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A Complex Continued Fraction Transformation and Its Ergodic Properties

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Introduction

In this paper we introduce a continued fraction algorithm T of complex numbers and investigate metrical properties of this algorithm. T is defined on the domain $X=\{z=x\alpha+y\bar{\alpha};\ -(1/2)\leqq x,\ y\leqq(1/2)\}$ $(\alpha=1+i)$ by $Tz=(1/z)-[1/z]_1$, where $[z]_1$ denotes $[x+(1/2)]\alpha+[y+(1/2)]\bar{\alpha}$ for a complex number $z=x\alpha+y\bar{\alpha}$. This map T induces a continued fraction expansion of $z\in X$,

$$z=\frac{1|}{|a_1}+\frac{1|}{|a_2}+\frac{1|}{|a_3}+\cdots$$

where each a_i is of the form $n\alpha+m\bar{\alpha}$ for some integers n and m. We give fundamental definitions and properties of this continued fraction algorithm T in §1.

To investigate approximation properties of continued fractions, the dual continued fraction

$$\frac{1|}{|a_n}+\frac{1|}{|a_{n-1}}+\cdots+\frac{1|}{|a_2}+\frac{1|}{|a_1}$$

plays an important role. In §2, we define the algorithm S which induces T-dual continued fraction. By using this algorithm S, we show that

$$\left|z-\frac{p_n}{q_n}\right|\leqq\frac{\sqrt{2}}{|q_n|}$$

for each $z\in X$ and $n\geqq 1$, where p_n/q_n denotes the n-th approximant introduced by T, and we also show that the value $\sqrt{2}$ is the best possible constant.

In §3 we construct the natural extension map R of T by combining

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T with S and introduce an absolutely continuous invariant measure for R. And in §4 we determine exact forms of absolutely continuous invariant measures for T and S by using this natural extension. This method of the dual algorithm and the natural extension was introduced by H. Nakada, Sh. Ito and the author [5] and has been used in several works which treat number theoretical transformations, for one-dimensional cases [7], [11], [14] and for multi-dimensional cases [3], [4], [9], [10].

A number of complex continued fraction algorithms are considered to discuss approximation theorems of complex numbers. Among them the most essential types of algorithms are that of A. Hurwitz [1] and that of R. Kaneiwa, I. Shiokawa and J. Tamura [2]. But it seems to be difficult to construct dual algorithms of these continued fraction algorithms, since Markov structures of these algorithms are very complicated. Our algorithm T has simple Markov structure, so we can construct dual algorithm.

Metrical properties of algorithms of Hurwitz and Kaneiwa, Shiokawa and Tamura were treated by H. Nakada [6] and I. Shiokawa [12]. Since these algorithms satisfy "Renyi's condition", we can apply the general theory of F-expansion in M. Waterman [15] and show that these algorithms have absolutely continuous invariant measures with bounded density functions and that they are exact with respect to these invariant measures.

On the contrary, our algorithms T and S do not satisfy "Renyi's condition", and so the density functions of the invariant measures become unbounded. This fact makes hard to investigate metrical properties of T and S. In §5, we show the ergodicity of T and S. Our method of proof is based on "local Renyi's condition", which was firstly considered in Schweiger [8]. But we can not apply the general theory of Schweiger, since it seems to be hard to verify the conditions of Schweiger. So we give our own proof. In §5 we also show several limit properties of T and S

In concluding these introductly remarks, we would like to thank Professors Shunji Ito, Michiko Yuri, Yuji Ito and Hitoshi Nakada for their interest on problem and valuable advice.

§1. Definition of a complex continued fraction transformation.

Every complex number z can be uniquely written in the form $z=x\alpha+y\bar{\alpha}$ for some real numbers x and y, where $\alpha=1+i$. Define the sets $\bar{I}$ and I by

$$\bar{I}=\{n\alpha+m\bar{\alpha};\ n \text{ and } m \text{ are integers}\}\,,$$
$$I=\bar{I}-\{0\}\,.$$

For any complex number z, let $[z]_1$ be the nearest point of $\bar{I}$ from z, that is,

$$[z]_1=\left[x+\frac{1}{2}\right]\alpha+\left[y+\frac{1}{2}\right]\bar{\alpha}$$

when z is written in the form $x\alpha+y\bar{\alpha}$.

The fundamental set X and the transformation T on X are defined by

$$X=\left\{z=x\alpha+y\bar{\alpha};\ -\frac{1}{2}\leqq x,\ y\leqq\frac{1}{2}\right\}\,,$$

$$Tz=\frac{1}{z}-\left[\frac{1}{z}\right]_1 \qquad \text{for} \quad z\in X\,.$$

If $T^k z\neq 0$ for all $k\leqq n-1$, then z is expanded in the form

$$z=\frac{1|}{|a_1}+\frac{1|}{|a_2}+\cdots+\frac{1|}{|a_n+T^n z}\,,$$

where $a_n=a_n(z)\in I$ $(n\geqq 1)$ are defined by

$$a_n=a_n(z)=\left[\frac{1}{T^{n-1}z}\right]_1\,.$$

As usual, we define p_n and $q_n\in I$ $(n\geqq -1)$ inductively by

$$p_{-1}=\alpha\,,\qquad p_0=0\,,\qquad p_n=a_n p_{n-1}+p_{n-2}\ (n\geqq 1)\,,$$
$$q_{-1}=0\,,\qquad q_0=\alpha\,,\qquad q_n=a_n q_{n-1}+q_{n-2}\ \ (n\geqq 1)\,,$$

and obtain the following formulae for all $n\geqq 1$:

$$q_n p_{n-1}-p_n q_{n-1}=2i(-1)^n\,,$$
$$\frac{p_n}{q_n}=\frac{1|}{|a_1}+\frac{1|}{|a_2}+\cdots+\frac{1|}{|a_n}\,,$$
$$(1)\qquad z=\frac{p_n+T^n z p_{n-1}}{q_n+T^n z q_{n-1}}\,,$$
$$\frac{q_{n-1}}{q_n}=\frac{1|}{|a_n}+\frac{1|}{|a_{n-1}}+\cdots+\frac{1|}{|a_1}\,.$$

We call p_n/q_n the n-th approximant of z with respect to T.

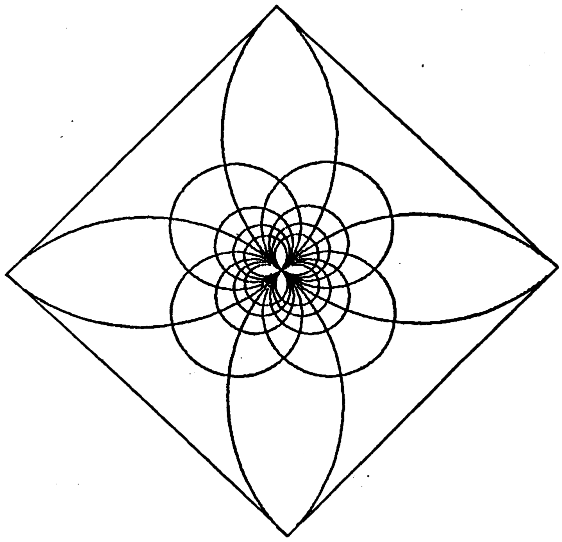

FIGURE 1. Fundamental T-cells $X(a)$ of rank 1

Now we define the set $A(n)$ of T-admissible sequences by

$$A(n)=\{a_1(z)a_2(z)\cdots a_n(z);\ z\in X\}\ .$$

For each $a_1a_2\cdots a_n\in A(n)$, define the subset $X(a_1a_2\cdots a_n)$ of X, which will be called a fundamental T-cell of rank n, by

$$X(a_1a_2\cdots a_n)=\{z\in X;\ a_k(z)=a_k \text{ for } 1\leqq k\leqq n\}\ .$$

For each n, the family of all fundamental T-cells of rank n becomes a partition of X, that is

$$X=\bigcup_{a_1\cdots a_n\in A(n)} X(a_1\cdots a_n)\ .$$

The fundamental T-cells of rank 1 are given in Fig. 1.

