$\sum_j \langle J_\kappa w, e_j\rangle\langle J_\alpha v, e_j\rangle=\langle J_\kappa w, J_\alpha v\rangle=-\langle J_\alpha J_\kappa w, v\rangle .$$

We now calculate c_P.

In case P is the trivial permutation, namely, $P(i)=i$ for $i=1, 2, 3, 4$, c_P is written c_e and we have

$$c_e=4^4 a_{\kappa\lambda\mu\nu}b^{\kappa\lambda\mu\nu}$$

because of the identity

$$\sum_i \langle J_\alpha J_\kappa e_i, e_i\rangle=-4\delta_{\alpha\kappa} .$$

Consider the case P fixes two of the numbers 1, 2, 3, 4, for example, the case $P(1)=1$, $P(2)=2$, $P(3)=4$, $P(4)=3$. As we have

$$\begin{aligned}&\sum_{i,j}\langle J_\gamma J_\mu e_i, e_j\rangle\langle J_\delta J_\nu e_j, e_i\rangle\\ &\quad=\sum_{i,j}\langle J_\gamma J_\mu e_i, e_j\rangle\langle e_j, J_\nu J_\delta e_i\rangle\\ &\quad=\sum_i\langle J_\gamma J_\mu e_i, J_\nu J_\delta e_i\rangle\\ &\quad=\sum_i\langle J_\delta J_\nu J_\gamma J_\mu e_i, e_i\rangle\\ &\quad=-\sum_i\langle J_\nu J_\delta J_\gamma J_\mu e_i, e_i\rangle+8\delta_{\delta\nu}\delta_{\gamma\mu}\end{aligned}$$

because of (2.3), and, as $b^{\alpha\beta\gamma\delta}$ are symmetric in the upper indices and satisfy (3.2), we get, in view of (2.3),

$$c_P=c_e/2 .$$

Let $n=n_1+\cdots+n_p$ be a partition of n and let us denote this by $(n_1, \cdots, n_p)$. By a subdivision of the set $S=\{1, \cdots, n\}$ subordinate to the partition $(n_1, \cdots, n_p)$ we mean a subdivision such that $S=S_1+\cdots+S_p$ where, for each $i(=1, \cdots, p)$ S_i is a subset of S with n_i elements. An element g of the symmetric group $\mathfrak{S}_n$ is said to be subordinate to the partition $(n_1, \cdots, n_p)$ if and only if g is the product $g_1\times\cdots\times g_p$ where, for each i, g_i acts as a cyclic permutation of length n_i on S_i for some subdivision of S subordinate to the partition.

The symmetric group S_4 has five types of elements corresponding to the partitions of 4, namely, (1, 1, 1, 1), (1, 1, 2), (1, 3), (2, 2), (4). For the first two types we have already calculated c_P.

In order to get c_P for the third type, namely, for the elements P of $\mathfrak{S}_4$ subordinate to the partition (1, 3), we use

$$\begin{aligned}
&\sum_{h,i,j} \langle J_\beta J_\lambda e_h, e_i\rangle\langle J_\gamma J_\mu e_i, e_j\rangle\langle J_\delta J_\nu e_j, e_h\rangle \\
&\quad=\sum_{h,i,j} \langle J_\beta J_\lambda e_h, e_i\rangle\langle J_\gamma J_\mu e_i, e_j\rangle\langle e_j, J_\nu J_\delta e_h\rangle \\
&\quad=\sum_{h,i} \langle J_\beta J_\lambda e_h, e_i\rangle\langle J_\gamma J_\mu e_i, J_\nu I_\delta e_h\rangle \\
&\quad=\sum_{h,i} \langle J_\beta J_\lambda e_h, e_i\rangle\langle e_i, J_\mu J_\gamma J_\nu J_\delta e_h\rangle \\
&\quad=\sum_{h} \langle J_\beta J_\lambda e_h, J_\mu J_\gamma J_\nu J_\delta e_h\rangle \\
&\quad=\sum_{h} \langle J_\delta J_\nu J_\gamma J_\mu J_\beta J_\lambda e_h, e_h\rangle
\end{aligned}$$

and the steps similar to those used for the second type. The result is $c_P=c_e/4$. Similarly we get $c_P=c_e/4$ for the fourth type and $c_P=c_e/8$ for the fifth type.

The number of permutations of each type is easily counted and the result is as follows: 1, 6, 8, 3, 6. Thus we get

$$\sum_P c_P=(1+6/2+8/4+3/4+6/8)c_e=(15/2)c_e$$

hence

$$\langle C_J^{(a)}, C_J^{(b)}\rangle=80a_{\kappa\lambda\mu\nu}b^{\kappa\lambda\mu\nu}\,. \tag{4.3}$$

Similarly we get

$$\langle C_I^{(a)}, C_I^{(b)}\rangle=80a_{\kappa\lambda\mu\nu}b^{\kappa\lambda\mu\nu}\,. \tag{4.4}$$

