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Sherman Transformations for Functions on the Sphere

Ryoko WADA

Sophia University
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Introduction

Let $S=S^d$ be the unit sphere in $\boldsymbol{R}^{d+1}$. It is well-known that functions and functionals f on S can be developed in the series of the spherical harmonics; $f=\sum_{n=0}^{\infty} f_n$, where f_n are spherical harmonic functions of degree n in $(d+1)$-dimensions. Certain function spaces or functional spaces on the sphere S can be characterized by the behavior of the sequence $\{\|f_n\|_2\}_{n=0,1,2,\cdots}$ (Lemma 1.1). On the other hand, T. O. Sherman [7] introduced the two transformations $f\rightarrow Ff(b, n)$ and $f\rightarrow F_*f(b, n)$ and studied the developments of functions or functionals on S using them.

In this paper we propose to replace his transformation F_* by a slightly different tranformation $F_\sharp$:

$$F_\sharp f(b, n)=F_* f_n(b, n) \quad \text{for} \quad f=\sum_{n=0}^{\infty} f_n \ .$$

Though $F_*f(b, n)$ is well-defined only for some differentiable functions, $F_\sharp f(b, n)$ can be defined for more general functions and functionals. And in the results of Sherman [7] we can replace $F_*f(b, n)$ by $F_\sharp f(b, n)$.

Here $Ff(b, n)$ and $F_\sharp f(b, n)$ are polynomials on the "equator" $B=\{s\in S;\ s\cdot a=0\}$, where $a=(0, 0, \cdots, 1)\in S$ denotes the "north pole".

The two transformations F and $F_\sharp$ define the mappings:

$$F: f \longrightarrow F(f)=\{Ff(\ , n)\}_{n=0,1,\cdots}\in \prod P_n(B)$$
$$F_\sharp: f \longrightarrow F_\sharp(f)=\{F_\sharp f(\ , n)\}_{n=0,1,\cdots}\in \prod P_n(B) \ ,$$

where $P_n(B)=\{g;$ a polynomial on B of degree at most $n\}$ and $\prod P_n(B)$ is the direct product of $P_n(B)$ $(n=0, 1, 2, \cdots)$. We call F the *Sherman transformation* and $F_\sharp$ the *modified Sherman transformation* respectively. Furthermore, we can consider that F and $F_\sharp$ are dual to each other in

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a sense (see Proposition 3.1).

The aim of this paper is to investigate the images of certain function spaces or functional spaces on S under the Sherman transformation F and the modified Sherman transformation $F_\sharp$. Among others we will determine the images of $\mathcal{O}(\tilde{S})$, $\mathrm{Exp}(\tilde{S})$ and their dual spaces under the transformations F and $F_\sharp$.

The plan of this paper is as follows: In § 1 we fix the notations and recall several results on spherical harmonic functions, which will be in need in the sequel. In § 2, we investigate the Sherman transformation F. Our main theorem in § 2 is Theorem 2.1. We study in § 3 the images of certain function or functional spaces on S by the modified Sherman transformation $F_\sharp$ (Theorem 3.2). Lastly in § 4 we discuss the relations between the Sherman transformation F and the Fantappié indicator.

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§ 1. Notations.

Let us fix notations.

Let d be a positive integer. $S=S^d$ denotes the unit sphere in $\boldsymbol{R}^{d+1}$: $S=\{x\in\boldsymbol{R}^{d+1};\ \|x\|=1\}$, where $\|x\|^2=x_1^2+x_2^2+\cdots+x_{d+1}^2$. The rotation invariant integral on S with $\int_S 1ds=1$ is denoted $\int_S f(s)ds$. $L^2(S)$ is the Hilbert space of L^2 functions with the inner product

$$(f\,|\,g)=(f,\bar{g})_S=\int_S f(s)\overline{g(s)}ds$$

and the norm $\|f\|_2=\|f\|_{S,2}=(f,\bar{f})_S^{1/2}$. We will use mainly the bilinear form

$$(f,g)_S=\int_S f(s)g(s)ds\ . \tag{1.1}$$

Let $H_{n,d}$ denote the space of spherical harmonics of degree n in $(d+1)$-dimensions. It is well known that $H_{n,d}\perp H_{m,d}$ $(n\neq m)$ with respect to the bilinear form $(f,g)_S$ and $L^2(S)$ is the direct sum of $H_{n,d}$, $n=0,1,\cdots$.

$$L^2(S)=\bigoplus_{n=0}^{\infty} H_{n,d}\ . \tag{1.2}$$

The orthogonal projection of $L^2(S)$ onto $H_{n,d}$ is given as follows:

$$f\in L^2(S)\longrightarrow f_n(s)=(\dim H_{n,d})\int_S f(s_1)P_{n,d}(s\cdot s_1)ds_1\ , \tag{1.3}$$

where

$$\dim H_{n,d}=N(n,d)=\begin{cases}1 & n=0\\ \dfrac{(2n+d-1)(n+d-2)!}{n!\,(d-1)!} & n\geqq 1\end{cases} \tag{1.4}$$

and $P_{n,d}$ is the Legendre polynomial of degree n and of dimension $d+1$. Namely,

$$P_{n,d}(r)=\left(-\frac{1}{2}\right)^n\frac{\Gamma(d/2)}{\Gamma(n+d/2)}(1-r^2)^{(2-d)/2}\frac{d^n}{dr^n}(1-r^2)^{(2n+d-2)/2}\,,\quad -1\leqq r\leqq 1 \quad \text{(Rodrigues' formula)}\,. \tag{1.5}$$

$$P_{n,d}(1)=1\,,\quad |P_{n,d}(r)|\leqq 1\,. \tag{1.6}$$

$\mathscr{D}(S)$ denotes the space of C^∞ functions on S equipped with the topology of uniform convergence on S in all derivatives.

Now we define the complex sphere $\tilde{S}$:

$$\tilde{S}=\{z\in \boldsymbol{C}^{d+1};\ z_1^2+z_2^2+\cdots+z_{d+1}^2=1\}\,.$$

We put for $r>1$

$$\tilde{S}(r)=\{z=x+iy\in\tilde{S};\ \|y\|<(r-1/r)/2\}\,.$$

It is clear that S is the real part of $\tilde{S}$:

$$S=\tilde{S}\cap \boldsymbol{R}^{d+1}$$

and that $\tilde{S}(r)$, $r>1$ forms the fundamental system of complex neighbourhoods of S in $\tilde{S}$. Let us denote by $\mathscr{O}(\tilde{S}(r))$ the space of holomorphic functions on $\tilde{S}(r)$ equipped with the topology of uniform convergence on every compact subset of $\tilde{S}(r)$. $\mathscr{A}(S)$ denotes the space of real-analytic functions on S. We have the linear isomorphism

$$\mathscr{A}(S)=\lim_{\substack{\longrightarrow\\ r>1}}\operatorname{ind}\mathscr{O}(\tilde{S}(r))$$

and we equip $\mathscr{A}(S)$ with the locally convex inductive limit topology of $\mathscr{O}(\tilde{S}(r))$, $r>1$.