Let us define U_j $(1\leqq j\leqq 4)$ by

$$U_1=\left\{z\in X;\ \left|z+\frac{\alpha}{2}\right|\geqq\frac{\sqrt{2}}{2}\right\}\ ,$$
$$U_2=-i\times U_1\ ,\qquad U_3=-i\times U_2\ ,\qquad U_4=-i\times U_3\ ,$$

and define $U(a)$ for each $a\in I$ by

$$U(\alpha)=U_1\ ,\qquad U(\bar{\alpha})=U_2\ ,\qquad U(-\alpha)=U_3\ ,\qquad U(-\bar{\alpha})=U_4\ ,$$
$$U(a)=X\qquad \text{if}\quad a\neq\alpha,\ \bar{\alpha},\ -\alpha,\ -\bar{\alpha}\ .$$

Then we have

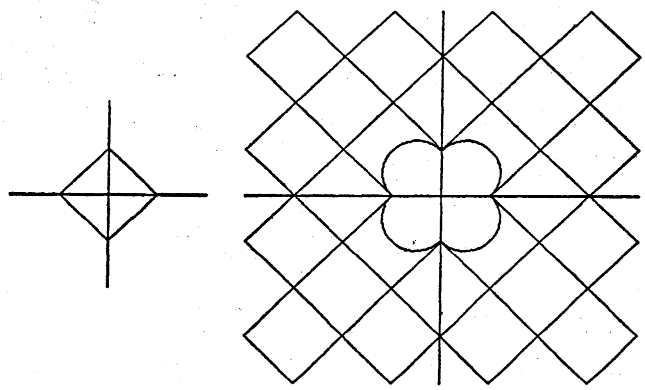

FIGURE 2. X and $X^{-1}=\bigcup_{a\in I}(a+U(a))$

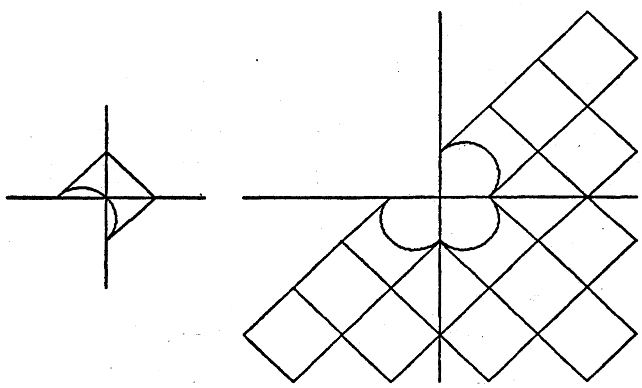

FIGURE 3. U_1 and $U_1^{-1}=\bigcup_{a\in I_1}(a+U(a))$

$$X^{-1}=\bigcup_{a\in I}(a+U(a))\ ,$$

where A^{-1} means the set $\{1/z;\ z\in A\}$ for each subset $A\subset X$. Moreover, if we define subset I_j $(1\leqq j\leqq 4)$ of I by

$$I_1=\{n\alpha+m\bar{\alpha};\ m\geqq 0\}\ ,$$
$$I_2=i\times I_1\ ,\qquad I_3=i\times I_2\ ,\qquad I_4=i\times I_3\ ,$$

then we obtain

$$U_j^{-1}=\bigcup_{a\in I_j}(a+U(a))\qquad(1\leqq j\leqq 4)\ .$$

These relations are shown in Fig. 2, 3. From these relations, it follows that

(2) $$T^n X(a_1a_2\cdots a_n)=U(a_n)\ ,$$

(3) $$A(n)=\{a_1a_2\cdots a_n;\ a_ka_{k+1}\in A(2)\text{ for }1\leqq k\leqq n-1\}\qquad(n\geqq 3)\ ,$$

(4) $$A(2)=\{a_1a_2;\ \text{if}\ a_1=\alpha_j\ \text{then}\ a_2\in I_j\ (1\leqq j\leqq 4)\}\ ,$$

where we denote $\alpha_1=\alpha$, $\alpha_2=\bar{\alpha}$, $\alpha_3=-\alpha$, $\alpha_4=-\bar{\alpha}$. From the relation (1) and (2), we see that the inverse map $\psi_{a_1\cdots a_n}$ of $T^n|_{X(a_1\cdots a_n)}$ is a 1-1 map of $U(a_n)$ onto $X(a_1\cdots a_n)$ given by

$$\psi_{a_1\cdots a_n}(z)=\frac{p_n+zp_{n-1}}{q_n+zq_{n-1}}\ ,$$

where $T^n|_{X(a_1\cdots a_n)}$ means the restriction of T^n on $X(a_1\cdots a_n)$. Since each $U(a_n)$ contains 0, we have that, for each $a_1\cdots a_n\in A(n)$,

$$\frac{p_n}{q_n}=\psi_{a_1\cdots a_n}(0)\in X(a_1\cdots a_n)\ .$$

§2. The dual transformation of T.

In this section we define the dual transformation S of T. Let us define the fundamental set Y and subsets V_j $(1\leqq j\leqq 8)$ of Y by

$$Y=\{w\in \boldsymbol{C};\ |w|\leqq 1\}\ ,$$
$$V_1=\{w\in Y;\ |w+\alpha|\geqq 1\}\ ,$$
$$V_2=-i\times V_1\ ,\quad V_3=-i\times V_2\ ,\quad V_4=-i\times V_3\ ,$$
$$V_5=V_1\cap V_2\ ,\quad V_6=-i\times V_5\ ,\quad V_7=-i\times V_6\ ,\quad V_8=-i\times V_7\ .$$

