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On the Isotropy Subgroup of the Automorphism Group of a Parahermitian Symmetric Space

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Introduction

Let (M, I, g) be a parahermitian symmetric space [2] which is identified with a co-adjoint orbit of a real simple Lie group with Lie algebra $\mathfrak{g}$. Such a manifold M can be expressed as an affine symmetric coset space $G/C(Z)$, where G is the analytic subgroup generated by $\mathfrak{g}$ in the simply connected Lie group corresponding to the complexification of $\mathfrak{g}$, and $C(Z)$ is the centralizer in G of an element $Z \in \mathfrak{g}$ satisfying the condition (C) (see §1).

The purpose of this paper is to study the isotropy subgroup $C(Z)$—the number of its connected components and the structure of the identity component. Our method here is classification-free. A main result is Theorem 3.7, which is efficiently used in Kaneyuki and Williams [3], [4], in applying the method of geometric quantization.

NOTATIONS.

G^0 the identity component of a Lie group G,
G_α the set of elements in G left fixed by an automorphism α of G,
$\boldsymbol{C}^*$ (resp. $\boldsymbol{R}^*$) the multiplicative group of non-zero complex (resp. real) numbers,
$\boldsymbol{R}^+$ the multiplicative group of positive real numbers,
i $=\sqrt{-1}$.

§1. Symmetric triples.

Let $\mathfrak{g}$ be a real simple Lie algebra and $\mathfrak{h}$ be a subalgebra of $\mathfrak{g}$ and σ be an involutive automorphism of $\mathfrak{g}$ such that $\mathfrak{h}$ is the set of σ-fixed elements in $\mathfrak{g}$. Then the triple $\{\mathfrak{g}, \mathfrak{h}, \sigma\}$ is called a simple symmetric triple. Suppose further that $\{\mathfrak{g}, \mathfrak{h}, \sigma\}$ satisfies the following condition (C) (which is equivalent to (C_3) in [2]):

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(C) there exists an element $Z \in \mathfrak{g}$ such that ad Z is a semisimple operator with eigenvalues 0, ± 1 only and that $\mathfrak{h}$ is the centralizer of Z in $\mathfrak{g}$.

We will denote by $\mathfrak{m}^{\pm}$ the ± 1 eigenspaces in $\mathfrak{g}$ of ad Z and put $\mathfrak{m}=\mathfrak{m}^{+}+\mathfrak{m}^{-}$. Then it is known [5] that the center $\mathfrak{z}(\mathfrak{h})$ of $\mathfrak{h}$ is of one or two dimension over $\boldsymbol{R}$. $\{\mathfrak{g}, \mathfrak{h}, \sigma\}$ is said to be *of the first category* or *of the second category*, according as $\dim \mathfrak{z}(\mathfrak{h})=1$ or 2. If $\{\mathfrak{g}, \mathfrak{h}, \sigma\}$ is of the second category, then $\mathfrak{g}$ has a structure of complex Lie algebra and σ is an involutive automorphism of $\mathfrak{g}$, regarded as the complex Lie algebra [5]; in particular, $\mathfrak{h}$ is a complex subalgebra of $\mathfrak{g}$. Let $\{\mathfrak{g}_0, \mathfrak{h}_0, \sigma\}$ be of the first category, and let $\mathfrak{g}$ and $\mathfrak{h}$ be the complexifications of $\mathfrak{g}_0$ and $\mathfrak{h}_0$, respectively. Then the triple $\{\mathfrak{g}, \mathfrak{h}, \sigma\}$ satisfies the condition (C) and it is of the second category, where σ is the $\boldsymbol{C}$-linear extension of the original σ on $\mathfrak{g}_0$.

§ 2. Symmetric triple of the second category.

