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On Logarithmic Canonical Divisors on Threefolds

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Introduction

The aim of this paper is to give a numerical criterion for the logarithmic canonical or the logarithmic anti-canonical divisor on a threefold to be ample. As a corollary we obtain a practical definition of logarithmic Fano threefolds. Let V be a non-singular projective variety over an algebraically closed field of characteristic zero and $D=D_1+\cdots+D_S$ a reduced divisor whose components are smooth and crossing normally on V. We consider here such a pair (V, D), which is called a non-singular pair of dimension $n=\dim V$. Let K_V, or in short K, denote a canonical divisor on V. Then $K+D$ (resp. $-K-D$) is called the logarithmic canonical divisor (resp. logarithmic anti-canonical divisor) on V (cf. [3, Chap. 11]). We prove the following

THEOREM. *Let (V, D) be a non-singular pair of dimension 3. Then* (i) *under the condition that $\kappa(K+D, V)\geqq 0$, $K+D$ is ample if and only if $K+D$ is numerically positive; i.e. $(K+D)\cdot C>0$ for all curves C on V,* (ii) *under the condition that $\kappa(-K-D, V)\geqq 0$, $-K-D$ is ample if and only if $-K-D$ is numerically positive.*

COROLLARY (cf. [4]). *Let (V, D) be as in the Theorem. Then (V, D) is a logarithmic Fano threefold if and only if the following two conditions are satisfied.*

(a) *The linear system $|-K-D|$ is non-empty.*

(b) *$-K-D$ is numerically positive.*

PROOF. The if part follows from the Theorem.

Let (V, D) be a logarithmic Fano threefold. Applying Norimatsu Vanishing ([5, Theorem 1]) we deduce

$$H^i(V, \mathcal{O}_V(-K-D))=0 \quad \text{for} \quad i>0$$

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and therefore

$$\dim H^0(V, \mathcal{O}_V(-K-D)) = \chi(\mathcal{O}_V(-K-D)) .$$

In order to calculate $D \cdot c_2(V)$ we consider $\chi(\mathcal{O}_V(-D))$. From Riemann-Roch and Norimatsu Vanishing,

$$\begin{aligned}\dim H^0(V, \mathcal{O}_V(-D)) &= \chi(\mathcal{O}_V(-D)) \\ &= 1/6(-D)^3 - 1/4(-D)^2 \cdot K + 1/12(-D) \cdot (K^2 + c_2)_V + \chi(\mathcal{O}_V) .\end{aligned}$$

Since $\chi(\mathcal{O}_V) = 1$ for V ([4, Corollary 2.2]), we obtain

$$D \cdot c_2 = 12 - D \cdot (D+K) \cdot (2D+K) .$$

It follows that

$$\chi(\mathcal{O}_V(-K-D)) = 1/2(-K-D)^3 + 1/2(-K-D)^2 \cdot D + 2 .$$

Hence $\dim H^0(V, \mathcal{O}_V(-K-D)) \geqq 3$. Q.E.D.

REMARKS. (1) The case $D=0$ in (i) of the theorem was a result of Wilson ([7, Proposition 2.3]).

(2) For two-dimensional non-singular pair (V, D), we can derive that $\kappa(K+D, V) \geqq 0$ (resp. $\kappa(-K-D, V) \geqq 0$) from the numerical positivity of $K+D$ (resp. $-K-D$) by the classification theory of divisors on surfaces ([6, Theorem 2]). Hence, if $\dim V = 2$, then the same assertions of the above theorem hold even if we omit the conditions $\kappa(K+D, V) \geqq 0$ in (i) and $\kappa(-K-D, V) \geqq 0$ in (ii).

§1. Proof of (i).

The "only if" part being obvious, we shall prove the "if" part. The proof follows the idea of Wilson ([7]).

Since $K+D$ is numerically effective, we have $(K+D)^2 \cdot S \geqq 0$ for all surfaces S (surfaces and curves are always irreducible in this paper). We first show that there is no surface S with $(K+D)^2 \cdot S = 0$.

Suppose that there exists such a surface S on V.

CLAIM 1. *S is a fixed component of $|m(K+D)|$, provided that $|m(K+D)| \neq \emptyset$ for $m > 0$.*

PROOF. There are only three possibilities:
(1) $A \cap S = \emptyset$ for some $A \in |m(K+D)|$,
(2) $A \cap S$ is a curve for some $A \in |m(K+D)|$,
(3) $A \supset S$ for all $A \in |m(K+D)|$.

The cases (1) and (2) are impossible by hypothesis. Q.E.D.

Now we have $|m(K+D)|=|B|+rS$, where B is an effective divisor not containing S. Note $r\geqq 1$ by Claim 1.

Thus $(K+D)\cdot B\cdot S+r(K+D)\cdot S^2=m(K+D)^2\cdot S=0$ and therefore $(K+D)\cdot S^2=-(K+D)\cdot B\cdot S\leqq 0$.