Define a partition $\{J_j;\ 1\leqq j\leqq 8\}$ of I by

$$J_1=\{n\alpha;\ n>0\}\ ,\quad J_2=-i\times J_1\ ,\quad J_3=-i\times J_2\ ,\quad J_4=-i\times J_3\ ,$$
$$J_5=\{n\alpha+m\bar{\alpha};\ n,\ m>0\}\ ,\quad J_6=-i\times J_5\ ,\quad J_7=-i\times J_6\ ,\quad J_8=-i\times J_7\ ,$$

and define $V(a)$ for each $a\in\bar{I}$ by

$$V(a)=\begin{cases} Y & \text{if}\ \ a=0\ ,\\ V_j & \textit{if}\ \ a\in J_j\quad (1\leqq j\leqq 8)\ ,\end{cases}$$

then we have the following partition of $\boldsymbol{C}$:

$$\boldsymbol{C}=\bigcup_{a\in\bar{I}}(a+V(a))\ . \qquad \text{(See Fig. 4.)}$$

The transformation S on Y is defined by

$$Sw=\frac{1}{w}-\left[\frac{1}{w}\right]_2$$

where $[w]_2$ is the point of $\bar{I}$ defined by

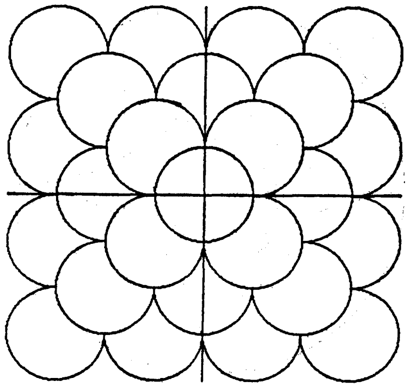

FIGURE 4. $C=\bigcup_{a\in \bar{I}}(a+V(a))$

$$[w]_2=a \qquad \text{if} \quad w\in a+V(a) .$$

In the same manner as for T, if we define $b_n=b_n(w)\in I$ and r_n, $s_n\in I$ by

$$b_n=b_n(w)=\left[\frac{1}{S^{n-1}w}\right]_2 \qquad (n\geqq 1) ,$$
$$r_{-1}=\alpha , \qquad r_0=0 , \qquad r_n=b_nr_{n-1}+r_{n-2} \quad (n\geqq 1) ,$$
$$s_{-1}=0 , \qquad s_0=\alpha , \qquad s_n=b_ns_{n-1}+s_{n-2} \quad (n\geqq 1) ,$$

then we have the expansion of $w\in Y$

$$w=\frac{1|}{|b_1}+\frac{1|}{|b_2}+\cdots+\frac{1}{|b_n+S^nw} .$$

We have also the following formulae:

$$s_nr_{n-1}-r_ns_{n-1}=2i(-1)^n ,$$
$$\frac{r_n}{s_n}=\frac{1|}{|b_1}+\frac{1|}{|b_2}+\cdots+\frac{1|}{|b_n} ,$$
$$w=\frac{r_n+S^nwr_{n-1}}{s_n+S^nws_{n-1}} ,$$
$$\frac{s_{n-1}}{s_n}=\frac{1|}{|b_n}+\frac{1|}{|b_{n-1}}+\cdots+\frac{1|}{|b_1} .$$

We call r_n/s_n the n-th approximant of w with respect to S.

Define the set $B(n)$ of S-admissible sequences and the fundamental S-cell $Y(b_1b_2\cdots b_n)$ of rank n by

$$B(n)=\{b_1(w)b_2(w)\cdots b_n(w);\ w\in Y\} ,$$
$$Y(b_1b_2\cdots b_n)=\{w\in Y;\ b_k(w)=b_k \text{ for } 1\leqq k\leqq n\} ,$$

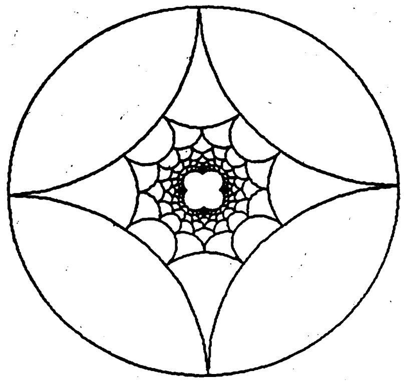

FIGURE 5. Fundamental S-cells $Y(a)$ of rank 1

then we have the following partition of Y:

$$Y = \bigcup_{b_1\cdots b_n \in B(n)} Y(b_1\cdots b_n) .$$

The fundamental S-cells of rank 1 are given in Fig. 5.

If we define subsets J'_j $(1 \leqq j \leqq 8)$ of I by

$$J'_1 = I - \{-\bar{\alpha}\} , \quad J'_2 = i \times J'_1 , \quad J'_3 = i \times J'_2 , \quad J'_4 = i \times J'_3 ,$$

$$J'_5 = J'_1 \cap J'_2 , \quad J'_6 = i \times J'_5 , \quad J'_7 = i \times J'_6 , \quad J'_8 = i \times J'_7 ,$$

then we have

$$Y^{-1} = \bigcup_{a \in I} (a + V(a)) ,$$

$$V_j^{-1} = \bigcup_{a \in J'_j} (a + V(a)) \qquad (1 \leqq j \leqq 8) .$$

These relations are shown in Fig. 6, 7, 8. So we obtain

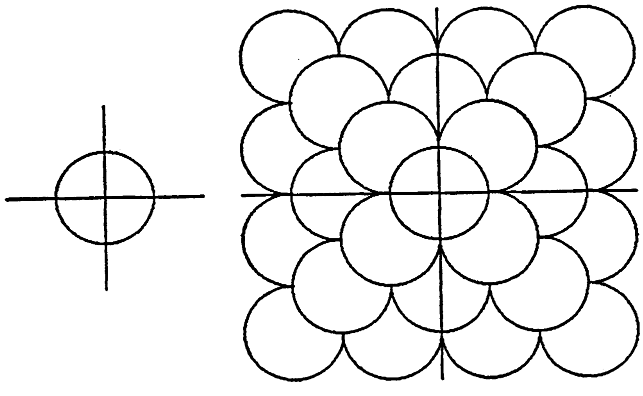

FIGURE 6. Y and $Y^{-1} = \bigcup_{a \in I} (a + V(a))$

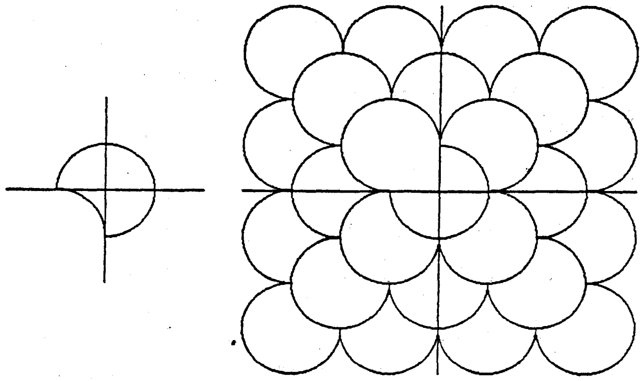

FIGURE 7. V_1 and $V_1^{-1} = \bigcup_{a \in J_1'} (a + V(a))$

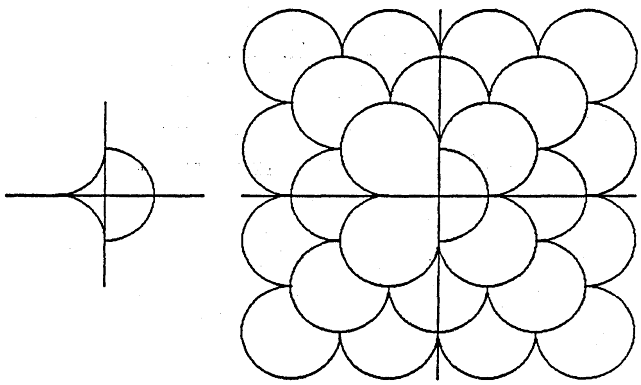

FIGURE 8. V_5 and $V_5^{-1} = \bigcup_{a \in J_5'} (a + V(a))$

$$S^n Y(b_1 \cdots b_n) = V(b_n)\ ,$$

(5) $$B(n) = \{b_1 \cdots b_n;\ b_k b_{k+1} \in B(2) \text{ for } 1 \leqq k \leqq n-1\} \qquad (n \geqq 3)\ ,$$

