

Hardy Spaces of 2-Parameter Brownian Martingales

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Introduction

In this note we shall study Hardy spaces and BMO spaces of martingales on the product Brownian spaces and their applications to function theory on the torus.

Hardy spaces and BMO spaces of martingales on the Brownian spaces were studied by N.Th.Varopoulos ([12], [13]) in connection with function theory on the unit circle T . If we will deal with H^p and BMO martingales related to function theory on the torus T^2 , then we need to consider ones of 2-parameter on the product Brownian spaces (cf. [14]). J. Brossard and L. Chevalier built stochastic integral theory for such martingales and defined Hardy spaces H^p . They generalized the Burkholder-Davis-Gundy inequality to the product Brownian spaces ([4]). We here write K^p instead of H^p . H. Sato defined 2-parameter BMO martingales and proved $(K^1)^* = BMO$ ([10]).

After some preliminaries in §1, in §2 we define Hilbert transforms H_j on K^p ($j=1, 2, 3; 0 < p < \infty$) modeled after Hilbert transforms for 1-parameter martingales defined by N.Th.Varopoulos ([13]). In this section we prove equivalence of K^1 -norm and $\|X\|_{L^1} + \sum_{j=1}^3 \|H_j X\|_{L^1}$ -norm (cf. Theorem 2.4). This extends a theorem of Varopoulos ([13, Theorem 3.2]). In §3 we study projections N and M introduced by N.Th. Varopoulos and obtain some results on them. Our main theorem is Theorem 3.5 which states that $\mathcal{H}^1(T^2)$ (resp. $BMO(T^2)$) is isomorphic to a closed complemented subspace of K^1 (resp. BMO). This theorem implies several results as corollaries. Two of these are a theorem of Sato ([10]) and a theorem of Gundy-Stein ([7]). In §4 we concern with H^∞ of 2-parameter holomorphic martingales considered as abstract Hardy algebras and their applications. In this section we obtain several results on H^∞ , which extend theorems of Varopoulos ([13]), for example, the density of H^∞ in H^p and that $\text{Log}(X) \in BMOA$ for every

non-zero H^1 -martingale X with positive real part. We prove also that these martingales X are outer functions.

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§ 1. Definitions and fundamental properties.

Let $(x_{t_j}^j; t_j \geq 0)$ and $(y_{t_j}^j; t_j \geq 0)$ be two independent 1-dimensional Brownian motions on a complete probability space $(\Omega_j, \mathcal{F}^j, p^j)$ such that $p^j(x_0^j = y_0^j = 0) = 1$, and let $\mathcal{F}_{t_j}^j$ be the σ -field generated by $\{x_s^j, y_s^j; s \leq t_j\}$ and all p^j -null sets $(t_j \geq 0; j = 1, 2)$. Then the family $(\mathcal{F}_{t_j}^j)_{t_j \geq 0}$ satisfies the right continuous condition, that is, $\bigcap_{s > t_j} \mathcal{F}_s^j = \mathcal{F}_{t_j}^j$ for every $t_j \geq 0$ ($j = 1, 2$) (cf. [8, p. 286]). For the simplicity we assume that \mathcal{F}^j coincides the σ -field generated by $\bigcup_{t_j \geq 0} \mathcal{F}_{t_j}^j$ ($j = 1, 2$).

Let us denote by (Ω, F, P) the completion of the product measure space $(\Omega_1 \otimes \Omega_2, \mathcal{F}^1 \otimes \mathcal{F}^2, p^1 \otimes p^2)$ and let $\mathcal{F}_{s,t}$ be the σ -field generated by $\mathcal{F}_{s,t}^1 \otimes \mathcal{F}_{s,t}^2$ and all P -null sets $(s, t \geq 0)$.

In this note we use the following notations introduced in [4] and [10].

$$L^p(L^2) = \left\{ \Phi: \Phi \text{ is } \mathcal{B}[(\mathbf{R}_+)^2] \otimes F\text{-measurable process and} \right.$$

$$\left. [\Phi]_p \equiv \left(E \left[\left(\int_0^\infty \int_0^\infty |\Phi_{s,t}|^2 ds dt \right)^{p/2} \right] \right)^{1/p} < \infty \right\} \quad (0 < p < \infty).$$

$$S = \left\{ \sum_{j=1}^n \varphi_j \chi[\sigma_1^j, \tau_1^j] \times]\sigma_2^j, \tau_2^j]: n \in N, \sigma_i^j, \tau_i^j \in \mathbf{R}_+, \sigma_i^j < \tau_i^j, \right.$$

$$\left. \varphi_j \in L^\infty(\Omega, \mathcal{F}_{\sigma_i^j, \tau_i^j}, P) \quad (j = 1, \dots, n; i = 1, 2) \right\}.$$

Let A^p be the closure of S with respect to the metric $[\Phi - \psi]_p$.

For each $\Phi \in A^p$ ($0 < p < \infty$) we can define the following stochastic integrals

$$\int_0^t \int_0^s \Phi dx^1 dx^2, \quad \int_0^t \int_0^s \Phi dx^1 dy^2, \quad \int_0^t \int_0^s \Phi dy^1 dx^2, \quad \text{and} \quad \int_0^t \int_0^s \Phi dy^1 dy^2,$$

$(s, t \geq 0)$ (see [4]).

In the case of 1-parameter we can also define A_j^p on Ω_j ($j = 1, 2$) in the same manner as A^p on Ω . For each $\varphi \in A_j^p$ we can define the following stochastic integrals

$$\int_0^s \varphi dx^j \quad \text{and} \quad \int_0^s \varphi dy^j \quad (j=1, 2) \quad (s \geq 0).$$

For each stochastic process $(X_{st}; s, t \geq 0)$, we use the following notations;

$$\Delta X_{st} = X_{st} - X_{s0} - X_{0t} + X_{00}, \quad X^* = \sup_{s,t} |X_{st}|.$$

For each $p \in]0, \infty[$, we set

$$K^p = \{X = (X_{st}; s, t \geq 0): X \text{ can be written as (1.1) below.}\}$$

X_{00} is a constant.

$$\begin{aligned} X_{s0} &= X_{00} + \int_0^s \varphi_1 dx^1 + \int_0^s \varphi_2 dy^1 \quad \text{for some } \varphi_1, \varphi_2 \in L^p_1, \\ (1.1) \quad X_{0t} &= X_{00} + \int_0^t \psi_1 dx^2 + \int_0^t \psi_2 dy^2 \quad \text{for some } \psi_1, \psi_2 \in L^p_2, \\ \Delta X_{st} &= \int_0^t \int_0^s \Phi_1 dx^1 dx^2 + \int_0^t \int_0^s \Phi_2 dx^1 dy^2 + \int_0^t \int_0^s \Phi_3 dy^1 dx^2 \\ &\quad + \int_0^t \int_0^s \Phi_4 dy^1 dy^2 \quad \text{for some } \Phi_1, \dots, \Phi_4 \in L^p. \end{aligned}$$

Here for $X \in K^p$, X has a unique representation (1.1) by [4, p. 105]. For each $X \in K^p$ written as (1.1), we put

$$\begin{aligned} \langle X, X \rangle &= \int_0^\infty (|\varphi_1|^2 + |\varphi_2|^2) ds + \int_0^\infty (|\psi_1|^2 + |\psi_2|^2) dt \\ &\quad + \sum_{j=1}^4 \int_0^\infty \int_0^\infty |\Phi_j|^2 ds dt, \end{aligned}$$

and $\|X\|_{K^p} = \|\langle X, X \rangle^{1/2}\|_{L^p} + |X_{00}|$, where $\|\cdot\|_{L^p}$ means usual $L^p(\Omega, \mathbf{F}, P)$ -norm (resp. -quasi norm) if $1 \leq p < \infty$ (resp. $0 < p < 1$). $(K^p, \|\cdot\|_{K^p})$ is a Banach space if $1 \leq p < \infty$.

BROSSARD-CHEVALIER'S THEOREM ([4]).

For each $p \in]0, \infty[$ and any $X \in K^p$ the following inequality holds.

$$(1.2) \quad c_p \|X^*\|_{L^p} \leq \|X\|_{K^p} \leq C_p \|X^*\|_{L^p},$$

where c_p and C_p are constants depending only on p .

Let $K^\infty = L^\infty \cap K^2$ and let $\|X\|_{K^\infty} = \|X_{\infty\infty}\|_{L^\infty}$.

We can prove the Doob's inequality with respect to positive (\mathcal{F}_{st}) -submartingales in a way similar to the proof of Theorem 11 in [7, p. 298]. Hence the K^p -norm is equivalent to the L^p -norm if $p \in]1, \infty]$. From this

fact and Proposition 1 in [4] it is true that $K^p = L^p$ for $p \in [2, \infty]$ by identifying $(X_{st}; s, t \geq 0)$ with X_{∞} (where $L^p = L^p(\Omega, \mathbf{F}, p)$).