Now we define $\mathscr{O}(\tilde{S})$ to be the space of holomorphic functions on $\tilde{S}$. The restriction mapping $\mathscr{O}(\tilde{S})\to\mathscr{A}(S)$ being injective, we identify $\mathscr{O}(\tilde{S})$ with a subspace of $\mathscr{A}(S)$. Remark that the restriction mapping

$$\rho:\ \mathscr{O}(\boldsymbol{C}^{d+1})\longrightarrow\mathscr{O}(\tilde{S})$$

is surjective.

Let us denote by $\operatorname{Exp}(\boldsymbol{C}^{d+1})$ the space of entire functions of exponential type on C^{d+1}. We define the subspace $\mathrm{Exp}(\tilde{S})$ of $\mathscr{O}(\tilde{S})$ to be the image of $\mathrm{Exp}(C^{d+1})$ under the restriction mapping ρ:

$$\mathrm{Exp}(\tilde{S})=\rho(\mathrm{Exp}(C^{d+1})) .$$

Summing up, we have defined the following function spaces on S:

$$(*) \qquad \mathrm{Exp}(\tilde{S}) \subsetneq \mathscr{O}(\tilde{S}) \subsetneq \mathscr{A}(S) \subsetneq \mathscr{D}(S) \subsetneq L^2(S) ,$$

where the inclusion mappings are linear continuous and with dense image.

$\mathscr{D}'(S)$, $\mathscr{A}'(S)$, $\mathscr{O}'(\tilde{S})$, and $\mathrm{Exp}'(\tilde{S})$ denote the dual spaces of $\mathscr{D}(S)$, $\mathscr{A}(S)$, $\mathscr{O}(\tilde{S})$, and $\mathrm{Exp}(\tilde{S})$ respectively. Using the bilinear form $(f, g)_S=\int_S f(s)g(s)ds$, we will identify $L^2(S)$ with its dual space. We have the following series of functional spaces on S:

$$(**) \qquad \mathrm{Exp}'(\tilde{S}) \supsetneq \mathscr{O}'(\tilde{S}) \supsetneq \mathscr{A}'(S) \supsetneq \mathscr{D}'(S) \supsetneq L^2(S) .$$

We can characterize the function or functional spaces in (*) or (**) by behaviors of the spherical harmonic development as follows.

LEMMA 1.1. (Morimoto [4]). *If f_n (resp. f'_n) is the n-th spherical harmonic component of f (resp. f'), then*

$$(1.7) \qquad f \in \mathrm{Exp}(\tilde{S}) \Longleftrightarrow \limsup_{n\to\infty}(n!\,\|f_n\|_2)^{1/n}<\infty ,$$

$$(1.8) \qquad f \in \mathscr{O}(\tilde{S}) \Longleftrightarrow \limsup_{n\to\infty}\|f_n\|_2^{1/n}=0 ,$$

$$(1.9) \qquad \begin{aligned} &f \in \mathscr{A}(S) \Longleftrightarrow \limsup_{n\to\infty}\|f_n\|_2^{1/n}<1 ,\\ &f=\sum_{n=0}^{\infty} f_n \ \textit{ converges in the topology of } \ \mathscr{A}(S) . \end{aligned}$$

$$(1.10) \qquad \begin{aligned} &f \in \mathscr{D}(S) \Longleftrightarrow \|f_n\|_2 \ \textit{ is rapidly decreasing as } \ n \longrightarrow \infty ,\\ &\sum_{n=0}^{\infty} f_n \ \textit{ converges to } f \textit{ in the topology of } \ \mathscr{D}(S) . \end{aligned}$$

$$(1.11) \qquad f \in L^2(S) \Longleftrightarrow \{\|f_n\|_2\}_{n=0,1,2,\dots} \in l^2 ,$$

$$(1.12) \qquad f' \in \mathscr{D}'(S) \Longleftrightarrow \|f'_n\|_2 \ \textit{ is slowly increasing as } \ n \longrightarrow \infty ,$$

$$(1.13) \qquad f' \in \mathscr{A}'(S) \Longleftrightarrow \limsup_{n\to\infty}\|f'_n\|_2^{1/n}\leqq 1 ,$$

$$(1.14) \qquad f' \in \mathscr{O}'(\tilde{S}) \Longleftrightarrow \limsup_{n\to\infty}\|f'_n\|_2^{1/n}<\infty ,$$

$$(1.15) \qquad f' \in \mathrm{Exp}'(\tilde{S}) \Longleftrightarrow \limsup_{n\to\infty}(\|f'_n\|_2/n!)^{1/n}=0 .$$

REMARK. If f_n is in $H_{n,d}$, then it is valid that

(1.16)
$$\|f_n\|_2 \leqq \|f_n\|_\infty \leqq \sqrt{N(n, d)}\|f_n\|_2 ,$$

where $\|f\|_\infty = \sup_{s\in S}|f(s)|$. So we may replace $\|\ \|_2$ by $\|\ \|_\infty$ in (1.7)–(1.10) and in (1.12)–(1.15).

The "north pole" $a \in S$ is fixed and $B=\{s\in S;\ s\cdot a=0\}\cong S^{d-1}$ denote the "equator". The rotation invariant integral on B with $\int_B 1db=1$ is denoted $\int_B f(b)db$.

If f is a measurable function on S, we have

(1.17)
$$\int_S f(s)ds = C_d \int_{-1}^{1}\int_B (1-r^2)^{(d-2)/2} f(ra+(1-r^2)^{1/2}b)dbdr ,$$

where

(1.18)
$$C_d = \frac{\Gamma((d+1)/2)}{\Gamma(d/2)\Gamma(1/2)}$$

$P_n(B)$ denotes the space of polynomials on B of degree at most n. $H_{j,d-1}$ denotes the space of spherical harmonics of degree n in d dimensions and we will identify $H_{j,d-1}$ with a subspace of $P_n(B)$ for $j=0, 1, 2, \cdots, n$. Then we have the direct sum decomposition of $P_n(B)$:

$$P_n(B) = \bigoplus_{j=0}^{n} H_{j,d-1} .$$

$H_{j,d-1}$ are orthogonal with respect to the bilinear form

$$(f, g)_B = \int_B f(b)g(b)db .$$

In particular, we have

$$\dim P_n(B) = \sum_{j=0}^{n} \dim H_{j,d-1} .$$

$H_{n,d}$ and $H_{j,d-1}$ $(j=0, 1, 2, \cdots, n)$ are related as follows:

LEMMA 1.2. (see for example Müller [6] p. 25 and Lemma 15). *Suppose* $s_{j,k}$, $1\leqq k\leqq \dim H_{j,d-1}$, *is a basis of* $H_{j,d-1}$, $j=0, 1, \cdots, n$. *Put*

$$f_{j,k}(ra+(1-r^2)^{1/2}b) = (1-r^2)^{j/2} P_{n-j,2j+d}(r) s_{j,k}(b) .$$

Then $f_{j,k}$, $j=0, 1, \cdots, n$, $k=1, 2, \cdots, \dim H_{j,d-1}$ *is a basis of* $H_{n,d}$.