Next we calculate the inner product $\langle C_I^{(a)}, C_J^{(b)}\rangle$ using (2.1) and (2.2). As we can write the inner product in the form

$$\langle C_I^{(a)}, C_J^{(b)}\rangle=(1/24)a^{\kappa\lambda\mu\nu}b^{\alpha\beta\gamma\delta}\sum_i^{*}\sum_P \langle J_\alpha I_\kappa e_{i_1}, e_{i_{P(1)}}\rangle\langle J_\beta I_\lambda e_{i_2}, e_{i_{P(2)}}\rangle$$
$$\langle J_\gamma I_\mu e_{i_3}, e_{i_{P(3)}}\rangle\langle J_\delta I_\nu e_{i_4}, e_{i_{P(4)}}\rangle ,$$

contribution to this inner product from permutations subordinate to partitions other than (2, 2) and (4) vanishes because of (2.2). On the other hand, as we have

$$\sum_j \langle J_\alpha I_\kappa e_i, e_j\rangle\langle J_\beta I_\lambda e_j, e_i\rangle=\langle J_\beta J_\alpha I_\lambda I_\kappa e_i, e_i\rangle$$

and, similarly,

$$\sum_{i,j,k}\langle J_\alpha I_\kappa e_h, e_i\rangle\langle J_\beta I_\lambda e_i, e_j\rangle\langle J_\gamma I_\mu e_j, e_k\rangle\langle J_\delta I_\nu e_k, e_h\rangle$$
$$=\langle J_\delta J_\gamma J_\beta J_\alpha I_\nu I_\mu I_\lambda I_\kappa e_h, e_h\rangle ,$$

we get, in view of (3.2),

$$\langle C_I^{(a)}, C_J^{(b)}\rangle=0 .$$

Thus we have proved the following lemma.

LEMMA 4. *Let* $C_J^{(a)}$ *and* $C_I^{(a)}$ *be defined by* (3.3) *and* (3.4) *respectively when* $a_{\kappa\lambda\mu\nu}x^\kappa x^\lambda x^\mu x^\nu$ *is a harmonic polynomial in* R^3. *Then these are elements of* $D_{4,4}^3$ *and the inner products satisfy*

$$\langle C_J^{(a)}, C_J^{(b)}\rangle=\langle C_I^{(a)}, C_I^{(b)}\rangle=80a_{\kappa\lambda\mu\nu}b^{\kappa\lambda\mu\nu} , \quad \langle C_I^{(a)}, C_J^{(b)}\rangle=0 .$$

§ 5. The subspaces D_I and D_J of $D_{4,4}^3$.

We can define inner products in the space H_4^3 of harmonic polynomials in R^3 in various ways. Here, and in the sequel, we take $\langle\ ,\ \rangle$ defined as follows: if $a=a_{\kappa\lambda\mu\nu}x^\kappa x^\lambda x^\mu x^\nu$, $b=b_{\kappa\lambda\mu\nu}x^\kappa x^\lambda x^\mu x^\nu$, then $\langle a, b\rangle=a_{\kappa\lambda\mu\nu}b^{\kappa\lambda\mu\nu}$.

From Lemma 4 we get the following theorem.

THEOREM 5.1. D_I *and* D_J *are linear subspaces of* $D_{4,4}^3$ *orthogonal to each other.* φ_I *and* φ_J *are homothetic mappings of* H_4^3 *into* $D_{4,4}^3$, *hence* $\dim D_I=\dim D_J=\dim H_4^3=9$, $\dim D_{4,4}^3\geqq 18$.

REMARK. It was proved by do Carmo and Wallach [3] that $\dim D_{4,4}^3\geqq 18$.

LEMMA 5.2.

$$gC_I^{(a)}=C_I^{(a)} \quad if \quad g\in O_J , \quad gC_J^{(a)}=C_J^{(a)} \quad if \quad g\in O_I .$$

Proof is easy since we have, for example,

$$\begin{aligned}&\langle I_\kappa(a+bJ_1+cJ_2+dJ_3)w,\ (a+bJ_1+cJ_2+dJ_3)v\rangle\\&\quad=\langle(a+bJ_1+cJ_2+dJ_3)I_\kappa w,\ (a+bJ_1+cJ_2+dJ_3)v\rangle\\&\quad=\langle I_\kappa w,\ v\rangle\end{aligned}$$

if $a^2+b^2+c^2+d^2=1$.

§ 6. The dimension of the space $D^3_{4,4}$.

6.1. A classification of the components of a tensor C belonging to $D^3_{4,4}$.

As it is pointed out in § 1 the necessary and sufficient condition for a tensor C of degree $2s$ to be an element of $D^m_{s,s}$ is that C satisfies the four conditions (i), (ii), (iii), (iv) for arbitrary vectors $v_1, \cdots, v_{2s}, v, w$ of R^{m+1}. Especially in the case of $D^3_{4,4}$, (iv) is equivalent to

$$C(v_1,\ v_2,\ v,\ v;\ v,\ v,\ v,\ v)=0\ . \tag{6.1.1}$$

From this we easily find that, if $C\in D^3_{4,4}$, then C satisfies the equations

$$\begin{aligned}&C(v,\ v,\ vv;\ v,\ v,\ v,\ v)=0\ ,\\&C(v_1,\ v,\ v,\ v;\ v,\ v,\ v,\ v)=0\ ,\\&C(v_1,\ v,\ v,\ v;\ v_2,\ v,\ v,\ v)=0\ ,\end{aligned}$$

and further

$$C(v_1,\ v_2,\ v_3,\ v;\ v,\ v,\ v,\ v)=0\ , \tag{6.1.2}$$

$$C(v_1,\ v_2,\ v,\ v;\ v_3,\ v,\ v,\ v)=0\ , \tag{6.1.3}$$

$$\begin{aligned}&C(v_1,\ v_2,\ v_3,\ v_4;\ v,\ v,\ v,\ v)\\&\quad=-4C(v_1,\ v_2,\ v_3,\ v;\ v_4,\ v,\ v,\ v)\\&\quad=6C(v_1,\ v_2,\ v,\ v;\ v_3,\ v_4,\ v,\ v)\ ,\end{aligned} \tag{6.1.4}$$

$$\begin{aligned}&C(v_1,\ v_2,\ v_3,\ v_4;\ v_5,\ v,\ v,\ v)\\&\quad=-C(v_1,\ v_2,\ v_3,\ v_5;\ v_4,\ v,\ v,\ v)\\&\qquad-3C(v_1,\ v_2,\ v_3,\ v;\ v_4,\ v_5,\ v,\ v)\\&\quad=3C(v_1,\ v_2,\ v_5,\ v;\ v_3,\ v_4,\ v,\ v)\\&\qquad+3C(v_1,\ v_2,\ v,\ v;\ v_3,\ v_4,\ v_5,\ v)\end{aligned} \tag{6.1.5}$$