Denote by \mathcal{P} the σ -field generated by

$$\{A \times]s_1, t_1] \times]s_2, t_2]: 0 \leq s_1 < t_1, 0 \leq s_2 < t_2, A \in \mathcal{F}_{s_1, s_2}\}.$$

Let $\mathbf{R}'_+ = \mathbf{R}_+ \setminus \{0\}$. Now, a mapping T on Ω into the power set of $(\mathbf{R}'_+)^2$ is called a random region if it satisfies

$$\{(\omega, s, t) \in \Omega \times (\mathbf{R}'_+)^2: (s, t) \in T(\omega)\} \in \mathcal{P}.$$

The "random region" was introduced by H. Sato ([10]). (He called it "region aleatoire".)

For any $X \in K^2$ written as (1.1) we set

$$(1.3) \quad \Delta X_T(\omega) = \int_0^\infty \int_0^\infty \left[\chi_{(s,t) \in T(\omega)} (\Phi_1 dx^1 dx^2 + \Phi_2 dx^1 dy^2 + \Phi_3 dy^1 dx^2 + \Phi_4 dy^1 dy^2) \right].$$

It is clear that $(\chi_{(s,t) \in T} \Phi_j; s, t \geq 0) \in A^2$ ($j=1, \dots, 4$) and we set

$$\langle \Delta X_T, \Delta X_T \rangle = \sum_{j=1}^4 \int_0^\infty \int_0^\infty \chi_{(s,t) \in T} |\Phi_j|^2 ds dt.$$

We can prove easily that $E[\Delta M_T \Delta N_{\infty}] = E[\Delta M_T \Delta N_T]$ for each $M, N \in K^2$ and random region T (see [10]). Denote by S the collection of all random regions.

Let $N \in K^2$. N is called a *BMO*-martingale (or simply *BMO*) if

$$\|N_{\infty}\|_{*0}^2 = \sup_s \|E[|N_{\infty} - N_{s0}|^2 | \mathcal{F}_s^1]\|_{L^\infty} < \infty,$$

$$\|N_{0\infty}\|_{0*}^2 = \sup_t \|E[|N_{0\infty} - N_{0t}|^2 | \mathcal{F}_t^2]\|_{L^\infty} < \infty,$$

and $\|\Delta N\|_{**} = \sup\{\|\Delta N_T\|_{K^2} / (P(T \neq \emptyset))^{1/2}: T \in S\} < \infty$, where we regard $0/0$ as 0 . And we put

$$\|N\|_* = \|N_{\infty}\|_{*0} + \|N_{0\infty}\|_{0*} + \|\Delta N\|_{**} + |N_{00}|.$$

Denote by *BMO* the collection of all *BMO*-martingales. Then we have easily $K^\infty \subset BMO \subset K^p$ ($p \neq \infty$) (cf. [10]).

This *BMO* was defined by H. Sato ([10]). He proved the following theorem.

FEFFERMAN'S INEQUALITY AND K^1 -*BMO* DUALITY ([10]).

(1.4) For every $X \in BMO$ and every $Y \in K^2$

$$|E[XY]| \leq C \|X\|_* \|Y\|_{K^1},$$

where C is a universal constant.

$$(1.5) \quad (K^1)^* = BMO .$$

Finally we prove the following as corollary of (1.4) and (1.5).

COROLLARY 1.1. *Suppose P is a bounded projection operator on all of K^1, K^2 and BMO and is self-adjoint on $L^2(=K^2)$. Then $P(K^1)^* = P(BMO)$, where $P(K^1)^* = \{\varphi \in (K^1)^* : \varphi((I-P)(K^1)) = 0\}$ (I is the identity mapping).*

PROOF. For every $\varphi \in P(K^1)^*$, there exists a unique $X \in BMO$ such that $\varphi(Y) = E[\bar{X}Y]$ ($Y \in K^2$) by (1.5). For any $Y \in K^2$ we have

$$E[Y(\overline{X-P(X)})] = E[(Y-P(Y))\bar{X}] = \varphi(Y-P(Y)) = 0 .$$

Hence $X = P(X) \in P(BMO)$. Conversely, take any $P(X) \in P(BMO)$, and let $\varphi(Y) = E[Y\overline{P(X)}]$ ($Y \in K^2$). Then by (1.4) we have $\varphi \in (K^1)^*$ and $\varphi(Y-P(Y)) = 0$ ($Y \in K^2$). Hence $\varphi \in P(K^1)^*$.

§ 2. Hilbert transforms on product Brownian spaces.

The Hilbert transforms of one parameter Brownian martingales were defined by N. Th. Varopoulos ([13]).

In the case of two parameter, we define the following Hilbert transforms H_j ($j=1, 2$) after the manner of Hilbert transforms for 1-parameter martingales:

$$\begin{aligned} H_j(dx^j) &= dy^j, \quad H_j(dy^j) = -dx^j, \quad (j=1, 2) \\ H_j(dx^k) &= 0, \quad H_j(dy^k) = 0, \quad (j \neq k; j, k=1, 2) . \end{aligned}$$

Further, the double Hilbert transform H_3 is defined by the composition of H_1 and H_2 , i.e. $H_3 = H_1 \circ H_2 = H_2 \circ H_1$, where \circ means usual composite of operators.

Then for every $X \in K^p$ written as (1.1), we obtain that

$$\begin{aligned} H_1(X) &= - \int \varphi_2 dx^1 + \int \varphi_1 dy^1 \\ &\quad + \iint (-\Phi_3 dx^1 dx^2 - \Phi_4 dx^1 dy^2 + \Phi_1 dy^1 dx^2 + \Phi_2 dy^1 dy^2) , \\ H_2(X) &= - \int \psi_2 dx^2 + \int \psi_1 dy^2 \\ &\quad + \iint (-\Phi_2 dx^1 dx^2 + \Phi_1 dx^1 dy^2 - \Phi_4 dy^1 dx^2 + \Phi_3 dy^1 dy^2), \quad \text{and} \end{aligned}$$

$$H_3(X) = \iint (\Phi_4 dx^1 dx^2 - \Phi_3 dx^1 dy^2 - \Phi_2 dy^1 dx^2 + \Phi_1 dy^1 dy^2) .$$

For each K^p ($0 < p \leq \infty$) we set

$$K_{10}^p = \{X \in K^p : X_{st} = X_{s0} \ (s, t \geq 0) \text{ and } X_{00} = 0\}$$

$$K_{01}^p = \{X \in K^p : X_{st} = X_{0t} \ (s, t \geq 0) \text{ and } X_{00} = 0\}$$

$$K_{11}^p = \{X \in K^p : X_{st} = \Delta X_{st} \ (s, t \geq 0)\} .$$

Then K^p is the direct sum $K^p = C \oplus K_{10}^p \oplus K_{01}^p \oplus K_{11}^p$ of C , K_{10}^p , K_{01}^p and K_{11}^p , namely, for every $(X_{st}) \in K^p$ we can identify X_{st} with $X_{00} \oplus [X_{s0} - X_{00}] \oplus [X_{0t} - X_{00}] \oplus [\Delta X_{st}]$ ($0 < p \leq \infty$). By the definition of the operator H_1 , H_2 and H_3 we have that $\langle H_1 X, H_1 X \rangle = \langle X, X \rangle$ for every $X \in K_{10}^p \oplus K_{11}^p$, $\langle H_2 X, H_2 X \rangle = \langle X, X \rangle$ for every $X \in K_{01}^p \oplus K_{11}^p$, and $\langle H_3 X, H_3 X \rangle = \langle X, X \rangle$ for every $X \in K_{11}^p$. Hence we get $\|H_j X\|_{K^p} \leq \|X\|_{K^p}$ ($X \in K^p$, $0 < p < \infty$, $j = 1, 2, 3$).

For every random region T and each $X \in BMO$, we have $\|H_j(\Delta X)_T\|_{K^2} = \|H_j(\Delta X_T)\|_{K^2} \leq \|\Delta X_T\|_{K^2}$. Thus $\|H_j X\|_* \leq \|X\|_*$ ($j = 1, 2, 3$).

Let I and 0 be the identity mapping and zero mapping respectively. Define $T_j = (I + iH_j)/2$ and $S_j = (I - iH_j)/2$ ($j = 1, 2$).

If $X \in K^p$, then $T_j X$ (resp. $S_j X$) is holomorphic (resp. antiholomorphic) in the variable t_j in the sense given at [13] ($j = 1, 2$). And T_1 and S_1 are bounded projection operators on all of $K_{10}^p \oplus K_{11}^p$ and $BMO \cap [K_{10}^2 \oplus K_{11}^2]$, since $H_1 \circ H_1 = -I$ on $K_{10}^p \oplus K_{11}^p$ ($0 < p < \infty$). Furthermore, T_1 and S_1 are self-adjoint on $K_{10}^2 \oplus K_{11}^2$, because $H_1^* = -H_1$ on $K_{10}^2 \oplus K_{11}^2$, where H_1^* is the adjoint of H_1 ($j = 1, 2$). Similarly, T_2 and S_2 are bounded projection operators on all of $K_{01}^p \oplus K_{11}^p$ and $BMO \cap [K_{01}^2 \oplus K_{11}^2]$, and T_2 and S_2 are self-adjoint on $K_{01}^2 \oplus K_{11}^2$.