Let $\{\mathfrak{g}, \mathfrak{h}, \sigma\}$ be a (complex) simple symmetric triple of the second category (which satisfies the condition (C)). The subalgebra $\mathfrak{h}$ is then reductive and can be written as the direct sum of complex ideals,

$$\mathfrak{h}=\mathfrak{s}+\mathfrak{z}(\mathfrak{h}) , \tag{2.1}$$

where $\mathfrak{s}$ is the commutator (semisimple) subalgebra $[\mathfrak{h}, \mathfrak{h}]$ of $\mathfrak{h}$ and $\mathfrak{z}(\mathfrak{h})$ is the center of $\mathfrak{h}$. Let us denote by G the simply connected (complex) Lie group generated by Lie $G=\mathfrak{g}$, and let H be the analytic subgroup of G corresponding to $\mathfrak{h}$. We extend σ to an involutive automorphism of G, which is denoted again by σ. Then H coincides with the set of σ-fixed elements in G [5], and so it is closed in G. Let us denote by S the (closed) analytic subgroup of G corresponding to $\mathfrak{s}$. Then we can write

$$H=SZ(H) , \tag{2.2}$$

where $Z(H)$ is the analytic subgroup of G corresponding to $\mathfrak{z}(\mathfrak{h})$.

LEMMA 2.1. $Z(H) \cong \boldsymbol{C}^{*}$.

PROOF. It is known [5] that the Lie algebra $\mathfrak{z}(\mathfrak{h})$ is generated by both elements Z in the condition (C) and iZ, which satisfies

$$\mathrm{ad}_{\mathfrak{m}}\, iZ=\begin{cases} i & \text{on } \mathfrak{m}^{+} , \\ -i & \text{on } \mathfrak{m}^{-} . \end{cases} \tag{2.3}$$

From this it follows that $\mathrm{Ad}_{\mathfrak{g}}Z(H)$ is isomorphic with C^*. On the other hand, G is complex simple and so its center is finite. Therefore $Z(H)$ is isomorphic with C^*.

Let τ be a Cartan involution of $\mathfrak{g}$ which commutes with σ. Then we have the corresponding Cartan decomposition

$$\mathfrak{g}=\mathfrak{k}+i\mathfrak{k}\ , \tag{2.4}$$

where $\mathfrak{k}$ is a maximal compact subalgebra of $\mathfrak{g}$. The subalgebra $\mathfrak{h}$ and consequently $\mathfrak{s}$ and $\mathfrak{z}(\mathfrak{h})$ are stable under τ. Therefore we have

$$\mathfrak{h}=\mathfrak{h}\cap\mathfrak{k}+\mathfrak{h}\cap i\mathfrak{k}\ , \tag{2.5}$$

$$\mathfrak{s}=\mathfrak{s}\cap\mathfrak{k}+\mathfrak{s}\cap i\mathfrak{k}\ , \tag{2.6}$$

$$\mathfrak{z}(\mathfrak{h})=\mathfrak{z}(\mathfrak{h})\cap\mathfrak{k}+\mathfrak{z}(\mathfrak{h})\cap i\mathfrak{k}\ . \tag{2.7}$$

(2.6) is a Cartan decomposition of $\mathfrak{s}$. Let K, K_S, K_Z be the analytic subgroups of G generated by $\mathfrak{k}$, $\mathfrak{s}\cap\mathfrak{k}$, $\mathfrak{z}(\mathfrak{h})\cap\mathfrak{k}$ which are maximal compact subgroups of G, S, $Z(H)$, respectively. Then we have

$$G=K\cdot P\ ,\quad K\cap P=(1)\ , \tag{2.8}$$

$$S=K_S\cdot P_S\ ,\quad K_S\cap P_S=(1)\ , \tag{2.9}$$

$$Z(H)=K_Z\cdot P_Z\ ,\quad K_Z\cap P_Z=(1)\ , \tag{2.10}$$

where $P=\exp i\mathfrak{k}$, $P_S=\exp(\mathfrak{s}\cap i\mathfrak{k})$ and $P_Z=\exp(\mathfrak{z}(\mathfrak{h})\cap i\mathfrak{k})$.