(6) $$B(2) = \{b_1 b_2;\ \text{if } b_1 \in J_j \text{ then } b_2 \in J_j'\ (1 \leqq j \leqq 8)\}\ ,$$

and the inverse map $\phi_{b_1 \cdots b_n}$ of $S^n|_{Y(b_1 \cdots b_n)}$ is a 1−1 map of $V(b_n)$ onto $Y(b_1 \cdots b_n)$ given by

$$\phi_{b_1 \cdots b_n}(w) = \frac{r_n + w r_{n-1}}{s_n + w s_{n-1}}\ .$$

Since $V(b_n)$ contains 0, we have

$$\frac{r_n}{s_n} = \phi_{b_1 \cdots b_n}(0) \in Y(b_1 \cdots b_n)\ .$$

Now we can show the duality of T and S.

LEMMA 1. *Let $a_1 \cdots a_n$ be a sequence of points of I. Then $a_1 \cdots a_n$*

is T-admissible if and only if $a_n \cdots a_1$ *is S-admissible.*

PROOF. From (3) and (5), it is sufficient to show the assertion in the case $n=2$. But from (4) and (6) it is easy to show that $a_1a_2 \in A(2)$ if and only if $a_2a_1 \in B(2)$. So we complete the proof.

From Lemma 1, we obtain the following two lemmas concerning the approximation by continued fraction expansions.

LEMMA 2. *Let* $a_1a_2\cdots a_n\cdots$ *be a T-admissible sequence obtained from* $z \in X$ *and let* p_n, q_n *be obtained from this sequence, then we have*

$$|q_n| \geqq \sqrt{2(n+1)} , \tag{7}$$

$$\left|z-\frac{p_n}{q_n}\right| \leqq \frac{\sqrt{2}}{|q_n|} , \tag{8}$$

$$\operatorname{diam} X(a_1\cdots a_n) \leqq \frac{2\sqrt{2}}{|q_n|} . \tag{9}$$

PROOF. If $a_1a_2\cdots a_n$ is T-admissible, then by Lemma 1, it follows that $a_na_{n-1}\cdots a_1$ is S-admissible. In term of r_n, s_n associated with this S-admissible sequence, we have

$$\frac{q_{n-1}}{q_n} = \frac{1\ |}{|\ a_n} + \frac{1\ |}{|\ a_{n-1}} + \cdots + \frac{1\ |}{|\ a_1} = \frac{r_n}{s_n} \in Y ,$$

so it follows $|q_{n-1}| \leqq |q_n|$. If we define the subset N of Y by

$$N = \{w \in Y;\ |w|=1 \text{ or } |w-\alpha_j|=1 \text{ for some } j\ (1 \leqq j \leqq 4)\} ,$$

then it is easy to show that $(q_n/q_{n+1}) \in N$ implies $(q_{n-1}/q_n) \in N$. By induction on n, it follows that $(q_{n-1}/q_n) \notin N$, that is, $|q_{n-1}/q_n| < 1$, for each n. Since $|q_n|^2$ is even number, we can show inductively that $|q_n|^2 \geqq 2(n+1)$. Thus we prove (7). From the relation (1) we have

$$z - \frac{p_n}{q_n} = \frac{2i(-1)^n T^n z}{q_n(q_n + T^n z q_{n-1})} .$$

And from the following equality

$$q_n + T^n z q_{n-1} = \frac{1}{T^{n-1}z}(q_{n-1} + T^{n-1} z q_{n-2}) ,$$

we obtain inductively that

(10) $$z-\frac{p_n}{q_n}=\frac{2i(-1)^n}{q_n q_0}\prod_{k=0}^{n} T^k z\ ,$$

which leads to (8). From (8) it is easy to show (9), so we complete the proof.

In the same manner, we can show the following

LEMMA 3. *Let $b_1 b_2 \cdots b_n \cdots$ be a S-admissible sequence obtained from $w \in Y$ and let r_n, s_n be obtained from this sequence, then we have*

$$|s_n| \geqq \sqrt{2(n+1)}\ ,$$
$$\left|w-\frac{r_n}{s_n}\right| \leqq \frac{\sqrt{2}}{|s_n|}\ ,$$
$$\operatorname{diam} Y(b_1 \cdots b_n) \leqq \frac{2\sqrt{2}}{|s_n|}\ .$$

REMARK. The estimate (8) is the best possible one, since in the case $z=1,\ -1,\ i,\ -i$ we have $|q_n+T^n z q_{n-1}|=2$ for each n.

There are several works which treat such estimates. L. Ford [16] showed the estimate

$$\left|z-\frac{p_n}{q_n}\right| \leqq \frac{1}{\sqrt{3}\,|q_n|^2}$$

for the continued fraction algorithm of Hurwitz. In this case p_n and q_n are taken from the set

$$\{n+mi;\ n \text{ and } m \text{ are integers}\}\ .$$

And Kaneiwa, Shiokawa and Tamura [13] showed the estimate

$$\left|z-\frac{p_n}{q_n}\right| \leqq \frac{1}{\sqrt[4]{13}\,|q_n|^2}$$

for their continued fraction algorithm, here p_n and q_n are taken from the set

$$\{n\zeta+m\bar{\zeta};\ n \text{ and } m \text{ are integers}\}\ \left(\zeta=\frac{1+\sqrt{3}\,i}{2}\right).$$

Our estimate is weaker than these estimates, but it should be noted that in our case p_n and q_n are restricted in the smaller set I.

§3. The natural extension of T.

In this section we define the natural extension of T by combining the transformation T and S.

Define the set Z and the transformation R on Z by

$$Z=\{(w, z)\in Y\times X;\, b_1(w)a_1(z)\in A(2)\}\ ,$$

$$\begin{aligned} R(w, z)&=(\phi_{a_1(z)}(w),\ Tz)\\ &=\left(\frac{1}{a_1(z)+w},\ \frac{1}{z}-a_1(z)\right) . \end{aligned}$$

Then we have the following

THEOREM 1. *R is the natural extension of T and the function $h(w, z)$ defined by*

$$h(w, z)=\frac{1}{|1+wz|^4}$$

is the density function of a finite absolutely continuous invariant measure of R.