For any operators A_1, A_2, A_3 and A_4 on K^p , we define the operator $A_1 \oplus A_2 \oplus A_3 \oplus A_4$ on K^p as follows:

$$\begin{aligned} A_1 \oplus A_2 \oplus A_3 \oplus A_4(X) &= A_1(X_{00}) + A_2((X_{s0} - X_{00})) \\ &\quad + A_3((X_{0t} - X_{00})) + A_4((\Delta X_{st})) \quad (X \in K^p) . \end{aligned}$$

Now we define following operators.

$$K^{aa} = I \oplus T_1 \oplus T_2 \oplus T_1 \circ T_2 ,$$

$$K^{ab} = 0 \oplus 0 \oplus 0 \oplus T_1 \circ S_2 ,$$

$$K^{ba} = 0 \oplus 0 \oplus 0 \oplus S_1 \circ T_2 \text{ and}$$

$$K^{bb} = 0 \oplus S_1 \oplus S_2 \oplus S_1 \circ S_2 \text{ (cf. Appendix I) .}$$

For each $X \in K^p$, we write simply $X^\varepsilon = K^\varepsilon(X)$ ($\varepsilon = aa, ab, ba, bb$). Let $(K^p)^\varepsilon = \{X^\varepsilon : X \in K^p\}$ and $BMO^\varepsilon = \{X^\varepsilon : X \in BMO\}$ ($0 < p < \infty$, $\varepsilon = aa, ab, ba, bb$).

Especially we put $H^p = (K^p)^{aa}$ ($0 < p < \infty$), $BMOA = BMO^{aa}$ and $H^\infty = H^2 \cap L^\infty$.

From the above observations we obtain the following

PROPOSITION 2.1. (1) K^ε is a bounded projection operator on K^p (resp. BMO) onto $(K^p)^\varepsilon$ (resp. BMO^ε), especially K^ε is self-adjoint on $L^2 = K^2$ ($0 < p < \infty$, $\varepsilon = aa, ab, ba, bb$), and $K^\varepsilon \circ K^\eta = 0$ if $\varepsilon \neq \eta$.

From (1) we can deduce (2) below.

(2) $(K^p)^\varepsilon$ (resp. BMO^ε) is a K^p -norm (resp. BMO -norm) closed subspace of K^p (resp. BMO) ($1 \leq p < \infty$, $\varepsilon = aa, ab, ba, bb$).

THEOREM 2.2. $[(K^1)^\varepsilon]^* = BMO^\varepsilon$ ($\varepsilon = aa, ab, ba, bb$), particularly, $(H^1)^* = BMOA$.

PROOF. The theorem is an immediate consequence of Corollary 1.1 and Proposition 2.1 (1).

LEMMA 2.3. Suppose $p \in]0, \infty[$. Then for every $X \in K^p$,

$$(2.1) \quad \|X^\varepsilon\|_{K^p} \approx \|(X^\varepsilon)^*\|_{L^p} \approx \sup_{s,t} \|X_{st}^\varepsilon\|_{L^p},$$

where \approx means an equivalence of norms.

If $p \in [1, \infty[$, then $\sup_{s,t} \|X_{st}^\varepsilon\|_{L^p} = \|X_{\infty\infty}^\varepsilon\|_{L^p}$ ($\varepsilon = aa, ab, ba, bb$).

PROOF. We have this lemma by the iteration of the argument in the proof of [6, Lemma 6.3]. Indeed, X^ε is a conformal martingale for each parameter. So $Z^\varepsilon = |X^\varepsilon|^{p/2}$ is a local submartingale for each parameter (cf. [6, Lemma 5.8]). By Proposition 2.1 and (1.2)

$$\sup_t E[(|X_{st}^\varepsilon|^{p/2})^2] \leq C \|X\|_{K^p}^p \quad (s \geq 0).$$

From this and a slight modification of [8, p. 292c], we can deduce that $(Z_{st}^\varepsilon)_{t \geq 0}$ is an L^2 -bounded submartingale. Similarly $(Z_{st}^\varepsilon)_{s \geq 0}$ is also an L^2 -bounded submartingale. Using the technique in [7, p. 298] we have the following inequality: For every $\alpha, \beta \in \mathbf{R}_+$,

$$E \left[\sup_{\substack{0 \leq s \leq \alpha \\ 0 \leq t \leq \beta}} |X_{st}^\varepsilon|^p \right] = E \left[\sup_{\substack{0 \leq s \leq \alpha \\ 0 \leq t \leq \beta}} (Z_{st}^\varepsilon)^2 \right] \leq CE[Z_{\alpha\beta}^{\varepsilon 2}] = E[|X_{\alpha\beta}^\varepsilon|^p].$$

Thus we obtain the lemma by Lebesgue's convergence theorem.

DEFINITION. For every $X \in K^2$, we define

$$\|X\|_{\mathscr{H}^1} = \|X\|_{L^1} + \sum_{j=1}^3 \|H_j X\|_{L^1}.$$

Denote by \mathscr{H}^1 the $\|\cdot\|_{\mathscr{H}^1}$ -norm closure of K^2 .

THEOREM 2.4. (*Varopoulos theorem on product Brownian spaces.*)

$$K^1 = \mathcal{H}^1 \quad \text{and} \quad \|\cdot\|_{K^1} \approx \|\cdot\|_{\mathcal{H}^1}.$$

PROOF. It is clear that

$$\|X\|_{\mathcal{H}^1} \approx \|X^{aa}\|_{L^1} + \|X^{ab}\|_{L^1} + \|X^{ba}\|_{L^1} + \|X^{bb}\|_{L^1}.$$

By Lemma 2.3 we have the following inequality for every $X \in K^2$.

$$\begin{aligned} \|X\|_{K^1} &\leq \|X^{aa}\|_{K^1} + \|X^{ab}\|_{K^1} + \|X^{ba}\|_{K^1} + \|X^{bb}\|_{K^1} \\ &\approx \|X^{aa}\|_{L^1} + \|X^{ab}\|_{L^1} + \|X^{ba}\|_{L^1} + \|X^{bb}\|_{L^1} \\ \text{and} \quad \|X^{aa}\|_{K^1} + \|X^{ab}\|_{K^1} + \|X^{ba}\|_{K^1} + \|X^{bb}\|_{K^1} &\leq C\|X\|_{K^1} \end{aligned}$$

(cf. Proposition 2.1 (1)). Thus $\|X\|_{K^1} \approx \|X\|_{\mathcal{H}^1}$.

By the routine duality argument and (1.5) we obtain the following theorem.

THEOREM 2.5. *Following conditions are equivalent.*

- (a) $X \in BMO$.
- (b) There exist $A_j \in L^\infty$ ($1 \leq j \leq 4$) such that

$$X = A_1^{aa} + A_2^{ab} + A_3^{ba} + A_4^{bb} \quad \text{and} \quad \|X\|_* \approx \sum_{j=1}^4 \|A_j\|_{L^\infty}.$$

- (c) There exist $B_j \in L^\infty$ ($1 \leq j \leq 4$) such that

$$X = B_1 + H_1 B_2 + H_2 B_3 + H_3 B_4 \quad \text{and} \quad \|X\|_* \approx \sum_{j=1}^4 \|B_j\|_{L^\infty}.$$

§ 3. Applications to function theory on the torus.

Let $D_j = \{z \in \mathbb{C}: |z| < 1\}$, $T_j = \{z \in \mathbb{C}: |z| = 1\}$, $D^2 = D_1 \times D_2$ and $T^2 = T_1 \times T_2$. dm_j denotes the normalized Lebesgue measure on T_j ($j=1, 2$) and dm denotes the product measure of dm_1 and dm_2 .

The Poisson kernel $P_j(z_j, w_j)$ and the conjugate Poisson kernel $Q_j(z_j, w_j)$ on D_j are of the forms

$$\begin{aligned} P_j(z_j, w_j) &= \operatorname{Re}((w_j + z_j)/(w_j - z_j)) \\ Q_j(z_j, w_j) &= \operatorname{Im}((w_j + z_j)/(w_j - z_j)) \quad \text{respectively,} \end{aligned}$$

where $z_j \in D_j$, $w_j \in T_j$ ($j=1, 2$).

For each $f \in L^1(T^2)$ ($= L^1(T^2, dm)$) and for every $z = (z_1, z_2) \in D^2$ we set

$$\begin{aligned}
 PP(f)(z) &= \int_{T^2} P_1(z_1, w_1)P_2(z_2, w_2)f(w)dm(w) , \\
 PQ(f)(z) &= \int_{T^2} P_1(z_1, w_1)Q_2(z_2, w_2)f(w)dm(w) , \\
 QP(f)(z) &= \int_{T^2} Q_1(z_1, w_1)P_2(z_2, w_2)f(w)dm(w) \quad \text{and} \\
 QQ(f)(z) &= \int_{T^2} Q_1(z_1, w_1)Q_2(z_2, w_2)f(w)dm(w) ,
 \end{aligned}$$

where $w=(w_1, w_2)$.

For each function u on D^2 , $R[u](w)$ denotes the formal radial limit of u , that is, $R[u](w)=\lim_{r \rightarrow 1} u(rw)$.