$$\begin{aligned}&C(v_1,\ v_2,\ v_3,\ v_4;\ v_5,\ v_6,\ v,\ v)\\&\quad=-C(v_1,\ v_2,\ v_3,\ v_5;\ v_4,\ v_6,\ v,\ v)\\&\qquad-C(v_1,\ v_2,\ v_3,\ v_6;\ v_4,\ v_5,\ v,\ v)\end{aligned} \tag{6.1.6}$$

$$
\begin{aligned}
&-2C(v_1, v_2, v_3, v; v_4, v_5, v_6, v)\\
=&C(v_1, v_2, v_5, v_6; v_3, v_4, v, v)\\
&+C(v_1, v_2, v, v; v_3, v_4, v_5, v_6)\\
&+2C(v_1, v_2, v_5, v; v_3, v_4, v_6, v)\\
&+2C(v_1, v_2, v_6, v; v_3, v_4, v_5, v)
\end{aligned}
$$

for any vectors $v_1, \cdots, v_6$ and v in $\boldsymbol{R}^4$. In short, any one of these equations is obtained by substituting $v+\lambda w$ for v in the equation or equations preceding that one and taking a suitable vector for w (see, for example, §7 of [5]).

Now we fix an orthonormal basis $\{e_1, e_2, e_3, e_4\}$ in $\boldsymbol{R}^4$ and use the notation $(abcd, efgh)$ defined by

$$(abcd, efgh)=C(e_a, e_b, e_c, e_d; e_e, e_f, e_g, e_h) \tag{6.1.7}$$

where $a, b, \cdots, g, h=1, 2, 3, 4$. The components such as $(bacd, efgh)$, $(efgh, abcd)$, $\cdots$ are identified with $(abcd, efgh)$ because of (i) and (ii). The equations given above suggest the following classification of the components.

Consider a component $(abcd, efgh)$. We say that this component belongs to the class $(\alpha, \beta, \gamma, \delta)$, $\alpha\geqq\beta\geqq\gamma\geqq\delta$, if a number appears α times, another number appears β times, and so on in $(a, b, \cdots, g, h)$. We delete 0 so that, for example, we say that the component $(1112, 1222)$ belongs to the class $(4, 4)$. (i), (ii) and (iv) are conditions within each of such classes. (6.1.2) and (6.1.3) show that every member of the class $(\alpha, \beta, \gamma, \delta)$ vanishes if $\alpha\geqq 5$.

6.2. Components of C in the classes $(4, 4)$, $(4, 2, 2)$ or $(2, 2, 2, 2)$.

Here and in the sequel we understand that h, i, j, k appearing in any one formula are different numbers taken from 1, 2, 3, 4, if it is not otherwise indicated. With this understanding we define α_{hi} and δ_{hij} by

$$(hhhh, iiii)=12\alpha_{hi}\,, \quad (hhhh, iijj)=12\delta_{hij}\,.$$

Then we get from (iii), namely,

$$(hhhh, iihh)+(hhhh, iiii)+(hhhh, iijj)+(hhhh, iikk)=0\,,$$

the equations

$$\alpha_{hi}+\delta_{hij}+\delta_{hik}=0$$

because of $(hhhh, iihh)=0$. It is easy to see that this system of equations is equivalent to the system of equations

$$2\delta_{hij}+\alpha_{hi}+\alpha_{hj}-\alpha_{hk}=0$$

because of $\alpha_{hi}=\alpha_{ih}$, $\delta_{hij}=\delta_{hji}$.

On the other hand we have from (iii)

$$(hhhh, iijj)+(iihh, iijj)+(jjhh, iijj)+(kkhh, iijj)=0 .$$

We also get from (6.1.4) $(hhhh, iijj)=6(hhii, hhjj)$, hence $(hhii, hhjj)=2\delta_{hji}$. Thus we get

$$12\delta_{hij}+2\delta_{ihj}+2\delta_{jhi}+(hhkk, iijj)=0$$

which is equivalent to

$$(6.2.1) \qquad (hhkk, iijj)=7\alpha_{hi}+7\alpha_{hj}+2\alpha_{ij}-6\alpha_{hk}-\alpha_{ik}-\alpha_{jk} .$$

But, as $(hhkk, iijj)$ is symmetric in h and k, we get $\alpha_{hi}+\alpha_{hj}-\alpha_{ik}-\alpha_{jk}=0$ and finally

$$(6.2.2) \qquad \alpha_{hj}=\alpha_{ik} .$$

Thus, if three parameters, for example, α_{12}, α_{13}, α_{14} are given, then all of α_{hi} and δ_{hij} are determined.

Next we want to show that all components in the class (4, 4), (4, 2, 2) or (2, 2, 2, 2) are determined.

