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Tangential Boundary Behavior of Green Potentials and Contractive Properties of L^p-capacities

Dedicated to Professor Yukio Kusunoki on the occasion of his 60th birthday

Hiroaki AIKAWA

Gakushuin University

Introduction

Let $H=\{x\in \boldsymbol{R}^n; x_n>0\}$, $n\geqq 2$, be the upper half space and G be the Green function for H. Wu [10] studied the tangential behavior of Green potentials $u(x)=\int_H G(x, y)\lambda(y)dy$ under a certain condition on λ. According to Wu, we shall use the following notation: Let $P=\{x; x_n=0\}$, $Q=\{x; x_n=1\}$ and $x'=(x_1, \cdots, x_{n-1})$. For $\gamma\geqq 1$, $a\in P$ and $b\in Q$, we denote by $\Gamma(\gamma, a, b)$ the arc in H joining b to a with tangency γ to the plane P so that if $x\in \Gamma(\gamma, a, b)$, then

$$x'-a'=x_n^{1/\gamma}(b'-a') .$$

For a positive number m let $R(\gamma, a, m)$ be the set $\{x; m|x'-a'|^\gamma<x_n<1\}$. If $\gamma\geqq 1$ and f is a function on H, we say that $f(x)$ has T_γ-limit l at $a\in P$ provided $f(x)$ tends to l as $x\to a$ inside $R(\gamma, a, m)$ for each $m>0$. We observe that $f(x)$ has T_1-limit l at a if and only if $f(x)$ has nontangential limit l at a. Wu [10; Theorem 1] proved

THEOREM A. *If $u\not\equiv\infty$ and*

$$\int_H \lambda(y)^p y_n^\beta dy<\infty \tag{1}$$

for some $p\geqq 1$ and β $(2p-n<\beta\leqq 2p-1)$, then corresponding to each γ $(1\leqq\gamma\leqq(n-1)/(\beta-2p+n))$, there is a set $V_\gamma\subset P$ with $(\beta-2p+n)\gamma$-dimensional Hausdorff measure zero, such that for each $a\in P\setminus V_\gamma$

(i) *in case $p>n/2$, u has T_γ-limit zero at a;*

(ii) *in case $1<p\leqq n/2$, the set $E_a=\{b\in Q; u(x)$ does not approach zero as $x\to a$ along $\Gamma(\gamma, a, b)\}$ has Hausdorff dimension at most $n-2p$;*

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(iii) *in case* $p=1$, $C_2(E_a)=0$ $(n\geqq 3)$ *or the logarithmic capacity of* E_a *is zero* $(n=2)$.

Here $C_\alpha(E)$ is defined by

$$C_\alpha(E)=\inf\left\{\mu(\boldsymbol{R}^n);\int|x-y|^{\alpha-n}d\mu(y)\geqq 1 \text{ on } E\right\}$$

with $0<\alpha<n$. Recently Mizuta [8; Theorem 9] dealt with the tangential behavior of Green potentials of order α and noted that (i), (ii) and (iii) are also valid for every $\gamma\geqq 1$ and every $a\in P$ in case $\beta\leqq 2p-n$. As to the case in (ii), however, the size of the exceptional set E_a can be improved. We shall characterize E_a by using the Bessel capacity. Our characterization is a natural extension of [7; Theorem 6]. In order to state our theorem we give the definition of the Bessel capacity, and more generally, L^p-capacity.

Let K be the totality of nonnegative nonincreasing lower semicontinuous functions on $[0,+\infty)$. We define L^p-capacities for $k\in K$ as follows:

$$C_{k,p}(E)=\inf\left\{\|f\|_p^p;\int k(|x-y|)f(y)dy\geqq 1 \text{ on } E,\ f\geqq 0\right\},\qquad \text{if}\quad p>1\,,$$

$$C_{k,1}(E)=\inf\left\{\mu(\boldsymbol{R}^n);\int k(|x-y|)d\mu(y)\geqq 1 \text{ on } E,\ \mu\geqq 0\right\}.$$

We note $C_\alpha=C_{k_\alpha,1}$ with $k_\alpha(t)=t^{\alpha-n}$.