The Hilbert transforms \tilde{H}_1, \tilde{H}_2 and \tilde{H}_3 are defined by

$$\tilde{H}_1 f = R[QP(f)], \tilde{H}_2 f = R[PQ(f)] \quad \text{and} \quad \tilde{H}_3 f = R[QQ(f)]$$

respectively ($f \in L^1(T^2)$). It is easy to check that $\tilde{H}_3 = \tilde{H}_1 \circ \tilde{H}_2 = \tilde{H}_2 \circ \tilde{H}_1$ on $L^p(T^2)$ ($1 < p \leq \infty$).

For every $f \in L^1(T^2)$ we set

$$f(z) = PP(f)(z) \quad (z \in D^2) ,$$

$$f(w_1, z_2) = P_2(f(w_1, \cdot))(z_2) = \int_{T^2} P_2(z_2, w_2)f(w_1, w_2)dm_2(w_2) \quad (w_1 \in T_1, z_2 \in D_2),$$

$$f(z_1, w_2) = P_1(f(\cdot, w_2))(z_1) = \int_{T^2} P_1(z_1, w_1)f(w_1, w_2)dm_1(w_1) \quad (z_1 \in D_1, w_2 \in T_2)$$

and

$$\Delta f(w_1, w_2) = f(w_1, w_2) - f(w_1, 0) - f(0, w_2) + f(0, 0) \quad ((w_1, w_2) \in T^2) .$$

$L^p(T^2)$ ($p \geq 1$) can be decomposed as follows:

$$L^p(T^2) = C \oplus L_{10}^p(T^2) \oplus L_{01}^p(T^2) \oplus L_{11}^p(T^2), \quad \text{where}$$

$$L_{10}^p(T^2) = \{f \in L^p(T^2): f(w_1, w_2) = f(w_1, 0) - f(0, 0) \text{ for all } (w_1, w_2) \in T^2\},$$

$$L_{01}^p(T^2) = \{f \in L^p(T^2): f(w_1, w_2) = f(0, w_2) - f(0, 0) \text{ for all } (w_1, w_2) \in T^2\} \text{ and}$$

$$L_{11}^p(T^2) = \{f \in L^p(T^2): f(w_1, w_2) = \Delta f(w_1, w_2) \text{ for all } (w_1, w_2) \in T^2\}.$$

Let $\tilde{T}_j = (I + i\tilde{H}_j)/2$ and $\tilde{S}_j = (I - i\tilde{H}_j)/2$ ($j=1, 2$). Then we define the following operators as well as in §2.

$$\tilde{K}^{aa} = I \oplus \tilde{T}_1 \oplus \tilde{T}_2 \oplus \tilde{T}_1 \circ \tilde{T}_2, \quad \tilde{K}^{ab} = 0 \oplus 0 \oplus 0 \oplus \tilde{T}_1 \circ \tilde{S}_2,$$

$$\tilde{K}^{ba} = 0 \oplus 0 \oplus 0 \oplus \tilde{S}_1 \circ \tilde{T}_2 \quad \text{and} \quad \tilde{K}^{bb} = 0 \oplus \tilde{S}_1 \oplus \tilde{S}_2 \oplus \tilde{S}_1 \circ \tilde{S}_2.$$

We write $f_\varepsilon = \tilde{K}^\varepsilon(f)$ and $L^p(T^2)_\varepsilon = \{f: f \in L^p(T^2)\}$ ($\varepsilon = aa, ab, ba, bb$).

Let $\tau_j = \inf\{t_j: |z_{t_j}^j| = 1\}$, where $z_{t_j}^j = x_{t_j}^j + (-1)^{1/2}y_{t_j}^j$ and $\bar{z}_{t_j}^j = x_{t_j}^j - (-1)^{1/2}y_{t_j}^j$ ($j=1, 2$). The following mappings M and N were introduced by N. Th.

Varopoulos ([14]):

For every $f \in C(T^2)$, let $Mf = f(z_{\tau_1}^1, z_{\tau_2}^2) \in L^\infty (= L^\infty(\Omega, F, P))$. M can be extended to an isometry from $L^p(T^2)$ to $L^p (= L^p(\Omega, F, P))$, $1 \leq p \leq \infty$. For every $X \in L^p$, NX is defined as follows.

$$NX(e^{i\theta}, e^{i\varphi}) = E[X | z_{\tau_1}^1 = e^{i\theta}, z_{\tau_2}^2 = e^{i\varphi}] .$$

Then $\|NX\|_{L^p} \leq \|X\|_{L^p}$ ($1 \leq p \leq \infty$). We denote $M \circ N$ by \tilde{N} . (It is clear that $N \circ M$ is identity mapping.)

For every $f \in L^p(T^2)$, let $(M_1f)(\omega_1, e^{i\varphi}) = f(z_{\tau_1}^1(\omega_1), e^{i\varphi})$ ($\omega_1 \in \Omega_1$) and let $(M_2f)(e^{i\theta}, \omega_2) = f(e^{i\theta}, z_{\tau_2}^2(\omega_2))$ ($\omega_2 \in \Omega_2$). For every $X \in L^p$, we set $(N_jX)(e^{i\theta}) = E_j[X | z_{\tau_j}^j = e^{i\theta}]$ ($j=1, 2$). It is clear that $N_jX(e^{i\theta}) \in L^p(\Omega_k)$ ($k \in \{1, 2\}$ and $k \neq j$). Let $\tilde{N}_j = M_j \circ N_j$.

Let $H^p(D^2)$ be the Hardy space of holomorphic functions (cf. [9, p. 50]) and, let $H^p(T^2) = \{R[f] : f \in H^p(D^2)\}$ ($1 \leq p \leq \infty$) and $\mathcal{H}^p(T^2) = \{f \in L^p(T^2) : \|f\|_{\mathcal{H}^p} < \infty\}$ ($1 \leq p < \infty$), where $\|f\|_{\mathcal{H}^p} = \|f\|_{L^p} + \sum_{j=1}^2 \|\tilde{H}_j f\|_{L^p}$.

For $f \in L^p(T^2)$, $n(f)$ denotes the radial maximal function of f , that is, $n(f)(e^{i\theta}, e^{i\varphi}) = \sup_{0 \leq r < 1} |PP(f)(re^{i\theta}, re^{i\varphi})|$. Let $h^p(T^2) = \{f \in L^p(T^2) : \|f\|_{h^p} < \infty\}$, where $\|f\|_{h^p} = \|n(f)\|_{L^p}$ ($1 \leq p < \infty$). H. Sato introduced *BMO*-functions on T^2 as follows (cf. [10]): $BMO(T^2) = \{f \in L^2(T^2) : Mf \in BMO\}$ and $\|f\|_* = \|Mf\|_*$. We put $BMOA(T^2) = BMO(T^2) \cap H^2(T^2)$. It follows that $\{f \in L^2(T) : Mf \in BMO\} = \left\{ f \in L^1(T) : \sup_I \int_I |f - f_I| dx / |I| < \infty \right\}$ (This is the usual *BMO* space) by the John-Nierenberg inequality and Theorem 1 in [1] (cf. [12, p. 216]).

LEMMA 3.1. If $p \in [1, \infty]$, then $NX \in H^p(T^2)$ for every $X \in H^p$.

REMARK: The following proof is due essentially to N. Th. Varopoulos [14].

PROOF. Notice that $z_{\tau_j \wedge t}^j = \int_0^t \chi_{\{s \leq \tau_j\}} dz_s^j$ is in H^∞ by Appendix I ($j=1, 2$). Let $Z_+ = \{n : n \geq 0 \text{ and is an integer.}\}$. Take any $(m, n) \notin Z_+ \times Z_+$. If $m < 0$ it is true that

$$\begin{aligned} NX^\wedge(m, n) &= \int_{T^2} w_1^{-m} w_2^{-n} NX(w) dm(w) \\ &= E[(z_{\tau_1}^1)^{(-m)} (z_{\tau_2}^2)^{(-n)} \tilde{N}X] \\ &= E[(z_{\tau_1}^1)^{(-m)} (z_{\tau_2}^2)^{(-n)} X] \\ &= E_2[(E_1(z_{\tau_1}^1))^{(-m)} (z_{\tau_2}^2)^{(-n)} E_1(X)] = 0 . \end{aligned}$$

If $n < 0$ we have that $NX^\wedge(m, n) = 0$ as above. By Theorem 2.1.4 in [9] it follows that $NX \in H^p(T^2)$.

N. Th. Varopoulos [14] showed that Mf is holomorphic martingale

if $f \in H^\infty(T^2) \cap C(T^2)$. We extend this result as follows by giving an alternative proof.

LEMMA 3.2. *Suppose $p \in [1, \infty]$. Then $Mf \in H^p$ for every $f \in H^p(T^2)$.*

PROOF. We first prove the lemma in the case of $p=1$. Let $f \in H^1(T^2)$. Then there exist $f_n \in H^2(T^2)$ ($n \in \mathbb{N}$) such that $\lim_{n \rightarrow \infty} \|f_n - f\|_{L^1} = 0$. Hence $\lim_{n \rightarrow \infty} \|Mf_n - Mf\|_{L^1} = 0$, and $Mf_n \in L^2$ implies $Mf_n \in K^2$ ($n \in \mathbb{N}$).