Thanks to Lemma 1.2, every $f_n \in H_{n,d}$ can be expressed in the form

$$f_n(ra+(1-r^2)^{1/2}b)=\sum_{j=0}^{n}(1-r^2)^{j/2}P_{n-j,2j+d}(r)s_j(b) \tag{1.19}$$

with $s_j \in H_{j,d-1}$, $j=0, 1, \cdots, n$. In this case, we have,

$$\|f_n\|_{S,2}^2=C_d\sum_{j=0}^{n}\alpha(n, j, d)\|s_j\|_{B,2}^2 , \tag{1.20}$$

where

$$\begin{aligned}\alpha(n, j, d)&=\int_{-1}^{1}(1-r^2)^{(2j+d-2)/2}(P_{n-j,2j+d}(r))^2dr \\ &=\frac{1}{C_{2j+d}N(n-j, 2j+d)} \\ &=\frac{\pi^{1/2}\Gamma(j+d/2)}{\Gamma(j+(d+1)/2)}\cdot\frac{(n-j)!\,(2j+d-1)!}{(2n+d-1)(n+j+d-2)!}\end{aligned} \tag{1.21}$$

and

$$\|s\|_{B,2}^2=\int_B |s(b)|^2 db . \tag{1.22}$$

Here we also have clearly that

$$\dim H_{n,d}=\dim P_n(B) . \tag{1.23}$$

§ 2. Sherman transformation F.

For any $f' \in \mathrm{Exp}'(\tilde{S})$, we define the *Sherman transform* Ff' of f' by

$$Ff'(b, n)=\langle f', e(b, n)\rangle ,$$

where $e(b, n)(s)=(a\cdot s+ib\cdot s)^n$ $(s \in S)$, $b \in B$ and $n \in Z_+=\{0, 1, 2, \cdots\}$. Then $Ff'(\ , n)$ belongs to $P_n(B)$ (cf. Sherman [7] p. 6) and by the definition we have

$$Ff(b, n)=\int_S f(s)e(b, n)(s)ds$$

for any $f \in L^2(S)$.

REMARK. If $f' \in \mathrm{Exp}'(\tilde{S})$ and f'_n is the n-th spherical component of f', then we have

$$Ff'(b, n)=\int_S f'_n(s)e(b, n)(s)ds=Ff'_n(b, n) ,$$

since we have $f'_m \in H_{m,d}$, $e(b, n) \in H_{n,d}$ and $H_{n,d} \perp H_{m,d}$ $(n \neq m)$.

The transformation F of $\mathrm{Exp}'(\tilde{S})$ into $\prod P_n(B)$

$$f' \in \mathrm{Exp}'(\tilde{S}) \longmapsto F(f') = \{Ff'(\ ,n)\}_{n \in Z_+}$$

is called the *Sherman transformation.*

In order to describe the images of the Sherman transformation F, let us define the following subspaces of $\prod P_n(B)$:

(2.1) $$L = \{\{g_n\}_{n\in Z_+} \in \prod P_n(B);\ \{\|g_n\|_{B,2}\}_{n\in Z_+} \in l_d^2\}\ ,$$

(2.2) $$L^+ = \{\{g_n\}_{n\in Z_+} \in \prod P_n(B);\ \{2^n n^{-1/4}\|g_n\|_{B,2}\}_{n\in Z_+} \in l_d^2\}\ ,$$

(2.3) $$L^- = \{\{g_n\}_{n\in Z_+} \in \prod P_n(B);\ \{2^{-n} n^{1/4}\|g_n\|_{B,2}\}_{n\in Z_+} \in l_d^2\}\ ,$$

(2.4) $$D = \{\{g_n\}_{n\in Z_+} \in \prod P_n(B);\ \|g_n\|_{B,2} \text{ is rapidly decreasing as } n \to \infty\}\ ,$$

(2.5) $$D^+ = \{\{g_n\}_{n\in Z_+} \in \prod P_n(B);\ 2^n\|g_n\|_{B,2} \text{ is rapidly decreasihg as } n \to \infty\}\ ,$$

(2.6) $$D^- = \{\{g_n\}_{n\in Z_+} \in \prod P_n(B);\ 2^{-n}\|g_n\|_{B,2} \text{ is rapidly decreasing as } n \to \infty\}\ ,$$

(2.7) $$D' = \{\{g_n\}_{n\in Z_+} \in \prod P_n(B);\ \|g_n\|_{B,2} \text{ is slowly increasing as } n \to \infty\}\ ,$$

(2.8) $$D'^+ = \{\{g_n\}_{n\in Z_+} \in \prod P_n(B);\ 2^n\|g_n\|_{B,2} \text{ is slowly increasing as } n \to \infty\}\ ,$$

(2.9) $$D'^- = \{\{g_n\}_{n\in Z_+} \in \prod P_n(B);\ 2^{-n}\|g_n\|_{B,2} \text{ is slowly increasing as } n \to \infty\}\ ,$$

(2.10) $$A(\eta) = \{\{g_n\}_{n\in Z_+} \in \prod P_n(B);\ \limsup_{n\to\infty} \|g_n\|_{B,2}^{1/n} < \eta\}\ ,$$

(2.11) $$A[\eta] = \{\{g_n\}_{n\in Z_+} \in \prod P_n(B);\ \limsup_{n\to\infty} \|g_n\|_{B,2}^{1/n} \leqq \eta\}\ ,$$

(2.12) $$O = A[0] = \{\{g_n\}_{n\in Z_+} \in \prod P_n(B);\ \limsup_{n\to\infty} \|g_n\|_{B,2}^{1/n} = 0\}\ ,$$

(2.13) $$O' = A(\infty) = \{\{g_n\}_{n\in Z_+} \in \prod P_n(B);\ \limsup_{n\to\infty} \|g_n\|_{B,2}^{1/n} < \infty\}\ ,$$

(2.14) $$E = \{\{g_n\}_{n\in Z_+} \in \prod P_n(B);\ \limsup_{n\to\infty} (n!\ \|g_n\|_{B,2})^{1/n} < \infty\}\ ,$$

(2.15) $$E' = \{\{g_n\}_{n\in Z_+} \in \prod P_n(B);\ \limsup_{n\to\infty} (\|g_n\|_{B,2}/n!)^{1/n} = 0\}\ ,$$

where $l_d^2 = \{\{a_n\}_{n\in Z_+};\ \sum_{n=0}^{\infty} N(n,d)|a_n|^2 < \infty\}$.