Let Γ and K_ν stand for the Gamma function and the modified Bessel function of the third kind of order ν, respectively. Then the Bessel capacity $B_{\alpha,p}$ with index (α,p) is defined by $B_{\alpha,p}=C_{g_\alpha,p}$, where

$$g_\alpha(t)=2^{-(n+\alpha-2)/2}\pi^{-n/2}\Gamma(\alpha/2)^{-1}t^{(\alpha-n)/2}K_{(n-\alpha)/2}(t)$$

(see e.g. [4; p. 279]). It is known that $g_\alpha(t)$ rapidly decreases to zero as $t\to\infty$, and that $g_\alpha(t)$ is comparable to $t^{\alpha-n}$ (resp. $-\log t$) for $0<\alpha<n$ (resp. $\alpha=n$) as $t\to 0$ ([4; (22), (23) and (24)]). Hence

(i) in case $0<\alpha<n$, $B_{\alpha,1}(E)=0$ if and only if $C_\alpha(E)=0$;

(ii) $B_{n,1}(E)=0$ if and only if the logarithmic capacity of E is zero.

We also have from [4; Theorems 20, 21 and 22]

(iii) in case $\alpha p>n$, $B_{\alpha,p}(E)=0$ if and only if $E=\varnothing$;

(iv) in case $1\leqq p\leqq n/\alpha$, if $B_{\alpha,p}(E)=0$, then the Hausdorff dimension of E is at most $n-\alpha p$;

(v) in case $1\leqq p<n/\alpha$, if the $(n-\alpha p)$-dimensional Hausdorff measure of E is zero, then $B_{\alpha,p}(E)=0$.

It is easy to see that the converse of each of (iv) and (v) is not necessarily true.

Our improvement of Theorem A is

THEOREM 1. *Let* $1 \leqq p \leqq n/2$ *and let* u, β, λ *and* γ *be as in Theorem A. Then there is a set* $V_\gamma \subset P$ *with* $(\beta-2p+n)\gamma$*-dimensional Hausdorff measure zero such that for each* $a \in P \setminus V_\gamma$, $B_{2,p}(E_a)=0$.

We observe that if $p=1$, then Theorem 1 is nothing but Theorem A (iii). In case $\beta \leqq 2p-n$, we also obtain from the proof of Theorem 1.

COROLLARY 1 (cf. [8; Theorem 9 (ii)]). *Let* $u \not\equiv \infty$ *and* λ *satisfy* (1) *with* $\beta \leqq 2p-n$. *For every* $\gamma \geqq 1$ *and every* $a \in P$

(i) *in case* $p > n/2$, u *has* T_γ*-limit zero at* a;

(ii) *in case* $1 \leqq p \leqq n/2$, $B_{2,p}(E_a)=0$.

The proof of Theorem 1 is closely related to contractive properties of L^p-capacities. Let T: $\boldsymbol{R}^n \to \boldsymbol{R}^n$ be a contraction mapping, i.e., $|Tx-Ty| \leqq |x-y|$ for all x and $y \in \boldsymbol{R}^n$. First we ask if the inequality $C_{k,p}(TE) \leqq C_{k,p}(E)$ hold.

It is not so difficult to prove that $C_{k,1}(TE) \leqq C_{k,1}(E)$, and it seems that Wu implicitly used this inequality in the proof of [10; Proposition 2]. Professor B. Fuglede kindly pointed out that if E is compact, $p=2$ and $k(|x|)=\int h(|x-y|)h(|y|)dy$ for some $h \in K$, then one can derive $C_{k,2}(TE) \leqq C_{k,2}(E)$ from $C_{k,2}(E)^2 = e_h(E)$ and $e_h(TE) \leqq e_h(E)$, where $e_h(E)=\max\left\{\mu(E); \int h(|x-y|)d\mu(x)d\mu(y) \leqq 1, \text{ supp } \mu \subset E\right\}$. The inequality $e_h(TE) \leqq e_h(E)$ can be proved in a way similar to that of Landkof [2; Theorem 2.9] by the aid of the selection theorem found in [9; Theorem 5.1].