Let $H^2(\mathcal{F}^1) = H^2 \cap (C \oplus K_{10}^2)$ and $H^2(\mathcal{F}^2) = H^2 \cap (C \oplus K_{01}^2)$. We can identify $H^2(\mathcal{F}^j)$ with $H^2(\Omega_j)$ defined by N. Th. Varopoulos [13] ($j=1, 2$). By [13, p. 97] we have

$$H^2(\mathcal{F}^j)^\perp = H_0^2(\mathcal{F}^j)^- = \{\bar{X} : X \in H^2(\mathcal{F}^j) \text{ and } E_j[X] = 0\}.$$

Take any $X \in H^2(\mathcal{F}^j)^\perp$. Then

$$E_j[(Mf_n)\bar{X}] = E_j[(Mf_n)(\tilde{N}_j\bar{X})] = \int_{T_j} (M_k f_n)(N_j\bar{X}) dm_j,$$

where $k \in \{1, 2\}$ and $k \neq j$. $M_k f_n \in H^2(T_j)$ by definition of f_n , and $\overline{N_j X} \in H^2(T_j)$ by Lemma 3.1. From this it follows that

$$E_j[(Mf_n)\bar{X}] = \int_{T_j} M_k f_n dm_j \int_{T_j} \overline{N_j X} dm_j = \int_{T_j} M_k f_n dm_j \cdot \overline{E_j[X]} = 0.$$

Hence $Mf_n \in H^2(\mathcal{F}^j)^{\perp\perp} = H^2(\mathcal{F}^j)$ ($j=1, 2$), and so $(E_j[Mf_n | \mathcal{F}_{t_j}^j])_{t_j}$ is holomorphic in the sense given at [13] ($j=1, 2$). So $Mf \in [H^2]_1 \subset H^1$ by Appendix I and Lemma 2.3, where we denote by $[H^2]_1$ the L^1 -norm closure of H^2 .

Next, suppose $p \in [1, \infty]$. Let $f \in H^p(T^2)$. By the above assertion we conclude that $Mf \in H^1 \cap L^p = H^p$ by Appendix I.

Following lemma was proved in Varopoulos [13] in the case of 1-parameter. We give an alternative proof and extend to the case of 2-parameter.

LEMMA 3.3. (1) $N(H_j X) = \tilde{H}_j(NX)$ for every $X \in K^2$ ($j=1, 2, 3$).
 (2) $M(\tilde{H}_j f) = H_j Mf$ for every $f \in L^2(T^2)$ ($j=1, 2, 3$).

PROOF. We first prove (1) for every $X = FG \in L^2_{\mathbb{R}}(\Omega_1) \otimes L^2_{\mathbb{R}}(\Omega_2)$, where $L^2_{\mathbb{R}}(\Omega_j)$ is the real part of $L^2(\Omega_j)$ ($j=1, 2$). By easy calculation we have that $X + iH_1 X = (F + iH_1 F)G$. Lemma 3.1 implies that $N_1(F + i(H_1 F)) \in H^2(T_1)$. Of course $N_1 F + i\tilde{H}_1 N_1 F \in H^2(T_1)$. Since any real valued function in $H^2(T_1)$ is constant, we have $N_1(H_1 F) - \tilde{H}_1(N_1 F) = c \in \mathbb{R}$. By definition of H_1 and \tilde{H}_1 , we have that $c=0$. Hence $NX + i\tilde{H}_1 NX = NX + iNH_1 X$, and so $N(N_1 X) = \tilde{H}_1(NX)$. In the same manner as the above we have

$N(H_2X) = \tilde{H}_2(NX)$. Thus $N(H_3X) = \tilde{H}_3(NX)$.

It is an immediate consequence of the above fact that (1) is true for every $X \in K^2$.

We can prove (2) by using the above method, Lemma 3.2 and that any real valued martingale in $H^2(\mathcal{F}^j)$ is constant (cf. [13] or Corollary 4.3) ($j=1, 2$).

The following corollary is clear from Lemma 3.3.

COROLLARY 3.4. (1) $\tilde{N}(H_jX) = H_j(\tilde{N}X)$ for every $X \in K^2$. ($j=1, 2, 3$).
 (2) $M(f)_\varepsilon = M(f)_\varepsilon'$ for every $f \in L^2(T^2)$ ($\varepsilon=aa, ab, ba, bb$).
 (3) $N(X^\varepsilon) = N(X)_\varepsilon$ and $\tilde{N}(X^\varepsilon) = \tilde{N}(X)_\varepsilon'$ for every $X \in K^2$ ($\varepsilon=aa, ab, ba, bb$).

Let us recall the definition of "isomorphic". Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be two Banach spaces. φ is called an isomorphism on X onto Y if φ is bijective linear operator on X to Y and is continuous and open. We say that X is isomorphic to Y if there exists an isomorphism on X onto Y .

Our main theorem is the following

THEORE 3.5. $\mathcal{H}^1(T^2)$ (resp. $BMO(T^2)$) is isomorphic to a closed complemented subspace of K^1 (resp. BMO). Indeed, \tilde{N} is a bounded projection operator on K^1 (resp. BMO) and M is an isomorphism on $\mathcal{H}^1(T^2)$ (resp. $BMO(T^2)$) onto $\tilde{N}(K^1)$ (resp. $\tilde{N}(BMO)$).

PROOF. That $\tilde{N} \circ \tilde{N} = \tilde{N}$ is clear. For every $X \in K^2$, we see that

$$\begin{aligned} \|\tilde{N}(X)\|_{K^1} &\leq C(\|\tilde{N}(X)\|_{L^1} + \sum_{j=1}^3 \|H_j(\tilde{N}X)\|_{L^1}) && \text{(by Theorem 2.4)} \\ &= C(\|\tilde{N}(X)\|_{L^1} + \sum_{j=1}^3 \|\tilde{N}(H_jX)\|_{L^1}) && \text{(by Corollary 3.4)} \\ &\leq C(\|X\|_{L^1} + \sum_{j=1}^3 \|H_jX\|_{L^1}) \\ &\leq c\|X\|_{K^1} && \text{(by Theorem 2.4),} \end{aligned}$$

where C and c are universal constants. Hence \tilde{N} is a bounded projection operator on K^1 . Take any $X \in BMO$ and any random region T . Then we have

$$\begin{aligned} \|\tilde{N}(\Delta X)_T\|_{K^2}^2 &= \|\tilde{N}(\Delta X)_T\|_{L^2}^2 = E[\tilde{N}(\Delta X)_T \overline{\tilde{N}(\Delta X)_T}] \\ &= E[\tilde{N}(\Delta X) \overline{\tilde{N}(\Delta X)_T}] = E[(\Delta X) \overline{\tilde{N}(\tilde{N}(\Delta X)_T)}] \\ &\leq C_1 \|\Delta X\|_{**} \|\tilde{N}(\tilde{N}(\Delta X)_T)\|_{K^1} && \text{(by (1.4))} \\ &\leq C_2 \|\Delta X\|_{**} \|\tilde{N}(\Delta X)_T\|_{K^1} && \text{(by the above result)} \end{aligned}$$

$$\begin{aligned}
 &= C_2 \|\Delta X\|_{**} E[\chi_{(T \neq \phi)} \cdot \langle \tilde{N}(\Delta X)_T, \tilde{N}(\Delta X)_T \rangle^{1/2}] \\
 &\leq C_2 \|\Delta X\|_{**} P(T \neq \phi)^{1/2} \|\tilde{N}(\Delta X)_T\|_{K^2},
 \end{aligned}$$

hence $\|\tilde{N}(\Delta X)\|_{**} \leq C_2 \|\Delta X\|_{**}$, where C_j denotes a universal constant ($j=1, 2$).

Since $\|X_{0\infty}\|_{0*} = \sup\{\|X_{0\infty} - X_{0T}\|_{L^2} / (P(T < \infty))^{1/2} : T \text{ is a stopping time.}\}$, we can prove that $\|\tilde{N}(X_{0\infty})\|_{0*} \leq C \|X_{0\infty}\|_{0*}$ by using the above method. In similar manner as the case of $X_{0\infty}$, we have that $\|\tilde{N}(X_{\infty 0})\|_{*0} \leq C \|X_{\infty 0}\|_{*0}$. So that, \tilde{N} is a bounded projection operator on BMO .

Hence $\tilde{N}(K^1)$ (resp. $\tilde{N}(BMO)$) is a closed complemented subspace of K^1 (resp. BMO).