PROOF. From the definition of Z and Lemma 1, we obtain the following two partitions of Z:

$$Z=\bigcup_{a\in I} V(a)\times X(a)=\bigcup_{a\in I} Y(a)\times U(a)\ .$$

For each $a\in I$, $R|_{V(a)\times X(a)}=\phi_a\times T|_{X(a)}$ is a 1-1 map of $V(a)\times X(a)$ onto $Y(a)\times U(a)$. Consequently we have that R is a 1-1 map of Z onto Z and that R is the natural extension of T. For each $(w, z)\in V(a)\times X(a)$, we have

$$\begin{aligned} |DR(w, z)|&=\left|\frac{d}{dw}\frac{1}{a+w}\right|^2\left|\frac{d}{dz}\left(\frac{1}{z}-a\right)\right|^2\\ &=\frac{1}{|a+w|^4|z|^4} \end{aligned}$$

where DR means the Jacobian of R. So $h(w, z)$ satisfies

$$\begin{aligned} |DR(w, z)|\,h(R(w, z))&=\frac{1}{|a+w|^4|z|^4}\,\frac{1}{\left|1+\dfrac{1}{a+w}\left(\dfrac{1}{z}-a\right)\right|^4}\\ &=\frac{1}{|1+wz|^4}=h(w, z) \end{aligned}$$

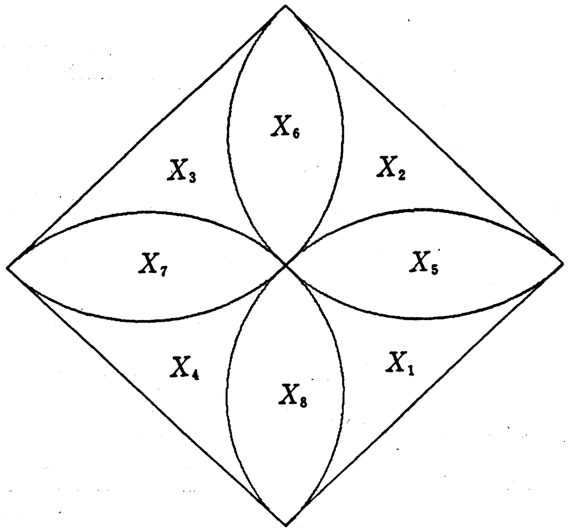

FIGURE 9. Partition $X=\cup_{j=1}^{8} X_j$

on each $V(a)\times X(a)$, which means that $h(w, z)$ is the density function of an R-invariant measure. It remains to show the finiteness of this invariant measure. Since Z has the following partition

$$Z=\bigcup_{j=1}^{8} V_j\times X_j$$

where X_j $(1\leqq j\lneqq 8)$ are defined by

$$X_j=\bigcup_{a\in J_j} X(a) \qquad \text{(See Fig. 9.)} ,$$

it is sufficient to show

$$\int_{V_j\times X_j} h(w, z)dm(w)dm(z) \tag{11}$$

for each j, where m is the Lebesgue measure on C. In the following, we prove the case of $j=2$ and 5 only, since the other case can be proved in the same manner. To prove them, we prepair the following

LEMMA 4. *Let $\zeta(z)=1/(1+wz)$, then for each measurable set $E\subset X$, we have*

$$\int_E h(w, z)dm(z)=\frac{m(\zeta(E))}{|w|^2} .$$

PROOF. It is easy to show that

$$|D\zeta|=\left|\frac{d\zeta}{dz}\right|^2=\frac{|w|^2}{|1+wz|^4} ,$$

so we obtain

$$\int_E h(w, z)dm(z)=\int_{\zeta(E)} \frac{1}{|w|^2}dm(\zeta)=\frac{m(\zeta(E))}{|w|^2} .$$

In the following we denote $z=x+yi$ and $w=u+vi$. We denote by $O(\beta, r)$ the disc of center β and radius r. Because $X_5\subset O(1/2, 1/2)$ and $\zeta(O(1/2, 1/2))$ is the disc of radius $|(1/2)w|/(|1+(1/2)w|^2-|(1/2)w|^2)=|w|/2(1+u)$, we have

$$(12) \qquad \int_{V_5\times X_5} h(w, z)dm(w)dm(z)\leqq\int_{V_5\times O(1/2,1/2)} h(w, z)dm(w)dm(z)$$
$$=\int_{V_5} \frac{m\left(\zeta\left(O\left(\frac{1}{2}, \frac{1}{2}\right)\right)\right)}{|w|^2}dm(w)=\int_{V_5} \frac{\pi}{4(1+u)^2}dm(w) .$$

Divide the last integral of (12) into integrals on $V_5\cap\{u<0\}$ and on $V_5\cap\{u\geqq 0\}$, then we have

$$\int_{V_5\cap\{u<0\}} \frac{\pi}{4(1+u)^2}dm(w)=\frac{\pi}{2}\int_{-1}^{0} \frac{du}{(1+u)^2}\int_0^{1-\sqrt{1-(1+u)^2}} dv$$
$$=\frac{\pi}{2}\int_{-1}^{0} \frac{1-\sqrt{1-(1+u)^2}}{(1+u)^2}du\leqq\frac{\pi}{2} ,$$
$$\int_{V_5\cap\{u\geqq 0\}} \frac{\pi}{4(1+u)^2}dm(w)\leqq\frac{\pi}{4}\int_{V_5\cap\{u\leqq 0\}} dm(w)=\frac{\pi^2}{8} .$$

Thus we obtain (11) for the case $j=5$. In the same manner we have

$$(13) \qquad \int_{V_2\times X_2} h(w, z)dm(w)dm(z)\leqq\int_{O(0,1)\times X_2} h(w, z)dm(w)dm(z)$$
$$=\int_{X_2} \frac{\pi}{(1-|z|^2)^2}dm(z) .$$

Let us divide the last integral of (13) into three integrals, those over $X_2\cap\{x>(1/2)\}$, $X_2\cap\{y>(1/2)\}$ and $X_2\cap\{x\leqq(1/2), y\leqq(1/2)\}$, respectively. Since $1-|z|^2\geqq 2x(1-x)$ on $X_2\cap\{x>(1/2)\}$, it follows that

$$\int_{X_2\cap\{x>(1/2)\}} \frac{\pi}{(1-|z|^2)^2} dm(z)\leqq\frac{\pi}{4}\int_{1/2}^{1} \frac{dx}{x^2(1-x)^2}\int_{\sqrt{(1/2)-(x-(1/2))^2}-(1/2)}^{1-x} dy$$
$$=\frac{\pi}{4}\int_{1/2}^{1} \frac{\frac{3}{2}-x-\sqrt{\frac{1}{2}-\left(x-\frac{1}{2}\right)^2}}{x^2(1-x)^2}dx\leqq\frac{\pi}{2}\int_{1/2}^{1} \frac{1}{x^2}dx=\frac{\pi}{2} .$$

In the same manner we can show

$$\int_{X_2\cap\{y>1/2\}} \frac{\pi}{(1-|z|^2)^2} dm(z) \leqq \frac{\pi}{2} ,$$

and also we can show

$$\int_{X_2\cap\{x\leqq(1/2),y\leqq(1/2)\}} \frac{\pi}{(1-|z|^2)^2} dm(z)$$

$$\leqq 4\pi \int_{X_2\cap\{x\leqq(1/2),y\leqq(1/2)\}} dm(z) \leqq \pi .$$

So we obtain (11) for the case $j=2$. Thus we complete the proof.

If we define the constant C by

$$C=\int_Z h(w, z)dm(w)dm(z) ,$$

then $(1/C)h(w, z)$ is the density function of an absolutely continuous R-invariant probability measure.