However whether $C_{k,p}(TE) \leqq C_{k,p}(E)$ holds or not for general $k \in K$, $p>1$ and T seems to be unknown. In this note we shall deal with the case when $p>1$ and k is a general function in K, but T is of the form $Tx=(T_1x_1, \cdots, T_nx_n)$, where T_i $(1 \leqq i \leqq n)$ is a contraction mapping from $\boldsymbol{R}$ to $\boldsymbol{R}$. We shall prove

THEOREM 2. *Let* p, k *and* T *be as above. Then*

$$C_{k,p}(TE) \leqq C_{k,p}(E) \qquad \text{for all} \quad E \subset \boldsymbol{R}^n . \tag{2}$$

As a simple corollary to Theorem 2, we have

COROLLARY 2. *Let* p *and* k *be as above. If* T *is an affine contraction mapping from* $\boldsymbol{R}^n$ *to* $\boldsymbol{R}^n$, *Then* (2) *holds.*

The work of Meyers [3] was brought to our attention by Professor

D. R. Adams. In that paper it was proved that (2) holds in case T is an orthogonal projection, which was also obtained in [5; Lemma 1]. Professor N. G. Meyers informed that about ten years ago he proved that (2) holds if T is a special mapping such as a linear contraction mapping or a certain nonlinear mapping. However his work has never been published.

§1. Proof of Theorem 2.

Throughout this section we fix a contraction mapping $T_1: \boldsymbol{R} \to \boldsymbol{R}$. The letters I and J will stand for an (open) interval and a (finite) union of intervals, respectively. We write $I_0 < I_1$ if the left end of I_1 is not smaller than the right end of I_0. A union J of intervals can be always written as $J = I_0 \cup \cdots \cup I_m$, where $I_{i-1} < I_i$. The length of J is denoted by $|J|$. We call J^* a gathered union of intervals of J (with respect to T_1) if $|J^*| = |J|$ and

$$\int_J k(|x-y|)dy \leqq \int_{J^*} k(|T_1 x - y|)dy \tag{3}$$

for all $k \in K$ and $x \in \boldsymbol{R}$. If J^* is an interval, then we call it a gathered interval of J. We shall prove later that every union of intervals has a gathered union of intervals (see Lemma 6).

We write $B(x, r) = \{y; |y-x| < r\}$. Let $w_0(x; I) = \min\{x-a, b-x\}$ for an interval $I = (a, b)$. If J is a union of intervals, then we put $w(x; J) = \min w_0(x; I)$, where the minimum is taken over all intervals I satisfying $|I| = |J|$ and

$$\int_J k(|x-y|)dy \leqq \int_I k(|x-y|)dy \qquad \text{for all} \quad k \in K\,,$$

or equivalently,

$$|J \cap B(x, r)| \leqq |I \cap B(x, r)| \qquad \text{for all} \quad r > 0\,. \tag{4}$$

Since $|J \cap B(x, r)| \leqq 2r$, $I = (x - |J|/2, x + |J|/2)$ satisfies (4), and hence

$$w(x; J) \leqq |J|/2\,. \tag{5}$$

It is convenient to give a different characterization of $w(x; J)$.

LEMMA 1. *Let* $r_0(x) = \max\{r; |J \cap B(x, r)| = 0\}$ *for* $x \notin J$ *and* $r_0(x) = 0$ *for* $x \in J$. *Then*

$$w(x; J) = \max\{|J \cap B(x, r)| - r;\ r \geqq r_0(x)\}\,. \tag{6}$$

PROOF. Let $\psi(c, r)=|(c-|J|, c)\cap B(0, r)|$ for $c\leqq|J|/2$ and $r>0$. We observe that $\psi(c, r)$ is a nondecreasing function of $c\in(-\infty, |J|/2]$ for every fixed $r>0$, and that if $|I|=|J|$, then

$$|I\cap B(x, r)|=\psi(w_0(x; I), r) . \tag{7}$$

Hence by (4) and (5)

$$w(x; J)=\min\{c; \psi(c, r)\geqq|J\cap B(x, r)| \text{ for all } r>0\} . \tag{8}$$

On the other hand, we obtain that if $0\leqq c\leqq|J|/2$, then

$$\psi(c, r)=\begin{cases} 2r & \text{for } 0<r\leqq c \\ r+c & \text{for } c\leqq r\leqq|J|-c \\ |J| & \text{for } r\geqq|J|-c , \end{cases} \tag{9}$$

and if $c<0$, then

$$\psi(c, r)=\begin{cases} 0 & \text{for } 0<r\leqq -c \\ r+c & \text{for } -c\leqq r\leqq|J|-c \\ |J| & \text{for } r\geqq|J|-c . \end{cases} \tag{10}$$

We infer from (8), (9) and (10) that

$$w(x; J)=\min\{c; r+c\geqq|J\cap B(x, r)| \text{ for all } r\geqq r_0(x)\} ,$$

which leads to (6).