We can easily deduce from (6.1.4) that all components of the class (4, 4) are given by

$$(6.2.3) \quad (hhhh, iiii)=12\alpha_{hi} , \quad (hhhi, hiii)=-3\alpha_{hi} , \quad (hhii, hhii)=2\alpha_{hi} .$$

We also get from (6.1.4) all components in the class (4, 2, 2) in the form

$$(6.2.4) \qquad \begin{aligned} &(hhhh, iijj)=6(\alpha_{hk}-\alpha_{hi}-\alpha_{hj}) ,\\ &(hhhi, hijj)=-(3/2)(\alpha_{hk}-\alpha_{hi}-\alpha_{hj}) ,\\ &(hhii, hhjj)=(hhij, hhij)=\alpha_{hk}-\alpha_{hi}-\alpha_{hj} \end{aligned}$$

as we have $\delta_{hij}=-(1/2)(\alpha_{hi}+\alpha_{hj}-\alpha_{hk})$. At the same time we get from (6.2.1)

$$(hhkk, iijj)=6\alpha_{hi}+6\alpha_{hj}-4\alpha_{hk} .$$

In order to get all components of the class (2, 2, 2, 2) we use (iii) and get $(hhjk, iijk)=-(hhjk, hhjk)-(hhjk, jjjk)-(hhjk, kkjk)$, hence $(hhjk, iijk)=\alpha_{hj}+\alpha_{hk}-4\alpha_{hi}$. Putting $v=e_h$, $v_1=v_4=e_i$, $v_2=v_5=e_j$, $v_3=v_6=e_k$ in (6.1.6), we get $(iijk, hhjk)=-(ijjk, hhik)-(ijkk, ijhh)-2(ijkh, ijkh)$, hence $2(hijk, hijk)=2(\alpha_{hi}+\alpha_{hj}+\alpha_{hk})$. Thus we have obtained all components in the class (2, 2, 2, 2) in the form

(6.2.5)
$$(hhii,\ jjkk)=6\alpha_{hj}+6\alpha_{hk}-4\alpha_{hi}\ ,$$
$$(hhjk,\ iijk)=\alpha_{hj}+\alpha_{hk}-4\alpha_{hi}\ ,$$
$$(hijk,\ hijk)=\alpha_{hi}+\alpha_{hj}+\alpha_{hk}\ .$$

6.3. Components of C in the class (3, 3, 1, 1).

In §6.2 we have proved $(hhhh,\ iiii)=(jjjj,\ kkkk)$, namely,

$$C(e_h,\ e_h,\ e_h,\ e_h;\ e_i,\ e_i,\ e_i,\ e_i)=C(e_j,\ e_j,\ e_j,\ e_j;\ e_k,\ e_k,\ e_k,\ e_k)\ .$$

Let us fix for a while the numbers $h,\ i,\ j,\ k$. As the orthonormal basis $\{e_1,\ e_2,\ e_3,\ e_4\}$ can be chosen arbitrarily in R^4, we can replace e_h by $'e_h=e_h\cos\theta+e_j\sin\theta$ and, at the same time, e_j by $'e_j='e_j\cos\theta-e_h\sin\theta$ and get

$$C('e_h,\ 'e_h,\ 'e_h,\ 'e_h;\ e_i,\ e_i,\ e_i,\ e_i)=C('e_j,\ 'e_j,\ 'e_j,\ 'e_j;\ e_k,\ e_k,\ e_k,\ e_k)\ .$$

Differentiating both members with respect to θ and putting $\theta=0$, we get

(6.3.1)
$$C(e_h,\ e_h,\ e_h,\ e_j;\ e_i,\ e_i,\ e_i,\ e_i)=-C(e_h,\ e_j,\ e_j,\ e_j;\ e_k,\ e_k,\ e_k,\ e_k)\ ,$$

that is,

(6.3.2)
$$(hhhj,\ iiii)=-(hjjj,\ kkkk)\ .$$

We can also replace e_i by $'e_i=e_i\cos\varphi+e_k\sin\varphi$ and at the same time e_k by $'e_k=e_k\cos\varphi-e_i\sin\varphi$ in (6.3.1). Thus we get

(6.3.3)
$$(hhhj,\ iiik)=(jjjh,\ kkki)\ .$$

Let us define $\beta_{hi,jk}$ by

(6.3.4)
$$(hhhi,\ jjjk)=9\beta_{hi,jk}\ .$$

Then we have $\beta_{hi,jk}=\beta_{jk,hi}$ from this definition, and $\beta_{hi,jk}=\beta_{ih,kj}$ from (6.3.3), hence

(6.3.5)
$$\beta_{hi,jk}=\beta_{ih,kj}=\beta_{jk,hi}=\beta_{kj,ih}\ .$$

Thus, if $\beta_{hi,jk}$ are known for $h=1$, then all of them are obtained.

On the other hand we get

$$(hhhi,\ iijk)+(hhhi,\ jjjk)+(hhhi,\ kkjk)=0\ ,$$
$$(hhhj,\ iiik)=-(hhhk,\ iiij)-3(hhhi,\ iijk)$$

from (iii) and (6.1.5). Thus we have

$$(hhhi,\ iijk)=-9(\beta_{hi,jk}+\beta_{hi,kj})\ ,$$
$$(hhhi,\ iijk)=-3(\beta_{hj,ik}+\beta_{hk,ij})\ ,$$

hence

$$3\beta_{hi,jk}-\beta_{hk,ij}=-3\beta_{hi,kj}+\beta_{hj,ik}\ .$$

Let us take up the following three among these equations,

$$\begin{aligned}
3\beta_{12,34}-\beta_{14,23}&=-3\beta_{12,43}+\beta_{13,24}\ ,\\
3\beta_{13,42}-\beta_{12,34}&=-3\beta_{13,24}+\beta_{14,32}\ ,\\
3\beta_{14,23}-\beta_{13,42}&=-3\beta_{14,32}+\beta_{12,43}\ .
\end{aligned}$$

Then we get

$$\begin{aligned}
&\beta_{12,34}=13e\ , &&\beta_{12,43}=-12e-4f+3g\ ,\\
&\beta_{13,42}=13f\ , &&\beta_{13,24}=-12f-4g+3e\ ,\\
&\beta_{14,23}=13g\ , &&\beta_{14,32}=-12g-4e+3f
\end{aligned} \tag{6.3.6}$$

where e, f, g are undetermined. Thus we get all $\beta_{hi,jk}$ using (6.3.5).