§4. Density function of invariant measure of T and S.

From Theorem 1, the density function $f(z)$ of an absolutely continuous invariant probability measure of T is given by

$$f(z)=\frac{1}{C}\int_{V_j} h(w, z)dm(w) \qquad \text{if} \quad z \in X_j \ (1\leqq j\leqq 8) .$$

In the following, we calculate explicitly the form of this function $f(z)$.

LEMMA 5. *If A and B are discs of radii r and s, boundaries of which intersect orthogonally with each other, then $m(A\cap B)$ is equal to*

$$d(r, s)=r^2 \tan^{-1}\frac{s}{r}+s^2 \tan^{-1}\frac{r}{s}-rs .$$

PROOF. Let O_1 and O_2 be centers of A and B, respectively, and let P and Q be points of intersection of boundaries of A and B. From the assumption of lemma, it follows that $\angle O_1PO_2=\angle O_1QO_2=(\pi/2)$, $\angle PO_1Q=2\tan^{-1}(s/r)$ and $\angle PO_2Q=2\tan^{-1}(r/s)$. So we obtain that $m(A\cap B)=d(r, s)$.

If we extend the definition of $d(r, s)$ as

$$d(r, s)=r|r| \tan^{-1}\frac{s}{|r|}+s|s| \tan^{-1}\frac{r}{|s|}-rs ,$$

then we can extend Lemma 5 to the case that one of r and s is negative or ∞. Here the disc of negative radius r means the complement of the disc of radius $|r|$, and the disc of radius ∞ means the half plane of C. The function $\tan^{-1}$ takes its value on $[0, \pi)$ as in Fig. 11, and we assume that $\infty^2 \tan^{-1}(a/\infty) - a\infty = 0$ and $\tan^{-1}(\infty/a) = (\pi/2)$.

Define functions $f_j(z)$ $(0 \leqq j \leqq 4)$ by

$$f_0(z) = \frac{1}{C}\frac{\pi}{(1-|z|^2)^2},$$
$$f_j(z) = \frac{1}{C} d\left(\frac{1}{1-|z|^2}, \frac{1}{|z-\bar{\alpha}_j|^2-1}\right) \qquad (1 \leqq j \leqq 4).$$

Then we have the following

THEOREM 2. *The density function $f(z)$ of an absolutely continuous invariant probability measure of T is given by*

$$f(z) = \begin{cases} f_0(z) - f_j(z) & z \in X_j (1 \leqq j \leqq 4), \\ f_0(z) - f_1(z) - f_2(z) & z \in X_5, \\ f_0(z) - f_2(z) - f_3(z) & z \in X_6, \\ f_0(z) - f_3(z) - f_4(z) & z \in X_7, \\ f_0(z) - f_4(z) - f_1(z) & z \in X_8, \end{cases} \tag{14}$$

PROOF. Let $\zeta(w) = 1/(1+wz)$. Since $\zeta(Y)$ is the disc of radius $|z|/(1-|z|^2)$, we obtain from Lemma 4 that

$$\frac{1}{C}\int_Y h(w, z) dm(w) = f_0(z).$$

For each j $(1 \leqq j \leqq 4)$, $Y - V_j = Y \cap O(-\alpha_j, 1)$, $\zeta(O(-\alpha_j, 1))$ is the disc of radius $|z|/(|1-\alpha_j z|^2 - |z|^2) = |z|/(|z-\bar{\alpha}_j|^2 - 1)$ and the boundaries of $\zeta(Y)$ and $\zeta(O(-\alpha_j, 1))$ intersect orthogonally. So from Lemma 5 we have

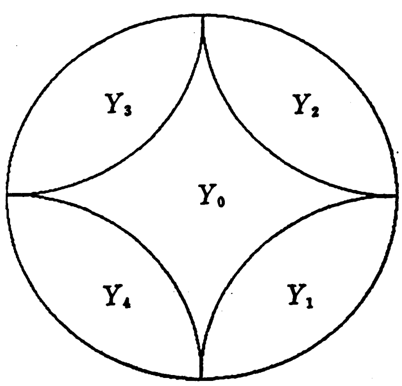

FIGURE 10. Partition $Y = \cup_{j=0}^4 Y_j$

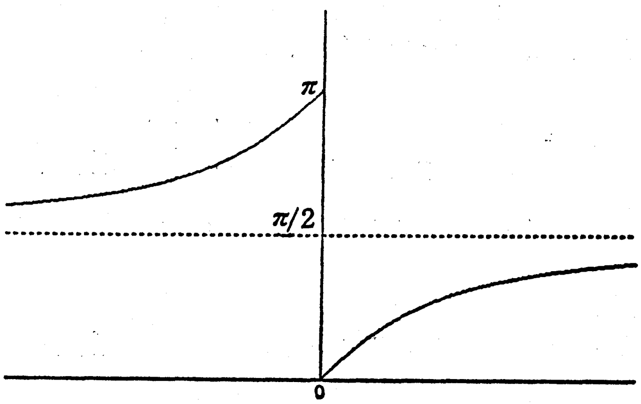

FIGURE 11. Graph of $\tan^{-1} x$

$$\int_{Y-V_j} h(w, z)dm(w)=f_j(z) \qquad (1\leqq j\leqq 4) .$$

From these relations we can show (14).

In the same manner we can obtain the density function of an absolutely continuous invariant probability measure of S. Define subsets Y_j of Y $(0\leqq j\leqq 4)$ by

$$Y_1=Y(\alpha)\,, \qquad Y_2=Y(\bar{\alpha})\,, \qquad Y_3=Y(-\alpha)\,, \qquad Y_4=Y(-\bar{\alpha})\,,$$
$$Y_0=Y-\bigcup_{j=1}^{4} Y_j\,, \qquad \text{(See Fig. 10.)}$$

then Z has another partition

$$Z=\bigcup_{j=0}^{4} Y_j\times U_j\,.$$

We define $g_j(w)$ $(0\leqq j\leqq 4)$ by

$$g_j(w)=\frac{1}{C}d(r_j, r'_j) \qquad (1\leqq j\leqq 4)\,,$$
$$g_0(w)=\sum_{j=1}^{4} g_j(w)-\frac{1}{C}\{d(r_1, r_2)+d(r_2, r_3)+d(r_3, r_4)+d(r_4, r_1)\}\,,$$

where r_j and r'_j $(1\leqq j\leqq 4)$ are given by

$$r_j=\frac{\sqrt{2}}{|w-\bar{\alpha}_j|^2-1}\,,$$
$$r'_j=\frac{\sqrt{2}}{1-|\alpha_j w-1|^2}\,.$$

Then we have the following

THEOREM 3. *The density function $g(w)$ of an absolutely continuous invariant probability measure of S is given by*

$$g(w)=\begin{cases} g_0(w) & w \in Y_0\ , \\ g_0(w)-g_j(w) & w \in Y_j\ (1 \leqq j \leqq 4)\ . \end{cases}$$

PROOF. In the same manner as in the proof of Theorem 2, we can show

$$\frac{1}{C}\int_{X-U_j} h(w, z)dm(z)=g_j(w) \qquad (1 \leqq j \leqq 4)\ ,$$

$$\frac{1}{C}\int_{X} h(w, z)dm(z)=g_0(w)\ .$$

So we obtain Theorem 3.