We have the following inclusion relations:

$$\begin{aligned} &E \subsetneqq O \subsetneqq A(1/2) \subsetneqq D^+ \subsetneqq L^+ \subsetneqq D'^+ \subsetneqq A[1/2] \subsetneqq A(1) \subsetneqq D \subsetneqq L \subsetneqq D' \\ &\subsetneqq A[1] \subsetneqq A(2) \subsetneqq D^- \subsetneqq L^- \subsetneqq D'^- \subsetneqq A[2] \subsetneqq O' \subsetneqq E'\ . \end{aligned}$$

REMARK. If $g_n \in P_n(B)$, then it is valid that

$$\|g_n\|_{B,2} \leqq \|g_n\|_B \leqq \sqrt{N(n, d)}\|g_n\|_{B,2} ,$$

where $\|g_n\|_B = \mathrm{sub}_{b\in B}|g_n(b)|$. So we may replace $\| \ \|_{B,2}$ by $\| \ \|_B$ in (2.4)-(2.15).

Our main theorem in this section is the following

THEOREM 2.1. *The Sherman transformation F is a linear one-to-one mapping of* $\mathrm{Exp}'(\tilde{S})$ *into* $\prod P_n(B)$, *which satisfies the following properties:*

$$L^+ \subset F(L^2(S)) \subset L , \tag{2.16}$$

$$D^+ \subset F(\mathscr{D}(S)) \subset D , \tag{2.17}$$

$$D'^+ \subset F(\mathscr{D}'(S)) \subset D' , \tag{2.18}$$

$$A(1/2) \subset F(\mathscr{A}(S)) \subset A(1) , \tag{2.19}$$

$$A[1/2] \subset F(\mathscr{A}'(S)) \subset A[1] , \tag{2.20}$$

(2.21) *F is a one-to-one mapping of $\mathscr{O}(\tilde{S})$ onto O* ,

(2.22) *F is a one-to-one mapping of $\mathscr{O}'(\tilde{S})$ onto O'* ,

(2.23) *F is a one-to-one mapping of* $\mathrm{Exp}(\tilde{S})$ *onto E* ,

(2.24) *F is a one-to-one mapping of* $\mathrm{Exp}'(\tilde{S})$ *onto E'* .

We need the following lemmas in order to prove the theorem.

LEMMA 2.2. (cf. Sherman [7] p. 25~). *If $f_n \in H_{n,d}$ is expressed in the form* (1.19), *we have*

$$Ff_n(b, n) = \sum_{j=0}^{n} \phi(n, j, d) s_j(b) , \tag{2.25}$$

where

$$\begin{aligned} \phi(n, j, d) &= \frac{2^{j+d-1}(i)^j \Gamma((d+1)/2)\Gamma(j+d/2)\Gamma(n+1)}{(2n+d-1)\pi^{1/2}\Gamma(n+j+d-1)} \\ &= (i/2)^j \frac{\Gamma((d+1)/2)}{\pi^{1/2}\Gamma(j+d/2)} \cdot \frac{n!}{(n-j)!} \alpha(n, j, d) \\ &= (i/2)^j \frac{\Gamma((d+1)/2)}{\Gamma(j+(d+1)/2)} \cdot \frac{n!}{(n-j)!} \cdot \frac{1}{N(n-j, 2j+d)} . \end{aligned} \tag{2.26}$$

LEMMA 2.3. *Consider $f_{n,j} \in H_{n,d}$ such that*

$$f_{n,j}(ra + (1-r^2)^{1/2}b) = (1-r^2)^{j/2} P_{n-j,2j+d}(r) s_j(b)$$

with $s_j \in H_{j,d-1}$, $j = 0, 1, \cdots, n$. Then we have

(2.27) $$\|Ff_{n,j}(\ ,n)\|^2_{B,2}$$
$$=\begin{cases}\dfrac{1}{N(n,d)}\|f_{n,0}\|^2_{S,2} & (j=0)\\[2mm] \dfrac{(d-1)!\,(n!)^2}{(2n+d-1)(n-j)!\,(n+d+j-2)!}\|f_{n,j}\|^2_{S,2} & (0\leqq j\leqq n)\\[2mm] \dfrac{(d-1)!\,(n!)^2}{(2n+d-1)!}\|f_{n,n}\|^2_{S,2} & (j=n)\,.\end{cases}$$

Furthermore, if $f'=\sum_{n=0}^{\infty}f'_n\in \mathrm{Exp}'(\tilde{S})$, $f'_n\in H_{n,d}$, $n=0, 1, 2, \cdots$, *then we have*

(2.28) $$C\cdot 2^{-n}n^{1/4}\|f'_n\|_{S,2}\leqq N(n,d)^{1/2}\|Ff'(\ ,n)\|_{B,2}\leqq\|f'_n\|_{S,2}\,,$$

where C is a constant.

PROOF. By Lemma 2.2, we have

$$\|Ff_{n,j}(\ ,n)\|_{B,2}=|\phi(n,j,d)|\ \|s_j\|_{B,2}\,.$$

On the other hand, (1.20) gives

$$\|f_{n,j}\|^2_{S,2}=C_d\alpha(n,j,d)\|s_j\|^2_{B,2}\,.$$

Now (1.21) and (2.26) give

(2.29) $$N(n,d)\frac{|\phi(n,j,d)|^2}{C_d\alpha(n,j,d)}=\frac{n!\,(n+d-2)!}{(n-j)!\,(n+d+j-2)!}\,.$$

Therefore we get

(2.30) $$N(n,d)\|Ff_{n,j}(\ ,n)\|^2_{B,j}=\frac{n!\,(n+d-2)!}{(n-j)!\,(n+d+j-2)!}\|f_{n,j}\|^2_{S,2}\,,$$

and (2.30) implies (2.27).

As we have $Ff'(b,n)=Ff'_n(b,n)$, we may assume $f'=f_n\in H_{n,d}$. Since f_n is given in the form $f_n=\sum_{j=0}^{n}f_{n,j}$ (cf. (1.19)), by Lemma 2.2, we have

$$\begin{aligned}\|Ff'(\ ,n)\|^2_{B,2}&=\sum_{j=0}^{n}|\phi(n,j,d)|^2\|s_j\|^2_{B,2}\\ &=\sum_{j=0}^{n}\|Ff_{n,j}(\ ,n)\|^2_{B,2}\,.\end{aligned}$$

From Lemma 1.2, we have

$$\|f_n\|^2_{S,2}=\sum_{j=0}^{n}\|f_{n,j}\|^2_{S,2}\,.$$

(2.30) gives

$$N(n, d)\|Ff'(\ ,n)\|_{B,2}^2=\sum_{j=0}^{n}\frac{n!\,(n+d-2)!}{(n-j)!\,(n+d+j-2)!}\|f_{n,j}\|_{S,2}^2 .$$

We also have

$$(2.31)\qquad (2n+d-2)!\geqq(n+j+d-1)!\,(n-j)!\geqq(n+d-2)!\,n! ,$$

$j=0, 1, 2, \cdots, n$, since

$$(n+j+d-2)!\,(n-j)!\geqq(n+(j-1)+d-2)!\,(n-(j-1))! ,$$

$j=1, 2, \cdots, n$. Therefore we get

$$\frac{n!\,(n+d-2)!}{(2n+d-2)!}\|f_n\|_{S,2}^2\leqq N(n, d)\|Ff'(\ ,n)\|_{B,2}^2\leqq\|f_n\|_{S,2}^2 .$$

By Stirling's formula, we obtain

$$\begin{aligned}\frac{n!\,(n+d-2)!}{(2n+d-2)!}&\sim(2\pi n)^{1/2}(n/(2n+d-2))^n((n+d-2)/(2n+d-2))^{n+d-3/2}\\ &\geqq C'\cdot 2^{-2n}n^{1/2} ,\end{aligned}$$

where C' is a constant. So we have (2.28). Q.E.D.