As a simple corollary to Lemma 1, we have

$$w(x; I)=w_0(x; I) \qquad \text{if } I \text{ is an interval} . \tag{11}$$

By using $w(x; J)$, we have a necessary and sufficient condition for J to have a gathered interval.

LEMMA 2. *Let I^* and J be an interval and a union of intervals such that $|I^*|=|J|$. Then I^* is a gathered interval of J if and only if*

$$w(x; J)\leqq w(T_1x; I^*) \tag{12}$$

for all $x\in \boldsymbol{R}$.

PROOF. We note that I^* is a gathered interval of J if and only if

$$|J\cap B(x, r)|\leqq|I^*\cap B(T_1x, r)| \tag{13}$$

for all $r>0$ and all $x\in \boldsymbol{R}$. If (13) holds, then it follows from (7) and (11) that

$$|J\cap B(x, r)|\leqq\psi(w(T_1x; I^*), r)$$

for all $r>0$ and all $x\in \boldsymbol{R}$, so that (8) leads to (12).

On the other hand, suppose that (12) holds for all $x\in \boldsymbol{R}$. Since $\psi(c, r)$ is a nondecreasing function of c on $(-\infty, |J|/2]$ for every r, we have

$$\psi(w(x; J), r)\leqq\psi(w(T_1x; I^*), r) \qquad \text{for all} \quad r>0 .$$

It follows from (7), (8) and (11) that (13) holds for all $r>0$ and $x\in \boldsymbol{R}$.

We shall prove later that (12) holds for all $x\in \boldsymbol{R}$ if (12) holds only for finitely many x's determined by J (see Lemma 4). For this purpose we evaluate $w(x; J)$ by writing $J=I_0\cup\cdots\cup I_m$ with $I_{i-1}<I_i$.

LEMMA 3. *For* $0\leqq i\leqq j\leqq m$ *let* $I_i=(c_i-p_i, c_i+p_i)$, $A_i=[c_{i-1}+p_{i-1}, c_i-p_i]$, $c(i, j)=2^{-1}(c_i-p_i+c_j+p_j)$, $J(i, j)=\cup_{l=i}^{j} I_l$, *and* $A(i, j)=\cup_{l=i+1}^{j} A_l$ *if* $i<j$, $A(i, j)=\varnothing$ *if* $i=j$. *Then* $w(x; J)$ *is equal to*

$$\max_{0\leqq i\leqq j\leqq m}\{2^{-1}(|J(i, j)|-|A(i, j)|)-|x-c(i, j)|\} . \tag{14}$$

PROOF. We note that if $i\leqq j$, then

$$c_j+p_j=c_i-p_i+|J(i, j)|+|A(i, j)| . \tag{15}$$

Hence

$$2^{-1}(|J(i, j)|-|A(i, j)|)-|x-c(i, j)|=|J(i, j)|-r(x, i, j) \tag{16}$$

with $r(x, i, j)=\max\{x-c_i+p_i, c_j+p_j-x\}$. We note that $r(x, i, j)$ is the minimum of the set of r such that $B(x, r)\supset J(i, j)$, so that $r(x, i, j)\geqq r_0(x)$ and

$$|J\cap B(x, r(x, i, j))|-r(x, i, j)\geqq|J(i, j)|-r(x, i, j) . \tag{17}$$

In view of (16), (17) and Lemma 1, we obtain that $w(x; J)$ is not smaller than (14).