§5. The ergodicity of T and S.

Let $A_0(n)$ be the set of T-admissible sequences $a_1a_2\cdots a_n$ which satisfy one of the following conditions:

(a) $a_n \neq \alpha,\ \bar{\alpha},\ -\alpha,\ -\bar{\alpha},\ 2,\ 2i,\ -2,\ -2i,$
(b-1) $a_{n-1}=2$ and $a_n \neq -2,\ -\alpha,\ -\bar{\alpha},$
(b-2) $a_{n-1}=-2$ and $a_n \neq 2,\ \alpha,\ \bar{\alpha},$
(b-3) $a_{n-1}=2i$ and $a_n \neq 2i,\ \alpha,\ -\bar{\alpha},$
(b-4) $a_{n-1}=-2i$ and $a_n \neq -2i,\ -\alpha,\ \bar{\alpha},$
(c-1) $a_{n-1}=\alpha$ and $a_n \neq \alpha,\ -\alpha,$
(c-2) $a_{n-1}=-\alpha$ and $a_n \neq \alpha,\ -\alpha,$
(c-3) $a_{n-1}=\bar{\alpha}$ and $a_n \neq \bar{\alpha},\ -\bar{\alpha},$
(c-4) $a_{n-1}=-\bar{\alpha}$ and $a_n \neq \bar{\alpha},\ -\bar{\alpha},$
(d-1) $a_{n-2}=\alpha,\ a_{n-1}=\pm\alpha$ and $a_n \neq -\alpha,$
(d-2) $a_{n-2}=-\alpha,\ a_{n-1}=\pm\alpha$ and $a_n \neq \alpha,$
(d-3) $a_{n-2}=\bar{\alpha},\ a_{n-1}=\pm\bar{\alpha}$ and $a_n \neq -\bar{\alpha},$
(d-4) $a_{n-2}=-\bar{\alpha},\ a_{n-1}=\pm\bar{\alpha}$ and $a_n \neq \bar{\alpha}.$

Then we have the following

LEMMA 6. *For each $a_1a_2\cdots a_n \in A_0(n)$, $\psi_{a_1a_2\cdots a_n}$ satisfies the following "Renyi's condition":*

$$\sup_{z \in U(a_n)} |\psi'_{a_1a_2\cdots a_n}(z)|^2 \leqq 5^4 \inf_{z \in U(a_n)} |\psi'_{a_1a_2\cdots a_n}(z)|^2\ . \tag{15}$$

PROOF. From the fact $(q_{n-1}/q_n)=1/(a_n+(q_{n-2}/q_{n-1}))$, we can show $|q_{n-1}/q_n|<(2/3)$ for each case (a)$\sim$(d-4) in the following manner: In the case (a) we have $|q_{n-1}/q_n| \leqq 1/(|a_n| - |q_{n-2}/q_{n-1}|) \leqq 1/(2\sqrt{2}-1) < (2/3)$. If $a_{n-1}=2$, then it follows that $(q_{n-2}/q_{n-1}) = 1/(2+(q_{n-3}/q_{n-2})) \in O(2/3, 1/3)$, so we have $|q_{n-1}/q_n| \leqq 1/(|a_n+(2/3)| - (1/3))$. In addition, if $a_n \neq -2, -\alpha, -\bar{\alpha}$, then $|a_n+(2/3)| \geqq (\sqrt{34}/3)$. So we obtain $|q_{n-1}/q_n| \leqq 3/(\sqrt{34}-1) < (2/3)$ in the case (b-1). If $a_{n-1}=\alpha$, then it follows that $(q_{n-2}/q_{n-1}) \in O(\bar{\alpha}, 1)$, so we have that $|q_{n-1}/q_n| \leqq 1/(|a_n+\bar{\alpha}| - 1)$. But in the case $a_{n-1}=\alpha$ we have $a_n \in I_1$, so if $a_n \neq \pm\alpha$ then $|a_n+\bar{\alpha}| \geqq 2\sqrt{2}$. Thus we obtain $|q_{n-1}/q_n| \leqq 1/(2\sqrt{2}-1) <$ $(2/3)$ in the case (c-1). If $a_{n-2}=\alpha$ and $a_{n-1}=\pm\alpha$, then $(q_{n-2}/q_{n-1}) \in O(2/3, 1/3)$ or $O((2/3)i, (1/3))$ according as $a_{n-1}=\alpha$ or $-\alpha$, respectively. So if $a_n \neq -\alpha$, we have $|q_{n-1}/q_n| \leqq 3/(\sqrt{34}-1) < (2/3)$, which show the assertion for the case (d-1). In the remaining cases, we can establish the assertion in the same manner.

From the relation

$$|\psi'_{a_1 a_2 \cdots a_n}(z)|^2 = \frac{1}{|q_n|^4 \left|1+z\dfrac{q_{n-1}}{q_n}\right|^4},$$

it follows that, in each case (a)$\sim$(d-4),

$$(16)\qquad \begin{cases} \sup\limits_{z \in U(a_n)} |\psi'_{a_1 \cdots a_n}(z)|^2 \leqq \dfrac{1}{|q_n|^4\left(1-\dfrac{2}{3}\right)^4}, \\[2ex] \inf\limits_{z \in U(a_n)} |\psi'_{a_1 \cdots a_n}(z)|^2 \geqq \dfrac{1}{|q_n|^4\left(1+\dfrac{2}{3}\right)^4}, \end{cases}$$

which means (15).

We call a fundamental T-cell $X(a_1 \cdots a_n)$ for $a_1 \cdots a_n \in A_0(n)$ a Renyi-cell.

Lemma 7. *Any fundamental T-cell is modulo a set of Lebesgue measure zero a disjoint union of Renyi-cells.*

Proof. Define sets $C(n)$ inductively by

$$C(n) = \{a_1 \cdots a_n \in A_0(n);\ a_1 \cdots a_k \notin A_0(k) \text{ for } 1 \leqq k \leqq n-1\},$$

and define X'_∞ and X_∞ by

$$X'_\infty = X(2, -2, 2, -2, \cdots) \cup X(2i, 2i, 2i, 2i, \cdots) \cup X(-2i, -2i, -2i, -2i, \cdots)$$
$$\cup X(\alpha, \alpha, -\alpha, -\alpha, \cdots) \cup X(\bar{\alpha}, \bar{\alpha}, -\bar{\alpha}, -\bar{\alpha}, \cdots),$$

$$X_\infty = \bigcup_{n=0}^{\infty} T^{-n} X'_\infty .$$

Then for each T-cell $X(a_1\cdots a_n)$, we can take some $X'\subset X_\infty$ which satisfies

$$X(a_1\cdots a_n)=\bigcup_{m=1}^{\infty}\bigcup_{b_1\cdots b_m\in C(m)} X(a_1\cdots a_n b_1\cdots b_m)\cup X' .$$

From (9) we have $m(X_\infty)=0$, so we obtain Lemma 7.

THEOREM 4. *T is ergodic with respect to the invariant measure given in Theorem* 2.