Suppose that $(K+D)\cdot S^2<0$. By Riemann-Roch on S, we have

$$(*)\qquad \chi(\mathcal{O}_S(n(K+D)))=1/2(n(K+D)\cdot(D-S)\cdot S)_V+\chi(\mathcal{O}_S)\ .$$

CLAIM 2. *$\chi(\mathcal{O}_S(n(K+D)))\geqq 1$ for sufficiently large n.*

PROOF. Case (1): $D=0$. In this case, we have

$$\chi(\mathcal{O}_S(nK))=-1/2\cdot nK\cdot S^2+\chi(\mathcal{O}_S)\ .$$

Since $K\cdot S^2<0$, $\chi(\mathcal{O}_S(nK))>0$ for sufficiently large n.

Case (2): $D\not\supset S$. In this case, $D\cdot S$ is effective and so $(K+D)\cdot D\cdot S\geqq 0$. By assumption, $(K+D)\cdot S^2<0$. Hence by (*), we obtain the result.

Case (3): $D=S$. Since $K_S\sim(K+D)|_S$, K_S is numerically positive. Hence there exists m such that $|mK_S|\neq\emptyset$, and clearly K_S is not numerically equivalent to 0. Thus $(K_S)^2{}_S>0$. But this contradicts our assumption to the fact that $(K_S)^2{}_S=(K+D)^2\cdot S=0$.

Case (4): $D=S+D'$, where $D'\not\supset S$. In this case, (*) can be rewritten as follows:

$$\chi(\mathcal{O}_S(n(K+D)))=1/2\cdot n(K+D)\cdot D'\cdot S+\chi(\mathcal{O}_S)\ .$$

If $D'\cdot S$ is a non-zero 1-cycle, then $(K+D)\cdot D'\cdot S>0$. Hence we are through. If $D'\cdot S=0$, then $\chi(\mathcal{O}_S(n(K+D)))=\chi(\mathcal{O}_S)\geqq 1$ since S turns out to be a smooth surface of general type in this case. Q.E.D.

CLAIM 3. $(K+D)\cdot S^2=0$.

PROOF. Suppose that $(K+D)\cdot S^2<0$. By Serre duality ([2, p. 244]),

$$h^2(S,\ n(K+D)|_S)=h^0(S,\ -(n-1)(K+D)|_S+(S-D)|_S)\ .$$

Since $(K+D)|_S$ is numerically positive, it follows that $|-(n-1)(K+D)|_S+(S-D)|_S)|=\emptyset$ for sufficiently large n. Thus by Claim 2,

$$h^0(S,\ n(K+D)|_S)\geqq\chi(\mathcal{O}_S(n(K+D)))>0$$

for sufficiently large n.

Let Γ be a curve defined by a non-zero section of $H^0(S,\ n(K+D)|_S)$. If $\Gamma\neq 0$, then $(K+D)\cdot\Gamma=n(K+D)^2\cdot S=0$. This contradicts the numerical positivity of $K+D$. If $\Gamma=0$, we have $(K+D)\cdot C=((K+D)|_S\cdot C)_S=0$, for any curve C on S, which also contradicts the hypothesis. Q.E.D.

Let $S_1, \cdots, S_r$ be all surfaces which satisfy $(K+D)^2\cdot S_i=0$. In this case, $(K+D)\cdot S_i^2=0$ by Claim 3. Now we have $|m(K+D)|=|D_m|+\sum_{i=1}^r r_iS_i$, where D_m is an effective divisor not containing any S_i. Since $(K+D)\cdot(D_m+\sum_{i=1}^r r_iS_i)\cdot S_i=0$ for each i and by the numerical positivity of $K+D$, we have $D_m\cap S_i=\varnothing$, for any i, and $S_i\cap S_j=\varnothing$, for any $i\neq j$.

Now recall the following theorem, due to T. Fujita·

THEOREM ([1, Theorem 1.10]). *Let L be a line bundle on an algebraic scheme V. Suppose that the restriction of L to the base locus of $|L|$ is ample. Then nL is base point free for sufficiently large n.*

CLAIM 4. *$Bs|nD_m|=\varnothing$ for $n\gg 0$.*

PROOF. Let B be an irreducible component of the set $Bs|D_m|$. We show that $D_m|_B$ is ample.

Case (1): $\dim B=2$. Let C be a curve on B. Since C doesn't meet any S_i, we have

$$(D_m|_B\cdot C)_B=\Big(m(K_V+D)-\sum_{i=1}^r r_iS_i\Big)\cdot C$$
$$=(K_V+D)\cdot C>0\ .$$

Moreover,

$$(D_m|_B)^2{}_B=m^2(K+D)^2\cdot B\ .$$

This must be positive, since otherwise B must coincide with one of S_i's. This contradicts the choice of D_m. Hence, by Nakai's criterion, $D_m|_B$ is ample in this case.