Take any $f \in \mathcal{H}^1(T^2)$. Then by using of Theorem 2.1.3 (c) in [9], we have that there exist $f_n \in C(T^2)$ ($n \in \mathbf{N}$) such that $\lim_{n \rightarrow \infty} \|f_n - f\|_{\mathcal{H}^1} = 0$. Theorem 2.4 and Lemma 3.3 imply that $\lim_{m, n \rightarrow \infty} \|Mf_n - Mf_m\|_{K^1} = 0$. Hence there exists a $X \in K^1$ such that $\lim_{n \rightarrow \infty} \|Mf_n - X\|_{K^1} = 0$. And the boundedness of \tilde{N} (see above) implies that

$$\lim_{n \rightarrow \infty} \|Mf_n - \tilde{N}X\|_{K^1} = \lim_{n \rightarrow \infty} \|\tilde{N}(Mf_n - X)\|_{K^1} = 0.$$

So $\lim_{n \rightarrow \infty} \|f_n - NX\|_{L^1} = 0$. Thus $f = NX$ and $Mf = \tilde{N}X \in \tilde{N}(K^1) \subset K^1$. Theorem 2.4 and Lemma 3.3 show that

$$\|f\|_{\mathcal{H}^1} = \lim_{n \rightarrow \infty} \|f_n\|_{\mathcal{H}^1} \approx \lim_{n \rightarrow \infty} \|Mf_n\|_{K^1} = \|Mf\|_{K^1}.$$

Hence M is isomorphism on \mathcal{H}^1 onto $\tilde{N}(K^1)$.

By definition of $BMO(T^2)$, it is clear that M is an isomorphism on $BMO(T^2)$ onto $\tilde{N}(BMO)$.

This theorem implies the following corollaries.

COROLLARY 3.6. (*H. Sato* [10])

$$\mathcal{H}^1(T^2)^* = BMO(T^2).$$

PROOF. It is clear that \tilde{N} is self-adjoint on $L^2(=K^2)$. Hence $\tilde{N}(K^1)^* = \tilde{N}(BMO)$ by Corollary 1.1 and Theorem 3.5. And $\mathcal{H}^1(T^2)$ (resp. $BMO(T^2)$) is isomorphic to $\tilde{N}(K^1)$ (resp. $\tilde{N}(BMO)$) by Theorem 3.5. Thus $\mathcal{H}^1(T^2)^* = BMO(T^2)$.

COROLLARY 3.7. $[\mathcal{H}^1(T^2)^\varepsilon]^* = BMO(T^2)^\varepsilon$ ($\varepsilon = aa, ab, ba, bb$). Especially, $H^1(T^2)^* = BMOA(T^2)$.

PROOF. $\tilde{N} \circ K^\varepsilon$ is a bounded projection operator on all of K^1, K^2 and BMO by Theorem 3.5 and Proposition 2.1. Thus the corollary is proved

in a similar way as the proof of Corollary 3.6.

COROLLARY 3.8. (A special case of a theorem of Gundy and Stein [7])

$h^1(\mathbf{T}^2)$ is isomorphic to $\mathcal{H}^1(\mathbf{T}^2)$.

PROOF. It is easy to check that $E[Mf \mid \mathcal{F}_{st}] = \tilde{f}(z_{r_1 \wedge s}^1, z_{r_2 \wedge t}^2)$ for every $f \in h^1(\mathbf{T}^2)$, where $\tilde{f} = PP(f)$ on D^2 and $\tilde{f} = f$ on \mathbf{T}^2 .

The following inequalities were proved in [7, p. 305, Lemme 4'].

$$(3.1) \quad c \|NX\|_{h^1} \leq \|\tilde{N}X^*\|_{L^1} \quad \text{for all } X \in K^1,$$

$$(3.2) \quad \|Mf^*\|_{L^1} \leq C \|f\|_{h^1} \quad \text{for all } f \in h^1(\mathbf{T}^2),$$

where c and C are universal constants. (In Appendix II we shall give an alternative proof of (3.1).)

From Theorem 2.3.1 in [9] and an easy calculation of the Poisson kernel we have that $\lim_{r \rightarrow 1} n(f_r - f) = 0$ a.e. dm for every $f \in h^1(\mathbf{T}^2)$, where $f_r(w) = f(rw)$ ($w \in \mathbf{T}^2$). Thus $C(\mathbf{T}^2)$ is dense in $h^1(\mathbf{T}^2)$ by Lebesgue's convergence theorem. Hence $Mf \in K^1$ for every $f \in h^1(\mathbf{T}^2)$ (cf. the proof of Theorem 3.5).

(1.2), (3.1) and (3.2) follows that $\|Mf\|_{K^1} \leq c_1 \|Mf^*\|_{L^1} \leq C_2 \|f\|_{h^1}$ for every $f \in h^1(\mathbf{T}^2)$, and $\|NX\|_{h^1} \leq c_3 \|\tilde{N}X^*\|_{L^1} \leq c_4 \|\tilde{N}X\|_{K^1}$ for every $X \in K^1$.

So M is an isomorphism on $h^1(\mathbf{T}^2)$ onto $\tilde{N}(K^1)$. Thus Theorem 3.5 prove the corollary.

From Corollary 3.6 we have the Fefferman-Stein decomposition of $BMO(\mathbf{T}^2)$, and Corollary 3.7 implies the following

COROLLARY 3.9. The following conditions (a) and (b) are equivalent.

(a) $f \in BMOA(\mathbf{T}^2)$.

(b) There exists an $h \in L^\infty(\mathbf{T}^2)$ such that

$$\begin{aligned} f(z_1, z_2) &= \tilde{K}^{aa}(h)(z_1, z_2) \\ &= (1/(2\pi i))^2 \int_{|\zeta_1|=1} \int_{|\zeta_2|=1} \left\{ \prod_{j=1}^2 (1/(\zeta_j - z_j)) \right\} h(\zeta_1, \zeta_2) d\zeta_1 d\zeta_2 \end{aligned}$$

for all $(z_1, z_2) \in D^2$, and $\|f\|_* \approx \|h\|_{L^\infty}$.

REMARK. This corollary is an extension of Theorem 5 (a) \Leftrightarrow (e) in [1].

§ 4. H^∞ as abstract Hardy algebras and their applications.

It is not hard to see that H^∞ is not weak * Dirichlet algebras. (See [11] for weak * Dirichlet algebras.) We begin by proving the following theorem.

THEOREM 4.1. (1) H^∞ is a weak * closed algebra of L^∞ .

(2) $H^\infty \ni 1$.

(3) $E[XY] = E[X]E[Y]$ for every $X, Y \in H^\infty$.

(4) $\log(|X_{00}|) \leq E[\log(|X|)]$ for every $X \in H^1$.

(5) $H^p \cap L^r = H^r$ ($1 \leq p \leq r \leq \infty$).

(6) $H^p = [H^\infty]_p = [H^\infty(\mathcal{F}^1) \otimes H^\infty(\mathcal{F}^2)]_p$ ($1 \leq p < \infty$), where we denote by $[H^\infty]_p$ (resp. $[H^\infty(\mathcal{F}^1) \otimes H^\infty(\mathcal{F}^2)]_p$) the L^p -norm closure of H^∞ (resp. $H^\infty(\mathcal{F}^1) \otimes H^\infty(\mathcal{F}^2)$) and we denote by $H^\infty(\mathcal{F}^1) \otimes H^\infty(\mathcal{F}^2)$ the linear span of $\{FG: F \in H^\infty(\mathcal{F}^1), G \in H^\infty(\mathcal{F}^2)\}$.

In order to prove Theorem 4.1 we need the following lemma.

LEMMA 4.2. Let $X^{(1)}, \dots, X^{(n)} \in H^1$ and let G be a domain in C^n such that $P\{M_{st} = (X_{st}^{(1)}, \dots, X_{st}^{(n)}) \in G \text{ for every } s, t \geq 0\} = 1$. If φ a function which is twice continuously differentiable on C^n and analytic on G , then we obtain the following (a) and (b).

(a) If $\sup_{s,t} |\varphi(M_{st})| \in L^2$, then $(\varphi(M_{st})) \in H^2$.

(b) If $X^{(j)} \in H^\infty$ ($1 \leq j \leq n$), then $(\varphi(M_{st})) \in H^\infty$.

PROOF. Let $Y^{(1)}, \dots, Y^{(n)} \in H^1(\mathcal{F}^j)$ and $p^j\{(Y_t^{(1)}, \dots, Y_t^{(n)}) \in G \text{ for every } t \geq 0\} = 1$. If $\sup_t |\varphi(Y_t^{(1)}, \dots, Y_t^{(n)})| \in L^1$, then Ito's formula and [8, p. 292 c)] give $(\varphi(Y_t^{(1)}, \dots, Y_t^{(n)})) \in H^1(\mathcal{F}^j)$ ($j=1, 2$). From this we have that $(\varphi(M_{st}))_{t \geq 0}$ (resp. $(\varphi(M_{st}))_{s \geq 0}$) is a holomorphic martingale for fixed $s \geq 0$ (resp. $t \geq 0$). By Proposition 1 in [4] and Appendix I we have (a). And (b) is clear by (a).

PROOF OF THEOREM 4.1. (2) is obvious. By Lemma 4.2 we obtain that H^∞ is an algebra and satisfies (3). By Proposition 2.1 (2) and Lemma 2.3, it is true that H^p is L^p -norm closed ($1 \leq p < \infty$). Since $H^\infty = H^2 \cap L^\infty$, H^∞ is weak * closed algebra by Krein-Smulian consequence ([2]). By Corollary 2.4.6 in [11] we have

$$(4.1) \quad \log |E_j[X]| \leq E_j[\log |X|],$$

for every $X \in H^1(\mathcal{F}^j)$ ($j=1, 2$).