PROOF OF THEOREM 2.1. By Lemma 1.2, for any $f_n\in H_{n,d}$, there exist $s_j\in H_{j,d-1}$ ($j=0, 1, 2, \cdots, n$) such that f_n is expressed in the form (1.19). Therefore, the mapping $f_n\to Ff_n(\ ,n)$ is a linear one-to-one mapping of $H_{n,d}$ onto $P_n(B)$. It implies that the Sherman transformation

$$F\colon f'\longrightarrow\{Ff'(\ ,n)\}_{n\in Z_+}$$

is a linear one-to-one mapping of $\mathrm{Exp}'(\tilde{S})$ into $\prod P_n(B)$. Using Lemma 1.1 and Lemma 2.3, we can obtain (2.16)-(2.24). Q.E.D.

REMARK 2.1. In connection with Theorem 2.1, (2.16)-(2.20), we give examples in which

$$\begin{aligned}&L^+\subsetneqq F(L^2(S))\subsetneqq L ,\quad D^+\subsetneqq F(\mathscr{D}(S))\subsetneqq D ,\quad D'^+\subsetneqq F(\mathscr{D}'(S))\subsetneqq D' ,\\ &A(1/2)\subsetneqq F(\mathscr{A}(S))\subsetneqq A(1) ,\quad\text{and}\quad A[1/2]\subsetneqq F(\mathscr{A}'(S))\subsetneqq A[1] .\end{aligned}$$

We consider $\{s_n\}_{n\in Z_+}\in\prod P_n(B)$ such that $s_n\in H_{n,d-1}$ and $\|s_n\|_{B,2}=(\rho/2)^n$ ($n\in Z_+$), where $1<\rho<2$. Then $\{s_n\}_{n\in Z_+}$ belongs to $A(1)$ since $\limsup_{n\to\infty}\|s_n\|_{B,2}^{1/n}=\rho/2<1$. If there exists $f'\in\mathscr{A}'(S)$ such that $F(f')=\{s_n\}_{n\in Z_+}$ and $f'=\sum_{n=0}^{\infty}f'_n$ ($f'_n\in H_{n,d}$), then we have, from (2.27),

$$\limsup_{n\to\infty}\|f'_n\|_{S,2}^{1/n}=\limsup_{n\to\infty}\left\{\frac{(2n+d-1)!}{(d-1)!\,(n!)^2}\right\}^{1/2n}\|s_n\|_{B,2}^{1/n}=\rho>1\ .$$

It implies that $f'\notin \mathscr{A}'(S)$ and $F(\mathscr{A}'(S))\not\supset A(1)$, therefore we obtain

$$F(L^2(S))\subsetneqq L\ ,\quad F(\mathscr{D}(S))\subsetneqq D\ ,\quad F(\mathscr{D}'(S))\subsetneqq D'\ ,$$
$$F(\mathscr{A}(S))\subsetneqq A(1)\ ,\quad \text{and}\quad F(\mathscr{A}'(S))\subsetneqq A[1]\ .$$

Next consider $f=\sum_{n=0}^{\infty}\eta^n P_{n,d}(\ \cdot a)$, where $1/2<\eta<1$. Then f lies in $\mathscr{A}(S)$ since $\limsup_{n\to\infty}\|\eta^n P_{n,d}(\ \cdot a)\|_{\infty}^{1/n}=\eta<1$. But we see by (2.27) and (1.16)

$$\limsup_{n\to\infty}\|Ff(\ ,n)\|_{B,2}^{1/n}=\limsup_{n\to\infty}\|\eta^n P_{n,d}(\ \cdot a)\|_{\infty}^{1/n}=\eta>1/2\ ,$$

which implies $F(f)\notin A[1/2]$. So we have $F(\mathscr{A}(S))\not\subset A[1/2]$, from which result

$$F(\mathscr{D}(S))\supsetneqq D^+\ ,\quad F(\mathscr{D}'(S))\supsetneqq D'^+\ ,\quad F(L^2(S))\supsetneqq L^+\ ,$$
$$F(\mathscr{A}(S))\supsetneqq A(1/2)\ ,\quad \text{and}\quad F(\mathscr{A}'(S))\supsetneqq A[1/2]\ .$$

REMARK 2.2. In connection with Theorem 2.1, (2.19) and (2.20), we give examples in which $A(\eta)\not\supset F(\mathscr{A}(S))$ (resp. $A[\eta]\not\supset F(\mathscr{A}'(S))$) and $A(\eta)\not\subset F(\mathscr{A}(S))$ (resp. $A[\eta]\not\subset F(\mathscr{A}'(S))$) for any η with $1/2<\eta<1$. We consider $f=\sum_{n=0}^{\infty}\eta_1^n P_{n,d}(\ \cdot a)$, where $\eta<\eta_1<1$. Then f belongs to $\mathscr{A}(S)$, since $\limsup_{n\to\infty}\|\eta_1^n P_{n,d}(\ \cdot a)\|_{\infty}^{1/n}=\eta_1<1$. But we have $F(f)\notin A[\eta]$, since $\limsup_{n\to\infty}\|Ff(\ ,n)\|_{B,2}^{1/n}=\eta_1>\eta$. So we get $F(\mathscr{A}(S))\not\subset A(\eta)$ and $F(\mathscr{A}'(S))\not\subset A[\eta]$.

Next consider $\{s_n\}_{n\in Z_+}\in\prod P_{n,d}(B)$ such that $s_n\in H_{n,d}$ and $\|s_n\|_{B,2}=\eta_2^n$ $(n=0,1,2,\cdots)$, where $1/2<\eta_2<\eta<1$. Then $\{s_n\}_{n\in Z_+}$ lies in $A(\eta)$ since $\limsup_{n\to\infty}\|s_n\|_{B,2}^{1/n}=\eta_2$. If there exists $f'\in\mathscr{A}'(S)$ such that $F(f')=\{s_n\}_{n\in Z_+}$ and $f'=\sum_{n=0}^{\infty}f'_n$ $(f'_n\in H_{n,d})$, then we have, from (2.27)

$$\limsup_{n\to\infty}\|f'_n\|_{S,2}^{1/n}=2\limsup_{n\to\infty}\|Ff'_n(\ ,n)\|_{B,2}^{1/n}=2\eta_2>1\ .$$

So f' does not belong to $\mathscr{A}'(S)$. It implies $\{s_n\}_{n\in Z_+}\notin F(\mathscr{A}'(S))$,

$$A(\eta)\not\subset F(\mathscr{A}(S))\quad\text{and}\quad A[\eta]\not\subset F(\mathscr{A}'(S))\quad(1/2<\eta<1)\ .$$

§ 3. The modified Sherman transformation $F_\sharp$.