Case (2): $\dim B\leqq 1$. Obvious.

By applying Fujita's theorem, we obtain Claim 4. Q.E.D.

Taking n and m as Claim 4, we have

$$nm(K+D)\sim nD_m+\sum_{i=1}^r nr_iS_i\ .$$

This can be also written as $nm(K+D)\sim D_{nm}+\sum_{i=1}^r r_i'S_i$.

CLAIM 5. $nD_m\sim D_{nm}$.

PROOF. First we show that

$$H^0(nD_m)\cong H^0\left(nD_m+\sum_{j=1}^{r} nr_jS_j\right) .$$

Let S' be an effective divisor with $S'\leqq\sum_{j=1}^{r} nr_jS_j$.

Fix some S_i, say S, we have an exact sequence

$$0\longrightarrow\mathcal{O}_V(nD_m+S')\longrightarrow\mathcal{O}_V(nD_m+S'+S)\longrightarrow\mathcal{O}_S(nD_m+S'+S)\longrightarrow 0 .$$

Since nD_m and the S_j are all disjoint, we have

$$(nD_m+S'+S)|_S\sim r'S|_S$$

for some $r'>0$.

Suppose that $r'S|_S$ is linearly equivalent to an effective curve Γ on S. Then we have

$$0<(K+D)\cdot\Gamma=r'(K+D)\cdot S^2=0 .$$

This is a contradiction.

Suppose next that $r'S|_S\sim 0$. Then for any curve C on S, we have

$$(K+D)\cdot C=(nD_m)\cdot C=0 .$$

This contradicts the hypothesis.

Thus, by induction on the number of components, we have

$$H^0(nD_m)\cong H^0\left(nD_m+\sum_{j=1}^{r} nr_jS_j\right) .$$

This implies that $|nm(K+D)|=|nD_m|+\sum_{j=1}^{r} nr_jS_j$. Since nD_m doesn't contain any of S_j's, nD_m coinsides with D_{nm}. Q.E.D.

Now we may assume that $Bs|D_m|=\emptyset$ and the m-th logarithmic canonical mapping (cf. [3, 11.6])

$$\Phi_{|m(K+D)|}: V\longrightarrow W$$

is a morphism, which we denote by ψ. It is clear that ψ contracts the surfaces S_i to points on W.

Note that W is a threefold; otherwise we have a curve Γ lying on a fiber of ψ, which meets neither $D_m=\psi^*L$, L being a hyperplane section of W, nor any S_i. This implies that $m(K+D)\cdot\Gamma=D_m\Gamma+\sum_{i=1}^{r} r_iS_i\cdot\Gamma=0$, a contradiction.

Take a general hyperplane section H on V. We know that H is a non-singular surface, $U=\psi(H)$ is a surface and $\psi|_H\colon H\to U$, contracts the reducible curves $H\cdot S_i$ to points on U. Hence, by [3, Theorem 8.5],

$(H\cdot S_i)^2_H = H\cdot S_i^2 < 0$. However, $H\cdot S_i$ are reducible curves and

$$m(K+D)\cdot H\cdot S_i = D_m\cdot H\cdot S_i + \sum_{j=1}^{r} r_j H\cdot S_j\cdot S_i = r_i H\cdot S_i^2 > 0\ ,$$

which contradicts the above inequality.

Thus we have shown that $(K+D)^2\cdot S>0$ for all surfaces S on V. Since $\chi(K+D, V)\geqq 0$, this gives also that $(K+D)^3>0$. Hence $K+D$ is ample by Nakai's criterion.

This completes the proof.

§2. Proof of (ii).

The proof in this case is quite similar to that in §1. We have only to replace $K+D$ by $-K-D$. But the proof of Claim 2 is slightly different.

Let V and D be as in the theorem and $-K-D$ satisfy the conditions of (ii) of the theorem. Let S be a surface (if exists) with $(-K-D)^2\cdot S=0$. Then we have

CLAIM 2′. *$\chi(\mathcal{O}_S(n(-K-D)))\geqq 1$ for sufficiently large n.*

PROOF. By the similar calculation as in Claim 2, we have

$$\chi(\mathcal{O}_S(n(-K-D))) = 1/2\cdot(n(-K-D)|_S\cdot(D-S)|_S)_S + \chi(\mathcal{O}_S)\ .$$

Assume that $(-K-D)\cdot S^2<0$. In the case where $D=0$ or $D\not\supset S$, the proof of the above statement is easy. But in the case where $D=S$, we have to show that $\chi(\mathcal{O}_S)>0$. Since $-K_S=(-K-D)|_S$ is numerically positive, we have $\kappa(-K_S, S)\geqq 0$ (see Remark (2) in Introduction). This implies that $(-K_S)^2{}_S>0$ and therefore $-K_S$ is ample. Hence S is a del Pezzo surface and therefore $\chi(\mathcal{O}_S)=1$. The rest of the proof is easy so we omit this.

Q.E.D.

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