Putting $v=e_h$, $v_1=v_4=v_5=e_j$, $v_2=e_i$, $v_3=e_k$ in (6.1.5) we get

$$(hhhj,\ jjik)=-(hhhj,\ jjik)-3(hhjj,\ hijk)\ ,$$

and putting $v=e_h$, $v_1=v_2=v_4=e_j$, $v_3=e_k$, $v_5=e_i$ in (6.1.5) we get

$$(hhhi,\ jjjk)=-(hhhj,\ ijjk)-3(hhij,\ hjjk)\ .$$

Thus we have

$$\begin{aligned}
(hhhi,\ jjjk)&=9\beta_{hi,jk}\ ,\\
(hhhj,\ ijjk)&=-3\beta_{hi,jk}-3\beta_{hk,ji}\ ,\\
(hhjj,\ hijk)&=2\beta_{hi,jk}+2\beta_{hk,ji}\ ,\\
(hhij,\ hjjk)&=-2\beta_{hi,jk}+\beta_{hk,ji}\ ,
\end{aligned} \tag{6.3.7}$$

which shows that all components in the class (3, 3, 1, 1) can be calculated if the numbers e, f, g are given.

6.4. Components of C in (4, 3, 1), (4, 2, 1, 1), (3, 3, 2) or (3, 2, 2, 1).

Let us define α_{hij} by

$$(hhhh,\ iiij)=12\alpha_{hij}\ . \tag{6.4.1}$$

Then we get, in view of (6.3.2),

$$\alpha_{hij}=-\alpha_{kji}\ . \tag{6.4.2}$$

All other components in the class (4, 3, 1) are then determined by (6.1.4) and we get

(6.4.3) $$(hhhi,\ hiij)=(hhhj,\ hiii)=-3\alpha_{hij}\ , \quad (hhii,\ hhij)=2\alpha_{hij}\ .$$

As we have, in view of (iii),

$$(hhhh,\ iijk)+(hhhh,\ jjjk)+(hhhh,\ jkkk)=0\ ,$$

we get

(6.4.4) $$(hhhh,\ iijk)=-12(\alpha_{hjk}+\alpha_{hkj})\ .$$

All other components in the class (4, 2, 1, 1) are then determined by (6.1.4) and we get

(6.4.5) $$\begin{aligned}(hhhi,\ hijk)&=(hhhj,\ hiik)=3(\alpha_{hjk}+\alpha_{hkj})\ ,\\ (hhii,\ hhjk)&=(hhij,\ hhik)=-2(\alpha_{hjk}+\alpha_{hkj})\ .\end{aligned}$$

If we replace e_j by $'e_j=e_j\cos\theta+e_i\sin\theta$ and at the same time e_i by $'e_i=e_i\cos\theta-e_j\sin\theta$ in (6.3.1), differentiate the resulting formula with respect to θ at $\theta=0$, then we get, in view of (6.1.2), $4(hhhj,\ iiij)=3(hijj,\ kkkk)$, hence

(6.4.6) $$(hhhj,\ iiij)=-9(\alpha_{kih}+\alpha_{khi})\ .$$

As we can deduce from (6.1.5)

$$\begin{aligned}2(iiij,\ jhhh)&=-3(iiih,\ jjhh)\ ,\\ (iiji,\ jhhh)&=-(iijj,\ ihhh)-3(iijh,\ ijhh)\ ,\\ 2(jjii,\ ihhh)&=-3(jjih,\ iihh)\ ,\end{aligned}$$

we get

(6.4.7) $$\begin{aligned}(hhhi,\ iijj)&=6(\alpha_{kih}+\alpha_{khi})\ ,\\ (hhii,\ hijj)&=-4(\alpha_{kih}+\alpha_{khi})\ ,\\ (hhij,\ hiij)&=\alpha_{kih}+\alpha_{khi}\ .\end{aligned}$$

Thus all components in the class (3, 3, 2) are obtained.

Now we want to determine the components in the class (3, 2, 2, 1). From $(hhhk,\ iijj)+(hhhk,\ jjjj)+(hhhk,\ kkjj)=0$ we get $(hhhk,\ iijj)=-12\alpha_{jhk}-6(\alpha_{ikh}+\alpha_{ihk})$, hence

(6.4.8) $$(hhhk,\ iijj)=-6(\alpha_{ihk}+\alpha_{jhk})$$

by virtue of (6.4.2). From $(hhhi,\ kiii)+(hhhi,\ kijj)+(hhhi,\ kikk)=0$ we get $(hhhi,\ kijj)=3\alpha_{ihk}+9(\alpha_{jkh}+\alpha_{jhk})$, hence

(6.4.9) $$(hhhi,\ ijjk)=9\alpha_{jhk}+6\alpha_{jkh}\ .$$

Similarly we get

(6.4.10) $$(hhii,\ hjjk)=4\alpha_{ihk}+6\alpha_{ikh}\ ,$$

(6.4.11) $$(hhik,\ hijj)=4\alpha_{ihk}+\alpha_{ikh}\ .$$

If we put in (6.1.5) $v=e_h$, $v_1=e_k$, $v_2=v_4=e_i$, $v_3=v_5=e_j$, then we get $(kiji, jhhh)=-(kijj, ihhh)-3(kijh, ijhh)$, hence

(6.4.12) $$(hhij,\ hijk)=-\alpha_{ihk}-\alpha_{jhk}$$

by virtue of (6.4.9). Thus we have obtained all components in the class (3, 2, 2, 1).

6.5. The dimension of the space $D^3_{4,4}$.

The results obtained above can be resumed in the following lemma.