If $f'=\sum_{n=0}^{\infty}f'_n\in\mathrm{Exp}'(\tilde{S})$ and $f'_n\in H_{n,d}$ is expressed in the form (1.19), then we will define

$$F_\sharp f'(b,n)=\sum_{j=0}^{n}\phi_\sharp(n,j,d)s_j(b)\ , \tag{3.1}$$

where

$$\phi_{\natural}(n, j, d)=C_d\frac{\alpha(n, j, d)}{N(n, d)}\cdot\frac{1}{\phi(n, j, d)} \tag{3.2}$$
$$=(-i)^j 2^{j+d-1}\frac{\Gamma((d+1)/2)(n-j)!\,\Gamma(j+d/2)}{(2n+d-1)\pi^{1/2}(n+d-2)!}$$
$$=\frac{(-2i)^j\Gamma(j+d/2)}{\Gamma(d/2)}\cdot\frac{(n-j)!}{n!}\cdot\frac{1}{N(n, d)}\,.$$

REMARK 3.1. The transformation $f\to F_{\natural}f(\ ,n)$ is a unique linear one-to-one mapping of $H_{n,d}$ onto $P_n(B)$, which satisfies the following formula:

$$\int_S f(s)g(s)\,ds=N(n, d)\int_B Ff(b, n)F_{\natural}g(b, n)\,db\,,$$

for any f and $g\in H_{n,d}$.

T. O. Sherman [7] introduced the transformation $f\to F_* f(b, n)$ on $\mathscr{D}(S)$. We can see that $F_{\natural}f'(b, n)$ is equal to $F_* f_n'(b, n)$.

The transformation $F_{\natural}$ of $\mathrm{Exp}'(\tilde{S})$ into $\prod P_n(B)$

$$f'\in \mathrm{Exp}'(\tilde{S})\longmapsto F_{\natural}(f')=\{F_{\natural}f'(\ ,n)\}_{n\in Z_+}$$

is called the modified Sherman transformation.

Following Proposition 3.1 shows that F and $F_{\natural}$ are dual to each other in a sence.

PROPOSITION 3.1. (Sherman [7] Theorem 3.8). *Let* $f\in\mathrm{Exp}(\tilde{S})$ (*resp.* $\mathscr{O}(\tilde{S})$, $\mathscr{A}(S)$, $\mathscr{D}(S)$, $L^2(S)$) *and* $f'\in\mathrm{Exp}'(\tilde{S})$ (*resp.* $\mathscr{O}'(\tilde{S})$, $\mathscr{A}'(S)$, $\mathscr{D}'(S)$, $L^2(S)$). *Then we have*

$$\langle f', f\rangle=\sum_{n=0}^{\infty} N(n, d)\int_B Ff(b, n)F_{\natural}f'(b, n)\,db\,, \tag{3.3}$$

$$\langle f', f\rangle=\sum_{n=0}^{\infty} N(n, d)\int_B Ff'(b, n)F_{\natural}f(b, n)\,db\,. \tag{3.4}$$

Our main theorem in this section is the following

THEOREM 3.2. *The modified Sherman transformation $F_{\natural}$ is a linear one-to-one mapping of* $\mathrm{Exp}'(\tilde{S})$ *into* $\prod P_n(B)$, *which satisfies the following properties*:

$$L\subset F_{\natural}(L^2(S))\subset L^-\,, \tag{3.5}$$

$$D\subset F_{\natural}(\mathscr{D}(S))\subset D^-\,, \tag{3.6}$$

$$D'\subset F_{\natural}(\mathscr{D}'(S))\subset D'^-\,, \tag{3.7}$$

(3.8) $$A(1)\subset F_{\sharp}(\mathscr{A}(S))\subset A(2)\ ,$$

(3.9) $$A[1]\subset F_{\sharp}(\mathscr{A}'(S))\subset A[2]\ ,$$

(3.10) $F_{\sharp}$ *is a one-to-one mapping of* $\mathscr{O}(\tilde{S})$ *onto* O ,

(3.11) $F_{\sharp}$ *is a one-to-one mapping of* $\mathscr{O}'(\tilde{S})$ *onto* O' ,

(3.12) $F_{\sharp}$ *is a one-to-one mapping of* $\mathrm{Exp}(\tilde{S})$ *onto* E ,

(3.13) $F_{\sharp}$ *is a one-to-one mapping of* $\mathrm{Exp}'(\tilde{S})$ *onto* E' .

We need the following lemma in order to prove the theorem.

LEMMA 3.3. *Consider* $f_{n,j}\in H_{n,d}$ *such that*

$$f_{n,j}(ra+(1-r^2)^{1/2}b)=(1-r^2)^{j/2}P_{n-j,2j+d}(r)s_j(b)$$

with $s_j\in H_{j,d-1}$, $j=0, 1, 2, \cdots, n$. *Then we have*

(3.14) $$\|F_{\sharp}f_{n,j}(\ ,n)\|_{B,2}^2=\begin{cases}\dfrac{1}{N(n,d)}\|f_{n,0}\|_{S,2}^2 & (j=0)\\[2mm] \dfrac{(d-1)!\,(n-j)!\,(n+d+j-2)!}{((n+d-2)!)^2(2n+d-1)}\|f_{n,j}\|_{S,2}^2 & (0\leqq j\leqq n)\\[2mm] \dfrac{(d-1)!\,(2n+d-2)!}{((n+d-2)!)^2(2n+d-1)}\|f_{n,n}\|_{S,2}^2 & (j=n)\ .\end{cases}$$

Furthermore, if $f'=\sum_{n=0}^{\infty}f'_n\in\mathrm{Exp}'(\tilde{S})$, $f'_n\in H_{n,d}$, $n=0, 1, 2, \cdots$, *then we have*

(3.15) $$\|f'_n\|_{S,2}\leqq N(n,d)^{1/2}\|F_{\sharp}f'(\ ,n)\|_{B,2}\leqq C\cdot 2^n n^{-1/4}\|f'_n\|_{S,2}\ ,$$

where C *is a constant.*

PROOF. (1.20) gives

$$\|f_{n,j}\|_{S,2}^2=C_d\alpha(n,j,d)\|s_j\|_{B,2}^2\ .$$

On the other hand, we have easily

$$\|F_{\sharp}f_{n,j}(\ ,n)\|_{B,2}=|\phi_{\sharp}(n,j,d)|\ \|s_j\|_{B,2}\ .$$

Now (3.2) and (2.29) give

$$\frac{N(n,d)|\phi_{\sharp}(n,j,d)|^2}{C_d\alpha(n,j,d)}=\frac{C_d\alpha(n,j,d)}{N(n,d)|\phi(n,j,d)|^2}=\frac{(n-j)!\,(n+d+j-2)!}{n!\,(n+d-2)!}\ .$$

Therefore we get

$$N(n, d)\|F_{\natural}f_{n,j}(\ ,n)\|_{B,2}^2=\frac{(n-j)!\,(n+d+j-2)!}{n!\,(n+d-2)!}\|f_{n,j}\|_{S,2}^2\,, \tag{3.16}$$

and (3.16) implies (3.14).