Lemma 2.2. *$\Gamma=S\cap Z(H)$ is a finite cyclic group.*

Proof. Let us take an arbitrary element $a\in\Gamma$. Then, by (2.9) and (2.10), a can be written as $a=k_1\exp X_1=k_2\exp X_2$, where $k_1\in K_S$, $X_1\in\mathfrak{s}\cap i\mathfrak{k}$, $k_2\in K_Z$ and $X_2\in\mathfrak{z}(\mathfrak{h})\cap i\mathfrak{k}$. Noting that $[X_1, X_2]=0$, we have $k_2^{-1}k_1=\exp(X_2-X_1)$. So, from (2.8) it follows that $k_2^{-1}k_1=\exp(X_2-X_1)=1$. Since $\exp\colon i\mathfrak{k}\to P$ is a diffeomorphism, we get $X_1=X_2\in\mathfrak{s}\cap\mathfrak{z}(\mathfrak{h})=(0)$. Thus we obtain $a=k_2\in K$. Since S and $Z(H)$ are closed in G, Γ is closed in the compact group K. Hence, from Lemma 2.1 it follows that Γ is a finite cyclic group.

Remark 2.3. By general theory of parabolic subgroups of complex semisimple Lie groups, the centralizer $\tilde{C}(Z)$ of Z in G is connected:

$$\tilde{C}(Z)=H\ . \tag{2.11}$$

Lemma 2.4. *S is simply connected.*

Proof. Let us put $M^+=\exp\mathfrak{m}^+\subset G$. Then the coset space G/H is the cotangent bundle of $G/H\cdot M^+=K/K\cap H$ which is a compact irreducible hermitian symmetric space (cf. Takeuchi [6]). $K^*:=K\cap H$ is a maximal compact subgroup of H. This implies that $\pi_1(H)\cong\pi_1(K^*)$. Let us consider the exact sequence of the homotopy groups:

$$\cdots\longrightarrow\pi_2(K^*)\longrightarrow\pi_2(K)\longrightarrow\pi_2(K/K^*)\longrightarrow\pi_1(K^*)\longrightarrow\pi_1(K)\longrightarrow\cdots. \tag{2.12}$$

Since K is simply connected, compact, semisimple, we have $\pi_2(K)=\pi_1(K)=0$ (E. Cartan [1]). Therefore $\pi_2(K/K^*)\cong\pi_1(K^*)$, K/K^* is a compact hermitian symmetric space, and so $\pi_1(K/K^*)=0$. Hence, by the Hurewicz isomorphism, we have $\pi_2(K/K^*)\cong H_2(K/K^*, \boldsymbol{Z})$. But it is well-known that $H_2(K/K^*, \boldsymbol{Z})\cong\boldsymbol{Z}$. Under the covering homomorphism of $S\times Z(H)$ onto $H=SZ(H)$, $\pi_1(S\times Z(H))$ is regarded as a subgroup of $\pi_1(H)\cong\pi_1(K^*)\cong H_2(K/K^*, \boldsymbol{Z})\cong\boldsymbol{Z}$. Also, by Lemma 2.1, we have $\pi_1(S\times Z(H))\cong\pi_1(S)\times\pi_1(Z(H))\cong\pi_1(S)\times\boldsymbol{Z}$. Therefore we should have $\pi_1(S)=0$.

LEMMA 2.5. *We have the decomposition*

$$H=\tilde{S}\cdot P_Z\,, \quad \text{(direct product)} \tag{2.13}$$

where $\tilde{S}=SK_Z$ *and* $P_Z\cong\boldsymbol{R}^+$.

PROOF. $\Gamma=S\cap Z(H)$ is a finite group and so it is contained in the maximal compact subgroup K_Z of $Z(H)$. Take an element $r\in SK_Z\cap P_Z$ and write $r=st$, where $s\in S$, $t\in K_Z$. Then $s=rt^{-1}\in S\cap Z(H)=\Gamma\subset K_Z$. Hence $r=st\in K_Z\cap P_Z=(1)$ (cf. (2.10)).

§ 3. Symmetric triple of the first category.