Take any $X \in H^1$. If $X_{00} = 0$, then (4) is clear. We assume $X_{00} \neq 0$. Then we have

$$\begin{aligned} \infty > E[|X|] &= E_2[E_1[|X|]] \geq E_2[E_1[\log |X|]] \\ &\geq E_2[\log |E_1[X|]|] \geq \log |E_2 E_1[X|]| \\ &= \log |X_{00}| > -\infty. \end{aligned}$$

Hence $\log |X_{00}| \leq E_2[E_1[\log |X|]] = E[\log |X|]$ by Fubini's theorem. So

we obtain (4). Finally we will prove (6). We first suppose $2 \leq p < \infty$. Let $X \in H^p$. For any $\varepsilon > 0$, there exist $a_n \in C$, $A^{(n)} \in L^p(\Omega_1)$ and $B^{(n)} \in L^p(\Omega_2)$ ($n=1, \dots, N$) such that

$$\left\| X - \sum_{n=1}^N a_n A^{(n)} B^{(n)} \right\|_{L^p} < \varepsilon.$$

Then we have that

$$\begin{aligned} & \left\| X - \sum_{n=1}^N a_n T_1(A^{(n)}) T_2(B^{(n)}) \right\|_{L^p} \\ &= \left\| K^{aa} \left(X - \sum_{n=1}^N a_n A^{(n)} B^{(n)} \right) \right\|_{L^p} \\ &\leq C \left\| K^{aa} \left(X - \sum_{n=1}^N a_n A^{(n)} B^{(n)} \right) \right\|_{K^p} \quad (\text{by (1.2)}) \\ &\leq C \|K^{aa}\|_p \left\| X - \sum_{n=1}^N a_n A^{(n)} B^{(n)} \right\|_{K^p} \quad (\text{by Proposition 2.1(1)}) \\ &\leq \varepsilon C \|K^{aa}\|_p, \end{aligned}$$

here and throughout this proof we denote by C various universal constants, and $\|K^{aa}\|_p = \sup\{\|K^{aa} Y\|_{K^p} : \|Y\|_{K^p} = 1\}$.

Let $R_m^k = \inf\{s : |T_1(A^{(k)})_s| \geq m\}$ and let $S_m^k = \inf\{t : |T_2(B^{(k)})_t| \geq m\}$, $m \in N$, $k=1, \dots, N$.

For every $k \in \{1, \dots, N\}$, there exists an $m \in N$ depending only on $\sum_{n=1}^N a_n A^{(n)} B^{(n)}$ such that

$$\begin{aligned} & \|T_1(A^{(k)}) - T_1(A^{(k)})_{R_m^k}\|_{L^p(\Omega_1)} \leq \varepsilon/N |a_k| \ell' \\ & \|T_2(B^{(k)}) - T_2(B^{(k)})_{S_m^k}\|_{L^p(\Omega_2)} \leq \varepsilon/N |a_k| \ell', \end{aligned}$$

where $\ell' = \max(1, \|T_1(A^{(k)})\|_{L^p(\Omega_1)}, \|T_2(B^{(k)})\|_{L^p(\Omega_2)} : k=1, \dots, N)$. Hence

$$\begin{aligned} & \left\| X - \sum_{n=1}^N a_n T_1(A^{(n)})_{R_m^n} T_2(B^{(n)})_{S_m^n} \right\|_{L^p} \\ &\leq \left\| X - \sum_{n=1}^N a_n T_1(A^{(n)}) T_2(B^{(n)}) \right\|_{L^p} \\ &\quad + \left\| \sum_{n=1}^N a_n T_1(A^{(n)}) T_2(B^{(n)}) - \sum_{n=1}^N a_n T_1(A^{(n)}) T_2(B^{(n)})_{S_m^n} \right\|_{L^p} \\ &\quad + \left\| \sum_{n=1}^N a_n T_1(A^{(n)}) T_2(B^{(n)})_{S_m^n} - \sum_{n=1}^N a_n T_1(A^{(n)})_{R_m^n} T_2(B^{(n)})_{S_m^n} \right\|_{L^p} \\ &\leq \varepsilon C \|K^{aa}\|_p + \varepsilon + \varepsilon = \varepsilon(2 + C \|K^{aa}\|_p). \end{aligned}$$

Since $\sum_{n=1}^N a_n T_1(A^{(n)})_{R_m^n} T_2(B^{(n)})_{S_m^n} \in H^\infty(\mathcal{S}^1) \otimes H^\infty(\mathcal{S}^2) \subset H^\infty$, we have that $H^p = [H^\infty(\mathcal{S}^1) \otimes H^\infty(\mathcal{S}^2)]_p = [H^\infty]^p$, if $2 \leq p < \infty$.

Suppose $1 \leq p < 2$. Let $X \in H^p$. By [13, p. 97] it is clear that $(X_{s_0} - X_{0_0})_s$ and $(X_{0_t} - X_{0_0})_t$ are in $[H^\infty(\mathcal{F}^1) \otimes H^\infty(\mathcal{F}^2)]_p$. By Appendix I, there exists $A \in \mathcal{A}^p$ such that $\Delta X = \iint A dz^1 dz^2$. By definition of \mathcal{A}^p there exists $A_n \in \mathcal{S}$ ($n \in \mathbb{N}$) such that $\lim_{n \rightarrow \infty} \|A - A_n\|_p = 0$. Hence we have

$$\begin{aligned} \left\| \Delta X - \iint A_n dz^1 dz^2 \right\|_{L^p} &\leq C \left\| \Delta X - \iint A_n dz^1 dz^2 \right\|_{K^p} \\ &\leq C \|A - A_n\|_p \longrightarrow 0 \quad (\text{as } n \longrightarrow \infty). \end{aligned}$$

Since $\iint A_n dz^1 dz^2 \in H^2$ we have that

$$H^p = [H^2]_p = [[H^\infty(\mathcal{F}^1) \otimes H^\infty(\mathcal{F}^2)]_2]_p = [H^\infty(\mathcal{F}^1) \otimes H^\infty(\mathcal{F}^2)]_p.$$

COROLLARY 4.3. *If $X \in H^1$ is real valued, then X is a constant function.*

PROOF. It is clear by [11, Lemma 3.2] and (4).

As application of results in this section stated above we study thereafter H^1 -martingales with positive real parts.

We put

$$H^+ = \{X: \text{Re } X \geq 0 \text{ a.s. and } \exp(-\alpha X) \in H^\infty \text{ for every } \alpha \geq 0\}$$

(cf. [2]).

THEOREM 4.4. *If $X \in H^1$, $\text{Re } X \geq 0$ and $X \neq 0$ a.s., then*

$$\begin{aligned} \text{Log}(X) = \log(|X|) + i \text{Arg } X &\in \text{BMOA} \quad (-\pi/2 \leq \text{Arg } X \leq \pi/2) \quad \text{and} \\ \log(|X_{0_0}|) = E[\log(|X|)] &> -\infty. \end{aligned}$$

Hence X is an outer function.

For the proof of this theorem we need following lemma.

LEMMA 4.5. *If $X \in H^1$ and $\exp(X) \in L^\infty$, then $\exp(X) \in H^\infty$*

PROOF. Since $|\exp(X_{s,t})| \leq \|\exp(X)\|_{L^\infty} < \infty$, we have that $\exp(X) \in H^1 \cap L^\infty = H^\infty$, by Lemma 4.2 and Theorem 4.1 (5).

LEMMA 4.6. *If $X \in H^1$ and $\text{Re}(X) \geq 0$ a.s., then $X \in H^+$.*

PROOF. $\exp(-\alpha X) \in L^\infty$ for any $\alpha \geq 0$. Hence $X \in H^+$ by Lemma 4.5.

LEMMA 4.7. *Suppose $p \in]0, \infty]$ and let $X \in H^p$. If $\text{Im}(X) \in \text{BMO}$, then $X \in \text{BMOA}$.*

PROOF. To prove the lemma, it suffices to show that $\Delta X \in BMOA$. Let $\Delta X = \iint A dz^1 dz^2$ for some $A \in L^p$ (cf. Appendix I). Then

$$\operatorname{Re}(\Delta X) = \iint (\operatorname{Re}(A))(dx^1 dx^2 - dy^1 dy^2) - \iint (\operatorname{Im}(A))(dy^1 dx^2 + dx^1 dy^2)$$

and

$$\operatorname{Im}(\Delta X) = \iint (\operatorname{Im}(A))(dx^1 dx^2 - dy^1 dy^2) + \iint (\operatorname{Re}(A))(dy^1 dx^2 + dx^1 dy^2).$$

Hence

$$\langle \operatorname{Re}(\Delta X), \operatorname{Re}(\Delta X) \rangle = \langle \operatorname{Im}(\Delta X), \operatorname{Im}(\Delta X) \rangle.$$

From this we have $\operatorname{Re}(\Delta X) \in K^2$. By the definition of *BMO*-norm we obtain

$$\|\operatorname{Re}(\Delta X)\|_{**} = \|\operatorname{Im}(\Delta X)\|_{**}.$$

Thus $\Delta X \in BMOA$.