LEMMA 6. *Suppose an orthonormal basis $\{e_1, e_2, e_3, e_4\}$ is fixed in $\boldsymbol{R}^4$ and the components of a tensor C belonging to $D^3_{4,4}$ are denoted by $(abcd, efgh)=C(e_a, e_b, e_c, e_d; e_e, e_f, e_g, e_h)$.* Then *$C$ is determined if the following* 18 *components are given*:

$$\begin{gathered}(1111, 2222), (1111, 3333), (1111, 4444)\ ,\\ (1112, 3334), (1113, 4442), (1114, 2223)\ ,\\ (1111, 2223), (1111, 3332), (1111, 2224), (1111, 4442)\ ,\\ (1111, 3334), (1111, 4443), (2222, 1113), (2222, 3331)\ ,\\ (2222, 1114), (2222, 4441), (3333, 1112), (3333, 2221)\ .\end{gathered}$$

Whether these components may be freely chosen or not is not proved above. But it has been proved by do Carmo and Wallach [3] that $\dim W_2 \geqq 18$ if $s\geqq 4$ and $m\geqq 3$. Due to this result the above 18 components are free from restriction. Thus $\alpha_{12}, \alpha_{13}, \alpha_{14}, e, f, g$ and α_{hij} satisfying (6.4.2) can be considered as parameters in Theorem 1.5 of [3].

§ 7. The main results.

We have shown that $\dim D^3_{4,4}=18$. On the other hand we have Lemma 4 and Theorem 5.1. Thus we have the following main theorem.

THEOREM 7.1. *Let $\{a_1, \cdots, a_9\}$ be an orthonormal basis of the space of harmonic polynomials of degree 4 in x^1, x^2, x^3. Then $\{\varphi_I(\rho a_1), \cdots, \varphi_I(\rho a_9), \varphi_J(\rho a_1), \cdots, \varphi_J(\rho a_9)\}$ is an orthonormal basis of $D^3_{4,4}$ if ρ is a suitable constant.*

Lemma 5.2 implies $O_J: D_I \to D_I$. As we have Lemma 1 and D_J is orthogonal to D_I by Theorem 7.1, we get O_J: $D_J \to D_J$. Similarly we have O_I: $D_J \to D_J$ and O_I: $D_I \to D_I$. In this way we get from Lemma 1, Lemma 5.2 and Theorem 7.1 the following theorem.

THEOREM 7.2. *D_I and D_J are invariant subspaces of $D^3_{4,4}$ under the action of $SO(4)$. Each element of D_I (resp. D_J) is invariant under the action of O_J (resp. O_I).*

§ 8. Relation between 18 parameters and the coefficients of the harmonic polynomial $a(x)$.

First we give an example. If a is given by

$$a(x, y, z)=8x^4-24x^2(y^2+z^2)+3(y^2+z^2)^2 , \tag{8.1}$$

we have, for $C_I^{(a)}$,

$$\begin{aligned}&\alpha_{12}=2/3 , \quad \alpha_{13}=\alpha_{14}=1/4 ,\\ &e=0 , \quad f=5/(12\cdot 13) , \quad g=5/(9\cdot 13) ,\\ &\alpha_{hij}=0 ,\end{aligned}$$

and, for $C_J^{(a)}$,

$$\begin{aligned}&\alpha_{12}=2/3 , \quad \alpha_{13}=\alpha_{14}=1/4 ,\\ &e=0 , \quad f=-5/(12\cdot 13) , \quad g=-5/(9\cdot 13) ,\\ &\alpha_{hij}=0 .\end{aligned}$$

These results can be obtained directly from (3.3) and (3.4). But we can take the following convenient way.

Taking unit vectors v and w we get from the definition of $C_I^{(a)}$

$$\begin{aligned}&C_I^{(a)}(v, v, v, v; w, w, w, w)\\ &\quad =a^{\kappa\lambda\mu\nu}\langle I_\kappa w, v\rangle\langle I_\lambda w, v\rangle\langle I_\mu w, v\rangle\langle I_\nu w, v\rangle ,\\ &C_I^{(a)}(v, v, v, v; I_\alpha v, I_\beta v, I_\gamma v, I_\delta v)\\ &\quad =a^{\kappa\lambda\mu\nu}\langle I_\kappa I_\alpha v, v\rangle\langle I_\lambda I_\beta v, v\rangle\langle I_\mu I_\gamma v, v\rangle\langle I_\nu I_\delta v, v\rangle\\ &\quad =a^{\kappa\lambda\mu\nu}\delta_{\kappa\alpha}\delta_{\lambda\beta}\delta_{\mu\gamma}\delta_{\nu\delta}\\ &\quad =a^{\alpha\beta\gamma\delta} ,\end{aligned}$$

and, similarly,

$$C_J^{(a)}(v, v, v, v; J_\alpha v, J_\beta v, J_\gamma v, J_\delta v)=a^{\alpha\beta\gamma\delta} .$$