PROOF. By the absolute continuity of this invariant measure, it is sufficient to show the ergodicity of T with respect to the Lebesgue measure. Let E satisfy $T^{-1}E=E$. Then for each Renyi-cell $X(a_1\cdots a_n)$ we have

$$\begin{aligned} m(E\cap X(a_1\cdots a_n)) &= m(T^{-n}E\cap X(a_1\cdots a_n)) \\ &= \int_{X(a_1\cdots a_n)} I_E(T^n z)dm(z) \\ &= \int_{U(a_n)} I_E(z')\,|\psi'_{a_1\cdots a_n}(z')|^2 dm(z') \\ &\geqq 5^{-4}m(U(a_n)\cap E)m(X(a_1\cdots a_n)) . \end{aligned}$$

Since $U(a_n)$ satisfies

$$U(a_n)\cap E\supset\psi_a(X)\cap T^{-1}E=\psi_a(E)$$

for either $a=2$ or -2, whichever is suitable, we have

$$m(U(a_n)\cap E)\geqq\int_E |\psi'_a(z)|\,dm(z)\geqq 3^{-4}m(E) .$$

So it follows that

$$m(E\cap X(a_1\cdots a_n))\geqq 15^{-4}m(E)m(X(a_1\cdots a_n)) . \tag{17}$$

By using Lemma 7, we obtain (17) for any fundamental T-cell. So we have

$$m(E\cap F)\geqq 15^{-4}m(E)m(F)$$

for all measurable sets F. If we take $F=E^c$, we obtain that $m(E)=0$ or $m(E^c)=0$, which completes the proof.

Since the natural extension of S is R^{-1}, we have the following

COROLLARY 1. *S is ergodic with respect to the invariant measure given in Theorem* 3.

In the rest of this section, we give several limit properties. From the definition of $f(z)$ we can easily show that $\log|z|$ is integrable with respect to the measure $d\mu(z)=f(z)dm(z)$. If we define

$$E=-\int_X \log|z|\, d\mu(z)\ ,$$

then from the ergodicity of T it follows, for almost all z,

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1}\log|T^k z|=-E\ . \tag{18}$$

Proposition 1. *For almost all $z\in X$, we have*

$$\lim_{n\to\infty}\frac{1}{n}\log|q_n(z)|=E\ , \tag{19}$$

$$\lim_{n\to\infty}\frac{1}{n}\log\left|z-\frac{p_n(z)}{q_n(z)}\right|=-2E \tag{20}$$

Proof. Since $p_{k+1}(Tz)=q_k(z)$, we obtain

$$\frac{\alpha}{q_n(z)}=\prod_{k=1}^{n}\frac{p_k(T^{n-k}z)}{q_k(T^{n-k}z)}\ ,$$

from which it follows that

$$\lim_{n\to\infty}\frac{1}{n}\log|q_n(z)|=-\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\log\left|\frac{p_k(T^{n-k}z)}{q_k(T^{n-k}z)}\right|\ . \tag{21}$$

From (10) we have

$$\left|\frac{z}{\frac{p_n(z)}{q_n(z)}}-1\right|\leqq\frac{\sqrt{2}}{|p_n(z)|}\ ,$$

so we can take such an absolute constant γ that

$$\left|\log|z|-\log\left|\frac{p_n(z)}{q_n(z)}\right|\right|\leqq\frac{\gamma}{\sqrt{n}}\ .$$

Then we obtain

$$\left|\frac{1}{n}\sum_{k=0}^{n-1}\log|T^k z|-\frac{1}{n}\sum_{k=0}^{n-1}\log\left|\frac{p_{n-k}(T^k z)}{q_{n-k}(T^k z)}\right|\right|$$
$$\leqq\frac{1}{n}\sum_{k=1}^{n}\frac{\gamma}{\sqrt{k}}\ .$$

So from (18) and (21) we can show (19), and from (10) we obtain (20).

THEOREM 5. *For almost all* $z \in X$, *we have*

$$\lim_{n\to\infty} \frac{1}{n} \log \mu(X(a_1(z)\cdots a_n(z)))$$
$$= \lim_{n\to\infty} \frac{1}{n} \log m(X(a_1(z)\cdots a_n(z)))$$
$$= -4E .$$

Thus the entropy of (T, μ) *is equal to* $4E$.

PROOF. By Shanon-McMillan-Breiman's Theorem, the limit of $-(1/n)\log \mu(X(a_1(z)\cdots a_n(z)))$ exist for almost all $z \in X$ and is equal to the entropy of (T, μ). Let $z \neq 1, -1, i, -i$ and define $\delta(z)$ by

$$\delta(z) = \min \{|z-1|, |z+1|, |z-i|, |z+i|\} .$$

Then for sufficiently large n ($n > (4/\delta(z))^2$), each element z' of $X(a_1(z)\cdots a_n(z))$ satisfies $\delta(z') \geqq (\delta(z)/2)$, so we can choose $0 < C_1 < C_2$ which satisfy $C_1 \leqq f(z') \leqq C_2$ for each $z' \in X(a_1(z)\cdots a_n(z))$. Thus we obtain

$$\lim_{n\to\infty} \frac{1}{n} \log \mu(X(a_1(z)\cdots a_n(z)))$$
$$= \lim_{n\to\infty} \frac{1}{n} \log m(X(a_1(z)\cdots a_n(z)))$$

for almost all $z \in X$.

Now let $z \in X_\infty$, then from the definition of X_∞, it follows that for infinitely many n, $X(a_1(z)\cdots a_n(z))$ becomes Renyi-cell. So from the Renyi's condition (16), we obtain

$$\frac{m(U_{a_n(z)})}{|q_n|^4\left(1+\frac{2}{3}\right)^4} \leqq m(X(a_1(z)\cdots a_n(z))) \leqq \frac{m(U_{a_n(z)})}{|q_n|^4\left(1-\frac{2}{3}\right)^4}$$

for infinitely many n. Thus we can show that $-(4/n)\log|q_n(z)|$ and $(1/n)\log m(X(a_1(z)\cdots a_n(z)))$ have the same limit for almost all $z \in X$, which complete the proof.

If we set $d\mu'(w) = g(w)dm(w)$ and define

$$E' = -\int_Y \log|w|\, d\mu'(w) ,$$

then we can show the same assertions for S. On the other hand, since the entropy of (S, μ') must be the same as that of (T, μ), we can conclude that $E=E'$. So we have the following

PROPOSITION 2. *For almost all $w \in Y$, we have*

$$\lim_{n\to\infty} \frac{1}{n} \log |s_n(w)| = E ,$$

$$\lim_{n\to\infty} \frac{1}{n} \log \left| w - \frac{r_n(w)}{s_n(w)} \right| = -2E .$$

THEOREM 6. *For almost all $w \in Y$, we have*

$$\begin{aligned} &\lim_{n\to\infty} \frac{1}{n} \log \mu'(Y(b_1(w)\cdots b_n(w))) \\ &= \lim_{n\to\infty} \frac{1}{n} \log m(Y(b_1(w)\cdots b_n(w))) \\ &= -4E . \end{aligned}$$

Thus the entropy of (S, μ') is equal to $4E$.

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