PROOF OF THEOREM 4.4. By Lemma 4.6 and [2, p. 94], there exist $h_k \in H^\infty$ ($k \in N$) such that

$$\operatorname{Re}(h_k) \geq 0, |X| \geq |h_k| \quad \text{and} \quad \lim_{k \rightarrow \infty} h_k = X \quad \text{a.s. .}$$

We put $X^{(n)} = X + (1/n)$ and $h_{k,n} = h_k + (1/n)$ ($n \in N$). Then $\operatorname{Re}(h_{k,n}) \geq 1/n$ ($n, k \in N$). By Runge's approximation theorem in function theory and Theorem 4.1 (1) we have that $\operatorname{Log}(h_{k,n}) \in H^\infty$ ($k, n \in N$). It is easy to check that $\log(|X|) \in L^1$ by using of Theorem 4.1 (4). Since

$$\begin{aligned} |\operatorname{Log}(h_{k,n})| &\leq \max(\log(n), \log(|X| + 1)) + \pi, \\ |\operatorname{Log}(X^{(n)})| &\leq \max(\log(|X| + 1), |\log(|X|)|) + \pi \end{aligned}$$

($n, k \in N$), we obtain that $\operatorname{Log}(X) \in H^1$ by Lebesgue's convergence theorem. Since $\operatorname{Im}(\operatorname{Log}(X)) \in L^\infty$, we have $\operatorname{Log}(X) \in BMOA$ by Lemma 4.7.

Now, we put $s_{k,n} = \max(\|\operatorname{Re}(h_{k,n})\|_{L^\infty}, \|\operatorname{Im}(h_{k,n})\|_{L^\infty})$, $\delta_n = 1/n$ and $\varepsilon_{k,n} = [\delta_n + (s_{k,n})^2]/\delta_n$ ($n, k \in N$). Then it follows that

$$\|h_{k,n} - \varepsilon_{k,n}\|_{L^\infty} \leq (-2\delta_n + (\varepsilon_{k,n})^2)^{1/2} < \varepsilon_{k,n} \quad (n, k \in N).$$

Hence, by §8 Corollary 4 in [3], there exist $g_{k,n} \in H^\infty$ ($k, n \in N$) such that

$$h_{k,n} = (\varepsilon_{k,n}) \exp(g_{k,n}).$$

We put $Y_{k,n} = (1/\varepsilon_{k,n}) \exp(-g_{k,n})$ ($k, n \in N$). By Lemma 4.5, we have that $Y_{k,n} \in H^\infty$ and $Y_{k,n} h_{k,n} = 1$ ($n, k \in N$). Theorem 4.1 (4) implies that

$$-\infty < \log(|E[h_{k,n}]|) = E[\log(|h_{k,n}|)] .$$

Since $|\log(|h_{k,n}|)| \leq |h_{k,n}| \leq |X| + 1$, it follows that

$$-\infty < \log(|E[X]|) = E[\log(|X|)] .$$

REMARK. In the case of 1-parameter, Varopoulos proved the Helson-Szegö's theorem by using of following result ([13]):

If $X \in H^1(\mathcal{F}^1)$, $X \neq 0$; $\text{Re}(X) \geq 0$ a.s. dP , then $\text{Log}(X) \in H^1(\mathcal{F}^1)$.

But the Helson-Szegö's theorem in general does not hold on the torus and product Brownian spaces. Indeed, we can construct $w \in L^\infty(T^2)$ satisfying the following (1), (2) and (3).

(1) $0 \leq w \leq 1$ a.s.dm.

(2) There exists a universal constant $C > 0$ such that

$$\sum_{j=1}^3 \int_{T^2} |H_j f|^2 w dm \leq C \int_{T^2} |f|^2 w dm \quad (f \in L^2(T^2)) .$$

(3) $\log(w) \notin \{g_0 + \sum_{j=1}^3 H_j g_j; g_k \in L^\infty(T^2), k=0, 1, 2, 3.\}$.

This w can be constructed by modifying a Fefferman's function in [5, p. 397].

Appendix I.

We shall here describe an explicit form of X^ε ($X \in K^p$, $\varepsilon = aa, ab, ba, bb$). Let X be written as (1.1). We put

$$\begin{aligned} a &= (\varphi_1 - i\varphi_2)/2, & a' &= (\varphi_1 + i\varphi_2)/2, & b &= (\psi_1 - i\psi_2)/2, \\ b' &= (\psi_1 + i\psi_2)/2, & A &= (\Phi_1 - i\Phi_2 - i\Phi_3 - \Phi_4)/4, \\ B &= (\Phi_1 + i\Phi_2 - i\Phi_3 + \Phi_4)/4, & C &= (\Phi_1 - i\Phi_2 + i\Phi_3 + \Phi_4)/4 \\ \text{and } D &= (\Phi_1 + i\Phi_2 + i\Phi_3 - \Phi_4)/4 . \end{aligned}$$

Then we have

$$\begin{aligned} X^{aa} &= X_{00} + \int adz^1 + \int b dz^2 + \iint Adz^1 dz^2, \\ X^{ab} &= \iint B dz^1 d\bar{z}^2, \quad X^{ba} = \iint C d\bar{z}^1 dz^2 \quad \text{and} \\ X^{bb} &= \int a' d\bar{z}^1 + \int b' d\bar{z}^2 + \iint D d\bar{z}^1 d\bar{z}^2 . \end{aligned}$$

Appendix II.

We shall here give an alternative proof of (3.1) in the proof of Corollary 3.8; there exists a universal constant C_1 such that

$$\|NX\|_{h^1} \leq C_1 \|\tilde{N}X^*\|_{L^1} \quad \text{for all } X \in K^1.$$

Let $Y \in K^2$ and let $P_1(NY)$ be analytic on D_1 . Put $f_1 = NY$. By Hardy-Littlewood's inequality on T_1 , we have that

$$\begin{aligned} & \int_0^{2\pi} \int_0^{2\pi} n(f_1)(e^{i\theta}, e^{i\varphi}) d\theta d\varphi \\ & \leq \int_0^{2\pi} \int_0^{2\pi} \left(\sup_{0 \leq r < 1} P_2 \left(\sup_{0 \leq s < 1} |P_1(f_1(\cdot, \cdot))(se^{i\theta})|^{1/2} \right) (re^{i\varphi}) \right)^2 d\theta d\varphi \\ & \leq c \int_0^{2\pi} \int_0^{2\pi} \left(\sup_{0 \leq r < 1} |P_1(f_1(\cdot, e^{i\varphi}))(re^{i\theta})|^{1/2} \right)^2 d\theta d\varphi \\ & \leq C \int_0^{2\pi} \int_0^{2\pi} |f_1(e^{i\theta}, e^{i\varphi})| d\theta d\varphi, \end{aligned}$$

where c and C are universal constants. Hence

$$\|n(f_1)\|_{L^1} \leq C \|f_1\|_{L^1}.$$

Let $X \in K^2$ and put $f = NX$. Since $\tilde{K}^{aa}(f)$, $\tilde{K}^{ab}(f)$, $\tilde{K}^{ba}(f)$ and $\tilde{K}^{bb}(f)$ are analytic on D_1 , we obtain

$$\|\tilde{K}^\varepsilon(f)\|_{h^1} \leq C \|\tilde{K}^\varepsilon(f)\|_{L^1} \quad (\varepsilon = aa, ab, ba, bb).$$

So we obtain

$$\begin{aligned} \|f\|_{h^1} & \leq \sum \{ \|\tilde{K}^\varepsilon(f)\|_{h^1} : \varepsilon = aa, ab, ba, bb \} \\ & \leq C \sum \{ \|\tilde{K}^\varepsilon(f)\|_{L^1} : \varepsilon = aa, ab, ba, bb \} \\ & = C \sum \{ \|M\tilde{K}^\varepsilon(f)\|_{L^1} : \varepsilon = aa, ab, ba, bb \} \\ & = C \sum \{ \|K^\varepsilon M(f)\|_{L^1} : \varepsilon = aa, ab, ba, bb \} \\ & \approx \|Mf\|_{K^1} \approx \|Mf^*\|_{L^1}. \end{aligned}$$

For every $X \in K^1$, there exist $X^{(n)} \in K^2$ ($n \in \mathbb{N}$) such that

$$\lim_{n \rightarrow \infty} \|X - X^{(n)}\|_{K^1} = 0$$

(cf. the definition of A^1). Hence

$$\begin{aligned} \lim_{n \rightarrow \infty} \|\tilde{N}X^* - \tilde{N}X^{(n)*}\|_{L^1} & \leq \lim_{n \rightarrow \infty} \|\tilde{N}(X - X^{(n)})^*\|_{L^1} \\ & \approx \lim_{n \rightarrow \infty} \|\tilde{N}(X - X^{(n)})\|_{K^1} \leq \|\tilde{N}\|_1 \lim_{n \rightarrow \infty} \|X - X^{(n)}\|_{K^1} = 0. \end{aligned}$$

Thus we obtain (3.1).

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