We can obtain (3.15) in the same way as in the proof of Lemma 2.3. Q.E.D.

PROOF OF THEOREM 3.2. It is easy to see that $F_{\natural}$ is a linear one-to-one mapping of $\mathrm{Exp}'(\tilde{S})$ into $\prod P_n(B)$. From Lemma 3.3 and Lemma 1.1, we can prove (3.5)–(3.13) in the same way as in the proof of Theorem 2.1. Q.E.D.

REMARK 3.2. In connection with Theorem 3.2, (3.5)–(3.9), we give examples in which $L\subsetneqq F_{\natural}(L^2(S))\subsetneqq L^-$, $D\subsetneqq F_{\natural}(\mathscr{D}(S))\subsetneqq D^-$, $D'\subsetneqq F_{\natural}(\mathscr{D}'(S))\subsetneqq D'^-$, $A(1)\subsetneqq F_{\natural}(\mathscr{A}(S))\subsetneqq A(2)$, and $A[1]\subsetneqq F_{\natural}(\mathscr{A}'(S))\subsetneqq A[2]$.

We consider $\{\rho^n\}_{n\in Z_+}\in\prod P_n(B)$, where $1<\rho<2$. Then $\{\rho^n\}_{n\in Z_+}$ belongs to $A(2)$. If there exists $f'\in\mathscr{A}'(S)$ such that $F_{\natural}(f')=\{\rho^n\}_{n\in Z_+}$ and $f'=\sum_{n=0}^{\infty}f'_n$ $(f'_n\in H_{n,d})$, then we have, from (3.14)

$$\begin{aligned}\limsup_{n\to\infty}\|f'_n\|_{S,2}^{1/n}&=\limsup_{n\to\infty}(N(n,d)^{1/2}\|F_{\natural}f'_n(\ ,n)\|_{B,2})^{1/n}\\&=\limsup_{n\to\infty}(N(n,d)^{1/2}\rho^n)^{1/n}=\rho>1\,.\end{aligned}$$

Therefore f' does not belong to $\mathscr{A}'(S)$. This implies $F_{\natural}(L^2(S))\subsetneqq L^-$, $F_{\natural}(\mathscr{D}(S))\subsetneqq D^-$, $F_{\natural}(\mathscr{D}'(S))\subsetneqq D'^-$, $F_{\natural}(\mathscr{A}(S))\subsetneqq A(2)$, and $F_{\natural}(\mathscr{A}'(S))\subsetneqq A[2]$.

Next consider

$$f_n(ra+(1-r^2)^{1/2}b_1)=(1-r^2)^{n/2}s_n(b_1)\,,$$

where $-1\leqq r\leqq 1$, $b_1\in B$, $s_n\in H_{n,d-1}$ and $\|s_n\|_{B,2}=\rho/2$ for $1<\rho<2$. Since $\limsup_{n\to\infty}\|f_n\|_{S,2}^{1/n}=\limsup_{n\to\infty}\|f_n\|_{\infty}^{1/n}=\limsup_{n\to\infty}\|s_n\|_{B}^{1/n}=\limsup_{n\to\infty}\|s_n\|_{B,2}^{1/n}=\rho/2<1$, f belongs to $\mathscr{A}(S)$. We obtain from (3.14),

$$\begin{aligned}\limsup_{n\to\infty}\|F_{\natural}f_n(\ ,n)\|_{B,2}^{1/n}&=\limsup_{n\to\infty}((2n+d-2)!/(n!\,(n+d-2)!\,N(n,d)))^{1/2n}\|f_n\|_{S,2}^{1/n}\\&=2\limsup_{n\to\infty}\|f_n\|_{S,2}^{1/n}=\rho>1\,.\end{aligned}$$

So $F_{\natural}(f)$ does not belong to $A[1]$. It implies $L\subsetneqq F_{\natural}(L^2(S))$, $D\subsetneqq F_{\natural}(\mathscr{D}(S))$, $D'\subsetneqq F_{\natural}(\mathscr{D}'(S))$, $A(1)\subsetneqq F_{\natural}(\mathscr{A}(S))$ and $A[1]\subsetneqq F_{\natural}(\mathscr{A}'(S))$.

REMARK 3.3. In connection with Theorem 3.2, (3.8) and (3.9) we give examples in which $A(\eta)\not\subset F_{\natural}(\mathscr{A}(S))$ (resp. $A[\eta]\not\subset F_{\natural}(\mathscr{A}'(S))$) and $A(\eta)\not\supset F_{\natural}(\mathscr{A}(S))$ (resp. $A[\eta]\not\supset F_{\natural}(\mathscr{A}'(S))$) for any η with $1<\eta<2$. We consider $f=\sum_{n=0}^{\infty} f_n$, such that $f_n(ra+(1-r^2)^{1/2}b_1)=(1-r^2)^{n/2}s_n(b_1)$, where $-1\leqq r\leqq 1$, $b_1\in B$, $s_n\in H_{n,d-1}$, and $\|s_n\|_{B,2}=(\eta_1/2)^n$, for $1<\eta<\eta_1<2$. Then f lies in $\mathscr{A}(S)$ since $\limsup_{n\to\infty}\|F_\sharp f_n(\ ,n)\|_{B,2}^{1/n}=2\limsup_{n\to\infty}\|f_n\|_{S,2}^{1/n}=\eta_1>\eta$. It implies $F_\sharp(f)\notin A[\eta]$ and we have $F_\sharp(\mathscr{A}(S))\not\subset A(\eta)$ and $F_\sharp(\mathscr{A}'(S))\not\subset A[\eta]$.

Next we consider $\{g_n\}_{n\in Z_+}\in\prod P_n(B)$ such that $g_n(b)\equiv\eta_2^n$, where $1<\eta_2<\eta<2$, $n\in Z_+$. Since $\limsup_{n\to\infty}\|g_n\|_{B,2}^{1/n}=\eta_2$, we see that $\{g_n\}_{n\in Z_+}$ belongs to $A(\eta)$. If there exists $f'=\sum_{n=0}^{\infty} f'_n\in\mathscr{A}'(S)$ such that $F_\sharp(f')=\{g_n\}_{n\in Z_+}$, we have, from (3.14),

$$\limsup_{n\to\infty}\|f'_n\|_{S,2}^{1/n}=\limsup_{n\to\infty}(N(n,d)^{1/2}\|g_n\|_{B,2})^{1/n}=\eta_2>1\ .$$

So f' does not belong to $\mathscr{A}'(S)$. Hence we obtain $\{g_n\}_{n\in Z_+}\notin F(\mathscr{A}'(S))$ and it means that $A[\eta]\not\subset F_\sharp(\mathscr{A}'(S))$ and $A(\eta)\not\subset F_\sharp(\mathscr{A}(S))$.