On the other hand we observe that $-r_0(x)$ equals 0 if $x\in J$,

$$\begin{aligned}
&2^{-1}|J(0, 0)|-|x-c(0, 0)| && \text{if} \quad x\leqq c_0-p_0 ,\\
&2^{-1}|J(m, m)|-|x-c(m, m)| && \text{if} \quad x\geqq c_m+p_m ,\\
&\max\{2^{-1}|J(i, i)|-|x-c(i, i)|,\ 2^{-1}|J(i+1, i+1)|-|x-c(i+1, i+1)|\} \\
& && \text{if} \quad x\in A(i, i+1) \quad \text{and} \quad 0\leqq i\leqq m-1 .
\end{aligned}$$

Hence it suffices to prove that $w(x; J)$ is not greater than (14) under the additional assumption that $w(x; J)>-r_0(x)$. Suppose that $|J\cap B(x, r)|-r$

attains the maximum at $r=r_1$. We may assume that there are two integers i and j such that $0\leqq i\leqq j\leqq m$ and

$$x-r_1=c_i-p_i\ , \qquad x+r_1\in \bar{I}_j$$

or

$$x-r_1\in \bar{I}_i\ , \qquad x+r_1=c_j+p_j\ .$$

Hence we have $r_1\geqq r_0(x)$ and

$$|J\cap B(x, r_1)|-r_1=|J(i, j)|-r(x, i, j)\ .$$

From Lemma 1 and (16) we obtain that $w(x; J)$ is not greater than (14). The lemma follows.

LEMMA 4. *Let J be as above. If I^* is an interval such that $|I^*|=|J|$, then the following statements are equivalent*:

(i) *I^* is a gathered interval of J.*

(ii) *For all $x\in \boldsymbol{R}$,* (12) *holds.*

(iii) *For $x=c(i, j)$, $0\leqq i\leqq j\leqq m$,* (12) *holds.*

(iv) *For i, j, $0\leqq i\leqq j\leqq m$,*

$$2^{-1}(|J(i, j)|-|A(i, j)|)\leqq w(T_1c(i, j); I^*)\ . \tag{18}$$

PROOF. We have proved in Lemma 2 that (i) and (ii) are equivalent. It is clear that (iii) follows from (ii). We observe from Lemma 3 that (iii) yields (iv). In order to complete the proof, we suppose (iv) and show (ii).

We infer from Lemma 3 that if $w(x; J)$ attains the local maximum at $x=x_0$, then $x_0=c(i, j)$ and $w(x_0; J)=2^{-1}(|J(i, j)|-|A(i, j)|)$ for some i and j. Let $x_0, \cdots, x_l$ $(x_0<\cdots<x_l)$ be the points at which $w(x; J)$ attains the local maxima. On account of (18), we obtain $w(x_j; J)\leqq w(T_1x_j; I^*)$ for $j=0, \cdots, l$.

We shall prove $w(x; J)\leqq w(T_1x; I^*)$ for $x_{j-1}\leqq x\leqq x_j$ and $1\leqq j\leqq l$. Note $|w(T_1x; I^*)-w(T_1x_j; I^*)|\leqq|T_1x-T_1x_j|$. Since T_1 is a contraction mapping, it follows that

$$w(T_1x; I^*)\geqq w(T_1x_j; I^*)-|T_1x-T_1x_j|\geqq w(T_1x_j; I^*)-|x-x_j|\geqq w(x_j; J)-|x-x_j|\ .$$

In the same way as above

$$w(T_1x; I^*)\geqq w(x_{j-1}; J)-|x-x_{j-1}|\ .$$

However Lemma 3 leads to

$$w(x; J)=\max\{w(x_{j-1}; J)-|x-x_{j-1}|,\ w(x_j; J)-|x-x_j|\}$$

for $x_{j-1} \leqq x \leqq x_j$. Hence $w(x; J) \leqq w(T_1 x; I^*)$.

By using

$$w(x; J) = w(x_0; J) - |x - x_0| \qquad \text{for} \quad x < x_0 ,$$
$$w(x; J) = w(x_l; J) - |x - x_l| \qquad \text{for} \quad x > x_l ,$$

we can prove $w(x; J) \leqq w(T_1 x; I^*)$ for $x < x_0$ or $x > x_l$. Thus the lemma follows.