Let $\{\mathfrak{g}_0, \mathfrak{h}_0, \sigma\}$ be a simple symmetric triple of the first category satisfying the condition (C). Let $\mathfrak{g}$ and $\mathfrak{h}$ be the complexifications of $\mathfrak{g}_0$ and $\mathfrak{h}_0$, respectively. Then, as is mentioned in § 1, $\{\mathfrak{g}, \mathfrak{h}, \sigma\}$ is a (complex) simple symmetric triple of the second category. All arguments in § 2 are then valid for $\{\mathfrak{g}, \mathfrak{h}, \sigma\}$ here. We will keep the notations in § 2.

LEMMA 3.1. *The element* $Z\in\mathfrak{g}_0$ *satisfying the condition* (C) *is contained in* $\mathfrak{z}(\mathfrak{h})\cap i\mathfrak{k}$; $\mathfrak{z}(\mathfrak{h})$ *is spanned by* Z *and* iZ *over* $\boldsymbol{R}$.

PROOF. (C) implies that the eigenvalues of $\operatorname{ad} Z$ on $\mathfrak{g}$ are also $0, \pm 1$. By (2.7) we can write $Z=Z'+Z''$, $Z'\in\mathfrak{z}(\mathfrak{h})\cap\mathfrak{k}$, $Z''\in\mathfrak{z}(\mathfrak{h})\cap i\mathfrak{k}$. We have $\operatorname{ad} Z=\operatorname{ad} Z'+\operatorname{ad} Z''$; here $\operatorname{ad} Z'$ is a semisimple operator and has purely imaginary eigenvalues only, and $\operatorname{ad} Z''$ is also semisimple with real eigenvalues only. So, $\mathrm{ad}\, Z'=0$ and consequently $Z'=0$. Hence we get $Z=Z''\in\mathfrak{z}(\mathfrak{h})\cap i\mathfrak{k}$. The second assertion is evident, since $\mathfrak{z}(\mathfrak{h})$ is the complexification of the center $\mathfrak{z}(\mathfrak{h}_0)$ of $\mathfrak{h}_0$.

Let G_0 be the analytic subgroup of G generated by $\mathfrak{g}_0$, and $C(Z)$ be the centralizer of Z in G_0. The conjugation θ of $\mathfrak{g}$ with respect to $\mathfrak{g}_0$ extends to an involutive automorphism of G which is denoted again by θ.

LEMMA 3.2. *We have*

$$C(Z)=H_\theta\,, \tag{3.1}$$

where H_θ denotes the set of elements in H left fixed by θ.

PROOF. Since G is simply connected, the set G_θ of θ-fixed elements in G is connected; so we have $G_\theta=G_0$. Let $\tilde{C}(Z)$ be the centralizer of Z in G. Then, by (2.11), we get $H=\tilde{C}(Z)$. Therefore $H_\theta=(\tilde{C}(Z))_\theta=\tilde{C}(Z)\cap G_0=C(Z)$.

Since Γ is a finite cyclic group, it is given by

$$\Gamma=\{1, z, z^2, \cdots, z^{m-1}\}\cong \boldsymbol{Z}_m\,. \tag{3.2}$$

Let 2Γ be the subgroup of the squares of elements in Γ. We will define a homomorphism ω of $C(Z)$ to $\Gamma/2\Gamma\cong\boldsymbol{Z}_2$. Let us take an arbitrary element $x\in C(Z)$ and write it in the form

$$x=ab, \quad \text{where} \quad a\in S \quad \text{and} \quad b\in Z(H)\,. \tag{3.3}$$

Then we have the following

LEMMA 3.3. *The element $y=a^{-1}\theta(a)=b\theta(b^{-1})$ is in Γ.*

PROOF. Since $x\in G_0$, we have $a^{-1}\theta(a)=b\theta(b^{-1})$. θ leaves $\mathfrak{h}$ stable and so does $\mathfrak{s}$. Since S is connected, we get $\theta(a)\in S$. By Lemma 3.1, $\theta(b^{-1})\in Z(H)$. So we get $a^{-1}\theta(a)=b\theta(b^{-1})\in S\cap Z(H)=\Gamma$.