§ 4. The Fantappié indicator.

Let K be a compact set in C^{d+1}. We define the DFS space $\mathscr{O}(K)$ of germs of holomorphic functions on K as follows:

$$\mathscr{O}(K)=\liminf_{W\supset K}\mathscr{O}(W)\ ,$$

where W is an open set of C^{d+1} containing K and $\mathscr{O}(W)$ is the space of holomorphic functions on W equipped with the topology of uniform convergence on every compact subset of W. $\mathscr{O}'(K)$ denotes the dual space of $\mathscr{O}(K)$.

The *Fantappié indicator* $\tilde{T}(\xi_0,\xi)$ for $T\in\mathscr{O}'(K)$ is defined as follows:

$$\tilde{T}(\xi_0,\xi)=\left\langle T_w,\ \frac{\xi_0}{\xi_0+\xi\cdot w}\right\rangle\ ,$$

where $\xi_0\in C$ and $\xi\cdot w=\sum_{j=1}^{d+1}\xi_j w_j$ for $\xi=(\xi_1,\xi_2,\cdots,\xi_{d+1})$, $w=(w_1,w_2,\cdots,w_{d+1})\in C^{d+1}$. $\tilde{T}$ is defined on the set of (ξ_0,ξ) such that $K\cap\{w\in C^{d+1};\ \xi_0+\xi\cdot w=0\}=\varnothing$.

For Fantappié indicator, see Martineau [3] and [4].

Now, we consider the following power series;

$$\mathscr{F}f'(b,z)=\sum_{n=0}^{\infty}Ff'(b,n)z^n\ ,\quad f'\in\mathscr{A}'(S)\ \text{ and }\ z\in C\ . \tag{4.1}$$

If we fix $z\in C$ with $|z|<1$, then we have

$$\sum_{n=0}^{\infty}Ff'(b,n)z^n=\sum_{n=0}^{\infty}\langle f',\ z^n e(b,n)\rangle\ .$$

Since $\limsup_{n\to\infty}\|z^n e(b,n)\|_\infty^{1/n}=|z|<1$, $\sum_{n=0}^{\infty}z^n e(b,n)$ lies in $\mathscr{A}(S)$ and

$\sum_{n=0}^{\infty} z^n e(b, n)$ converges in $\mathscr{A}(S)$ by Lemma 1.1, (1.9). Hence we can see

$$\sum_{n=0}^{\infty} \langle f', z^n e(b, n)\rangle = \left\langle f', \sum_{n=0}^{\infty} z^n e(b, n)\right\rangle .$$

On the other hand, we have

$$\sum_{n=0}^{\infty} z^n e(b, n)(s) = \sum_{n=0}^{\infty} z^n (a\cdot s + ib\cdot s)^n = \frac{1}{1 - z(a+ib)\cdot s} .$$

So we have

$$\left\langle f'_s, \frac{1}{1-z(a+ib)\cdot s}\right\rangle = \mathscr{F}f'(b, z) \tag{4.2}$$

if $|z|<1$. The left hand side of (4.2) is the Fantappié indicator for $f'(\xi_0=1, \xi=-z(a+ib)$, and $K=S)$. Therefore the Sherman transformation is closely related with the Fantappié indicator. We show some properties of $\mathscr{F}f'(b, z)$ using Theorem 2.1.

THEOREM 4.1. (1) *If* $f' \in \mathscr{A}'(S)$, $\mathscr{F}f'(b, z)$ *is holomorphic in* $\{z \in \boldsymbol{C}; |z|<1\}$.

(2) *If* $f \in \mathscr{D}(S)$, $\mathscr{F}f(b, z)$ *is holomorphic in* $\{z \in \boldsymbol{C}; |z|<1\}$ *and all derivatives of* $\mathscr{F}f(b, z)$ *are continuous on* $\{z \in \boldsymbol{C}; |z|\leqq 1\}$.

(3) *If* $f \in \mathscr{A}(S)$, *there exists* $\varepsilon>0$ *such that* $\mathscr{F}f(b, z)$ *is holomorphic in* $\{z \in \boldsymbol{C}; |z|<1+\varepsilon\}$, *where* ε *depends on* f.

(4) *If* $f \in \mathscr{O}(\tilde{S})$, $\mathscr{F}f(b, z)$ *is an entire function in* z.

(5) *If* $f \in \mathrm{Exp}(\tilde{S})$, $\mathscr{F}f(b, z)$ *is an entire function of exponential type in* z.

We need the following lemmas in order to prove the theorem.

LEMMA 4.2. *If* $a_n \in \boldsymbol{C}$ $(n \in \boldsymbol{Z}_+)$ *and* $|a_n|$ *is rapidly decreasing as* $n\to\infty$, *then the power series* $g(z)=\sum_{n=0}^{\infty} a_n z^n$ *is holomorphic in* $\{z \in \boldsymbol{C}; |z|<1\}$ *and* $(d^k/dz^k)g(z)$ *is continuous on* $\{z \in \boldsymbol{C}; |z|\leqq 1\}$ *for any* $k \in \boldsymbol{Z}_+$.

LEMMA 4.3. (Boas [1]). $F(z)=\sum_{n=0}^{\infty} a_n z^n$ *is an entire function of exponential type* $\leqq M$ *if and only if* $\limsup_{n\to\infty} (n!\, |a_n|)^{1/n} \leqq M$.

PROOF OF THEOREM 4.1. We denote by ρ_b the convergence radius of $\mathscr{F}f'(b, z)=\sum_{n=0}^{\infty} Ff'(b, n)z^n$. Then we have

$$\rho_b = \frac{1}{\limsup_{n\to\infty} |Ff'(b, n)|^{1/n}} \geqq \frac{1}{\limsup_{n\to\infty} \|Ff'(\ , n)\|_B^{1/n}} .$$

Hence, $\mathscr{F}f'(b, z)$ is holomorphic in $\{z \in \boldsymbol{C}; |z|<1/(\limsup_{n\to\infty} \|Ff'(\ , n)\|_B^{1/n})\}$.

This fact and Theorem 2.1 imply (1), (3) and (4) of the theorem. We can prove (2) by Theorem 2.1 and Lemma 4.2. (5) is implied by Theorem 2.1 and Lemma 4.3. Q.E.D.

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Present Address:
DEPARTMENT OF MATHEMATICS
SOPHIA UNIVERSITY
KIOI-CHO, CHIYODA-KU, TOKYO 102