Then, putting $v=e_1$, $v=e_2$, or $v=e_3$, we get

$$
\begin{aligned}
a^{1111} &= C_I^{(\alpha)}(1111, 2222) = C_J^{(\alpha)}(1111, 2222) ,\\
a^{2222} &= C_I^{(\alpha)}(1111, 3333) = C_J^{(\alpha)}(1111, 3333) ,\\
a^{3333} &= C_I^{(\alpha)}(1111, 4444) = C_J^{(\alpha)}(1111, 4444) ,\\
a^{1112} &= -C_I^{(\alpha)}(1111, 2223) = -C_J^{(\alpha)}(1111, 2223) ,\\
a^{1113} &= -C_I^{(\alpha)}(1111, 2224) = C_J^{(\alpha)}(1111, 2224) ,\\
a^{1222} &= -C_I^{(\alpha)}(1111, 2333) = -C_J^{(\alpha)}(1111, 2333) ,\\
a^{1333} &= -C_I^{(\alpha)}(1111, 2444) = C_J^{(\alpha)}(1111, 2444) ,\\
a^{2223} &= C_I^{(\alpha)}(1111, 3334) = -C_J^{(\alpha)}(1111, 3334) ,\\
a^{2333} &= C_I^{(\alpha)}(1111, 3444) = -C_J^{(\alpha)}(1111, 3444) ,\\
a^{1112} &= C_I^{(\alpha)}(2222, 1114) = -C_J^{(\alpha)}(2222, 1114) ,\\
a^{1113} &= -C_I^{(\alpha)}(2222, 1113) = -C_J^{(\alpha)}(2222, 1113) ,\\
a^{1222} &= C_I^{(\alpha)}(2222, 1444) = -C_J^{(\alpha)}(2222, 1444) ,\\
a^{1333} &= -C_I^{(\alpha)}(2222, 1333) = -C_J^{(\alpha)}(2222, 1333) ,\\
a^{2223} &= -C_I^{(\alpha)}(3333, 1112) = -C_J^{(\alpha)}(3333, 1112) ,\\
a^{2333} &= -C_I^{(\alpha)}(3333, 1222) = -C_J^{(\alpha)}(3333, 1222) .
\end{aligned}
$$

From the above equations we can calculate parameters α_{hi} and α_{hij}.

In order to calculate parameters e, f, g, we take

$$a^{\alpha\alpha\alpha\alpha} = C_I^{(\alpha)}(v, v, v, v; I_\alpha v, I_\alpha v, I_\alpha v, I_\alpha v) ,$$

put $v = e_i \cos\theta + e_j \sin\theta$ and differentiate twice with respect to θ. Then, putting $\theta = 0$, we get

$$
\begin{aligned}
&3C_I^{(\alpha)}(v, v, w, w; I_\alpha v, I_\alpha v, I_\alpha v, I_\alpha v)\\
&\quad +3C_I^{(\alpha)}(v, v, v, v; I_\alpha v, I_\alpha v, I_\alpha w, I_\alpha w)\\
&\quad +8C_I^{(\alpha)}(v, v, v, w; I_\alpha v, I_\alpha v, I_\alpha v, I_\alpha w)\\
&\quad -2C_I^{(\alpha)}(v, v, v, v; I_\alpha v, I_\alpha v, I_\alpha v, I_\alpha v) = 0
\end{aligned}
$$

where $v = e_i$, $w = e_j$. Let us fix $i = 1$. Then putting $j = 2, 3, 4$ and $\alpha = 2, 3, 1$, we get

$$
\begin{aligned}
2C_I^{(\alpha)}(1111, 3333) &= 3C_I^{(\alpha)}(1122, 3333) + 3C_I^{(\alpha)}(1111, 3344) + 8C_I^{(\alpha)}(1112, 3334) ,\\
2C_I^{(\alpha)}(1111, 4444) &= 3C_I^{(\alpha)}(1133, 4444) + 3C_I^{(\alpha)}(1111, 2244) + 8C_I^{(\alpha)}(1113, 4442) ,\\
2C_I^{(\alpha)}(1111, 2222) &= 3C_I^{(\alpha)}(1144, 2222) + 3C_I^{(\alpha)}(1111, 2233) + 8C_I^{(\alpha)}(1114, 2223) .
\end{aligned}
$$

Similarly, we get

$$
\begin{aligned}
2C_J^{(\alpha)}(1111, 3333) &= 3C_J^{(\alpha)}(1122, 3333) + 3C_J^{(\alpha)}(1111, 3344) - 8C_J^{(\alpha)}(1112, 3334) ,\\
2C_J^{(\alpha)}(1111, 4444) &= 3C_J^{(\alpha)}(1133, 4444) + 3C_J^{(\alpha)}(1111, 2244) - 8C_J^{(\alpha)}(1113, 4442) ,
\end{aligned}
$$

$$2C_J^{(a)}(1111,\ 2222)=3C_J^{(a)}(1144,\ 2222)+3C_J^{(a)}(1111,\ 2233)-8C_J^{(a)}(1114,\ 2223)\ .$$

Thus we can express 18 parameters in terms of $a^{\kappa\lambda\mu\nu}$ as follows, where we understand by $\pm$ (resp. $\mp$)$+$ for I and $-$ for J (resp. $-$ for I and $+$ for J):

$$\begin{aligned}
&\alpha_{12}=a^{1111}/12\ , \quad \alpha_{13}=a^{2222}/12\ , \quad \alpha_{14}=a^{3333}/12\ ,\\
&\alpha_{123}=-a^{1112}/12\ , \quad \alpha_{124}=\mp a^{1113}/12\ , \quad \alpha_{134}=\pm a^{2223}/12\ ,\\
&\alpha_{132}=-a^{1222}/12\ , \quad \alpha_{142}=\mp a^{1333}/12\ , \quad \alpha_{143}=\pm a^{2333}/12\ ,\\
&\alpha_{213}=-a^{1113}/12\ , \quad \alpha_{214}=\pm a^{1112}/12\ , \quad \alpha_{312}=-a^{2223}/12\ ,\\
&\alpha_{231}=-a^{1333}/12\ , \quad \alpha_{241}=\pm a^{1222}/12\ , \quad \alpha_{321}=-a^{2333}/12\ ,\\
&\beta_{12,34}=(\mp 3\alpha_{12}\pm 5\alpha_{13}\pm 3\alpha_{14})/6\ ,\\
&\beta_{13,42}=(\mp 3\alpha_{13}\pm 5\alpha_{14}\pm 3\alpha_{12})/6\ ,\\
&\beta_{14,23}=(\mp 3\alpha_{14}\pm 5\alpha_{12}\pm 3\alpha_{13})/6\ .
\end{aligned}$$

§ 9. A subgroup of $SO(4)$ and some elements of $D^3_{4,4}$.