We define a map ω by putting

$$\omega(x)=[y]\,, \tag{3.4}$$

where $[y]$ denotes the equivalence class of y in $\Gamma/2\Gamma$. It can be verified (cf. Proof of Theorem 3.5 below) that $\omega(x)$ is well-defined, that is, $\omega(x)$ does not depend on the choices of a and b in (3.3).

LEMMA 3.4. *$\omega: C(Z)\to\Gamma/2\Gamma$ is a homomorphism.*

PROOF. Take two elements $x=ab$, $x_1=a_1b_1$, where a, $a_1\in S$ and b, $b_1\in Z(H)$. Then, since $a^{-1}\theta(a)\in\Gamma\subset Z(H)$, we have $a^{-1}\theta(a)\cdot a_1^{-1}\theta(a_1)=$

$a_1^{-1}(a^{-1}\theta(a))\theta(a_1)=(aa_1)^{-1}\theta(aa_1)$, which implies $\omega(x)\omega(x_1)=\omega(xx_1)$. Let π be the natural projection of Γ onto $\Gamma/2\Gamma$. We denote $\pi^{-1}(\omega(C(Z)))$ by Γ_0, which is a subgroup of Γ. Note that $\omega(C(Z))=\Gamma_0/2\Gamma$.

THEOREM 3.5. *$S_\theta Z(H)_\theta$ is a normal subgroup of $C(Z)$, and ω induces the following isomorphism:*

$$\mathrm{C}(Z)/S_\theta Z(H)_\theta \cong \Gamma_0/2\Gamma \ . \tag{3.5}$$

PROOF. Let us take an element $x\in S_\theta Z(H)_\theta$. We can write x in two ways: $x=ab=a_1b_1$, where $a\in S_\theta$, $b\in Z(H)_\theta$, $a_1\in S$, $b_1\in Z(H)$. Then we get $a_1^{-1}a=b_1b^{-1}\in S\cap Z(H)=\Gamma$, and so we have $a=a_1z^l$, $b_1=bz^l$. Γ is a finite subgroup of $Z(H)$ and so it is contained in a unique maximal compact subgroup K_Z of $Z(H)$. Since $K_Z=\exp \boldsymbol{R}iZ$, we have $\theta(\gamma)=\gamma^{-1}$ for $\gamma\in\Gamma$. Noting this in mind, $a_1^{-1}\theta(a_1)=(az^{-l})^{-1}\theta(az^{-l})=z^la^{-1}\theta(a)z^l=z^{2l}\in 2\Gamma$, which implies $S_\theta Z(H)_\theta\subset \mathrm{Ker}\,\omega$.

Conversely, let us take $x\in\mathrm{Ker}\,\omega$, and write it in the form of (3.3). Then $y=a^{-1}\theta(a)=b\theta(b^{-1})\in 2\Gamma$; we write $y=y_1^2$, $y_1\in\Gamma$. Then we put $x=a_1b_1$, where $a_1=ay_1$ and $b_1=y_1^{-1}b$. Then we have $\theta(a_1)=\theta(ay_1)=\theta(a)\theta(y_1)=\theta(a)y_1^{-1}=ayy_1^{-1}=ay_1=a_1$, which implies $a_1\in S_\theta$. Analogously we have $\theta(b_1)=b_1$, which implies $b_1\in Z(H)_\theta$. These arguments show that $\mathrm{Ker}\,\omega\subset S_\theta Z(H)_\theta$.

Let φ be the isomorphism of $Z(H)$ onto $\boldsymbol{C}^*$ given in Lemma 2.1, and let us consider the conjugation $\tilde\theta=\varphi\theta\varphi^{-1}$ of $\boldsymbol{C}^*$. Then it is easily verified that $\tilde\theta$ is the restriction of the usual complex conjugation of $\boldsymbol{C}$ to $\boldsymbol{C}^*$. Therefore $\varphi(Z(H)_\theta)=(\varphi(Z(H)))_{\tilde\theta}=\boldsymbol{C}^*_{\tilde\theta}=\boldsymbol{R}^*$. We will give the structure of the subgroup $S_\theta Z(H)_\theta$.

LEMMA 3.6. *If m in* (3.2) *is odd, then*

$$S_\theta Z(H)_\theta\cong S_\theta\times Z(H)_\theta\cong S_\theta\times\boldsymbol{R}^*\ ; \tag{3.6}$$

if m is even, then $S_\theta Z(H)_\theta$ is connected and

$$S_\theta Z(H)_\theta\cong S_\theta\times\boldsymbol{R}^+\ . \tag{3.7}$$

PROOF. Suppose first that m is odd. Then it is enough to show $S_\theta\cap Z(H)_\theta=(1)$. From the fact mentioned just before the lemma, it follows that

$$\begin{aligned}\varphi(S_\theta\cap Z(H)_\theta)&=\varphi(\Gamma\cap Z(H)_\theta)\\&=\varphi(\Gamma)\cap\varphi(Z(H)_\theta)=\varphi(\Gamma)\cap\boldsymbol{R}^*\ .\end{aligned}\tag{3.8}$$

$\varphi(\Gamma)$ here, isomorphic to $\boldsymbol{Z}_m$, is a cyclic subgroup of $U(1)$, and so $\varphi(\Gamma)$ is the group of the m-th roots of unity. Since m is odd, $-1\in\boldsymbol{R}^*$ is not contained in $\varphi(\Gamma)$. Therefore $\varphi(\Gamma)\cap \boldsymbol{R}^*=(1)$, which implies $S_\theta\cap Z(H)_\theta=(1)$.

Let us consider next the case where m is even. In this case, the group $\varphi(\Gamma)$ contains -1, and so $\varphi(\Gamma)\cap \boldsymbol{R}^*=\{\pm 1\}$. On the other hand, since $Z(H)_\theta$ is a central subgroup in $C(Z)$, we have the isomorphisms

$$S_\theta Z(H)_\theta \cong (S_\theta\times Z(H)_\theta)/S_\theta\cap Z(H)_\theta \cong (S_\theta\times \boldsymbol{R}^*)/\{\pm 1\}\ . \tag{3.9}$$

The natural projection π of $S_\theta\times \boldsymbol{R}^*$ onto $(S_\theta\times \boldsymbol{R}^*)/\{\pm 1\}$ induces an isomorphism of $S_\theta\times \boldsymbol{R}^+$ onto $(S_\theta\times \boldsymbol{R}^*)/\{\pm 1\}$.

THEOREM 3.7. *Let $\{\mathfrak{g}_0, \mathfrak{h}_0, \sigma\}$ be a simple symmetric triple of the first category satisfying the condition* (C). *Let G_0 be the analytic subgroup, generated by $\mathfrak{g}_0$, of the simply connected Lie group corresponding to the complexification of $\mathfrak{g}_0$. Let $C(Z)$ be the centralizer of Z in G_0 whose identity component is denoted by $C^0(Z)$, and let S_0 be the analytic subgroup of $C(Z)$ generated by the commutator subalgebra $\mathfrak{s}_0=[\mathfrak{h}_0, \mathfrak{h}_0]$. Then, if m in* (3.2) *is odd, then*

$$C(Z)\cong S_0\times \boldsymbol{R}^*\ . \tag{3.10}$$

If m in (3.2) *is even, then*

$$C^0(Z)\cong S_0\times \boldsymbol{R}^+\ ; \tag{3.11}$$

furthermore, in this case, we have $[C(Z)\colon C^0(Z)]=1$ or 2, according as $\Gamma_0=2\Gamma$ or $\Gamma_0=\Gamma$.

PROOF. By Lemma 2.4, S is simply connected and so S_θ is connected [5]. Therefore we have $S_\theta=S\cap G_0=S_0$. Suppose first that m is odd. Then $2\Gamma=\Gamma$ and so $\Gamma_0=2\Gamma$. By Theorem 3.5 and Lemma 3.6, we have the first assertion. The other case follows also from Theorem 3.5 and Lemma 3.6.

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