The result of the infinitesimal action of J_1 on $C_J^{(a)}$ is, as $a^{\kappa\lambda\mu\nu}$ are symmetric,

$$\begin{aligned}
&\mathscr{S}_w\mathscr{S}_v{}'a^{\kappa\lambda\mu\nu}\langle J_\kappa w_1,\ v_1\rangle\langle J_\lambda w_2,\ v_2\rangle\langle J_\mu w_3,\ v_3\rangle\langle J_\nu w_4,\ v_4\rangle\\
&\quad =4\mathscr{S}_w\mathscr{S}_v a^{\kappa\lambda\mu\nu}(\langle J_\kappa J_1 w_1,\ v_1\rangle+\langle J_\kappa w_1,\ J_1 v_1\rangle)\\
&\qquad\qquad \langle J_\lambda w_2,\ v_2\rangle\langle J_\mu w_3,\ v_3\rangle\langle J_\nu w_4,\ v_4\rangle\ .
\end{aligned}$$

The derivative $'a$ is obtained as follows. As we have

$$\langle J_\kappa J_1 w,\ v\rangle+\langle J_\kappa w,\ J_1 v\rangle=\langle (J_\kappa J_1-J_1 J_\kappa)w,\ v\rangle\ ,$$

we get for the right-hand-side

$$\begin{aligned}
&8\mathscr{S}_w\mathscr{S}_v(a^{3\lambda\mu\nu}\langle J_2 w_1,\ v_1\rangle-a^{2\lambda\mu\nu}\langle J_3 w_1,\ v_1\rangle)\\
&\qquad\qquad \langle J_\lambda w_2,\ v_2\rangle\langle J_\mu w_3,\ v_3\rangle\langle J_\nu w_4,\ v_4\rangle\ ,
\end{aligned}$$

hence

$${}'a^{\kappa\lambda\mu\nu}=2(\delta^\kappa_2 a^{3\lambda\mu\nu}+\delta^\lambda_2 a^{3\kappa\mu\nu}+\delta^\mu_2 a^{3\kappa\lambda\nu}+\delta^\nu_2 a^{3\kappa\lambda\mu}-\delta^\kappa_2 a^{2\lambda\mu\nu}-\delta^\lambda_3 a^{2\kappa\mu\nu}-\delta^\mu_3 a^{2\kappa\lambda\nu}-\delta^\nu_3 a^{2\kappa\lambda\mu})\ .$$

Thus we have

$$\begin{aligned}
&{}'a^{1111}=0\ , \quad {}'a^{2222}=8a^{2223}\ , \quad {}'a^{3333}=-8a^{2333}\ ,\\
&{}'a^{1112}=2a^{1113}\ , \quad {}'a^{1113}=-2a^{1112}\ ,\\
&{}'a^{1222}=6a^{1223}\ , \quad {}'a^{1333}=-6a^{1233}\ ,
\end{aligned}$$

$$'a^{2223}=6a^{2233}-2a^{2222}\ ,\quad 'a^{2333}=2a^{3333}-6a^{2233}\ ,$$
$$'a^{1122}=4a^{1123}\ ,\quad 'a^{1133}=-4a^{1123}\ ,$$
$$'a^{2233}=4a^{2333}-4a^{2223}\ ,$$
$$'a^{1233}=2a^{1333}-4a^{1223}\ ,\quad 'a^{1223}=4a^{1233}-2a^{1222}\ ,$$
$$'a^{1123}=2a^{1133}-2a^{1122}\ .$$

If $C_J^{(a)}$ is invariant under the action of the one-parameter group generated by J_1, namely, if $'a^{\kappa\lambda\mu\nu}$ vanish, then $a^{\kappa\lambda\mu\nu}$ must be such that

$$a^{2222}=a^{3333}=3a^{2233}\ ,\quad a^{1122}=a^{1133}\ ,$$

and other $a^{\kappa\lambda\mu\nu}$ vanish except a^{1111}. As we have $\sum_\kappa a^{\kappa\kappa\mu\nu}=0$, we get

$$a^{1111}=8\alpha\ ,\quad a^{2222}=a^{3333}=3\alpha\ ,$$
$$a^{1122}=a^{1133}=-4\alpha\ ,\quad a^{2233}=\alpha\ ,$$

where α is arbitrary.

The same result is obtained when $C_I^{(a)}$ is invariant under the action of the one-parameter group generated by I_1.

Any element C of $D_{4,4}^3$ can be written as

$$C=C_J^{(a)}+C_I^{(b)}\ .$$

On the other hand we have Theorem 7.2. Thus we have the following theorem.

THEOREM 9.1. *Let C be an invariant element of $D_{4,4}^3$ under the action of the subgroup $G_{1,1}$ of $SO(4)$ generated by J_1 and I_1. Then C is a linear combination of $C_J^{(a)}$ and $C_I^{(a)}$ where a is given by* (8.1). *The converse is also true.*

For a standard minimal immersion h of order 4 the associated tensor $h_{4,4}$ is invariant under the action of $SO(4)$. Thus we have the following corollary.

COROLLARY 9.2. *If for an isometric minimal immersion f of order 4 the associated tensor $f_{4,4}$ is such that $f_{4,4}-h_{4,4}$ is a linear combination of $C_J^{(a)}$ and $C_I^{(a)}$ where a is given by* (8.1), *then $f_{4,4}$ is invariant under the action of $G_{1,1}$. Consequently, for any $g\in G_{1,1}$, we have $f(gu)=\psi(g)f(u)$, $u\in S^3(1)$, where $\psi(g)$ is an element of the isometry of $S^{24}(r)$.*

PROOF. If we put $f(gu)='f(u)$, then f and $'f$ belong to the same equivalence class and consequently there exists an element $\psi(g)$ of $SO(25)$ [3].

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