Suppose that $I^* = (a, b)$ satisfies (18). Since $w(x; I^*) = \min\{x - a, b - x\}$, it follows that $2^{-1}(|J(i, j)| - |A(i, j)|) \leqq \min\{T_1 c(i, j) - a, b - T_1 c(i, j)\}$, so that

$$(19) \qquad \begin{aligned} a &\leqq T_1 c(i, j) - 2^{-1}(|J(i, j)| - |A(i, j)|) , \\ b &\geqq T_1 c(i, j) + 2^{-1}(|J(i, j)| - |A(i, j)|) . \end{aligned}$$

Let $d(J) = \min\{|I^*|;\ I^* \text{ is an interval satisfying (18) for all } i, j, 0 \leqq i \leqq j \leqq m\}$. From the above observation, we have

$$d(J) = \max\{d'(i, j, i', j');\ 0 \leqq i \leqq j \leqq m,\ 0 \leqq i' \leqq j' \leqq m\} ,$$

where $d'(i, j, i', j') = T_1 c(i, j) - T_1 c(i', j') + 2^{-1}(|J(i, j)| - |A(i, j)| + |J(i', j')| - |A(i', j')|)$. Changing the roles of $\{i, j\}$ and $\{i', j'\}$, we obtain that

$$d(J) = \max\{d(i, j, i', j');\ 0 \leqq i \leqq j \leqq m,\ 0 \leqq i' \leqq j' \leqq m\} ,$$

where $d(i, j, i', j') = |T_1 c(i, j) - T_1 c(i', j')| + 2^{-1}(|J(i, j)| - |A(i, j)| + |J(i', j')| - |A(i', j')|)$. It follows from Lemma 4 that J has a gathered interval if and only if $d(J) \leqq |J|$, or equivalently,

$$(20) \qquad d(i, j, i', j') \leqq |J|$$

for all $i, j, i', j', 0 \leqq i \leqq j \leqq m, 0 \leqq i' \leqq j' \leqq m$.

For a subset $S \subset \{0, \cdots, m\}$ we write $J(S) = \cup_{j \in S} I_j$. By a partition of $\{0, \cdots, m\}$ we mean a mutually disjoint family $\{S_1, \cdots, S_l\}$ such that $\{0, \cdots, m\} = S_1 \cup \cdots \cup S_l$. The number l is called the length of the partition $\{S_1, \cdots, S_l\}$.

LEMMA 5. *Let $\{S_1, S_2\}$ be a partition of $\{0, \cdots, m\}$. Suppose that $J(S_1)$ and $J(S_2)$ have gathered intervals I'^* and I''^*. If $I'^* \cap I''^* \neq \varnothing$, then J has a gathered interval.*

PROOF. It is sufficient to prove (20) for all $\{i, j, i', j'\} \subset \{0, \cdots, m\}$ satisfying $i \leqq j$ and $i' \leqq j'$. First suppose that $i, j \in S_1$ and $i', j' \in S_2$. Since $I'^* \cap I''^* \neq \varnothing$, we infer from (19) that

$$d(i, j, i', j') \leqq \max\{b', b''\} - \min\{a', a''\} < |J| ,$$

where $I'^* = (a', b')$ and $I''^* = (a'', b'')$.

Secondly suppose that $\{i, j, i', j'\} \subset S_1$. Since $J(S_1)$ has a gathered interval, $d(i, j, i', j') \leqq |J(S_1)| < |J|$.

Thirdly suppose that $\{i, j, i'\} \subset S_1$ and $j' \in S_2$. Since $c(i', j') = c(i', j'-1) + 2^{-1}(|I_{j'}| + |A_{j'}|)$, we have

$$|T_1c(i', j') - T_1c(i, j)| \leqq |T_1c(i', j'-1) - T_1c(i, j)| + 2^{-1}(|I_{j'}| + |A_{j'}|) .$$

Hence

$$\begin{aligned} d(i, j, i', j') &\leqq |T_1c(i', j'-1) - T_1c(i, j)| + 2^{-1}(|I_{j'}| + |A_{j'}|) \\ &\quad + 2^{-1}(|J(i, j)| - |A(i, j)| + |J(i', j')| - |A(i', j')|) \\ &= d(i, j, i', j'-1) + |I_{j'}| . \end{aligned}$$

Repeating this, we obtain

$$d(i, j, i', j') \leqq d(i, j, i', l) + |I_{j'}| + \cdots + |I_{l+1}| ,$$

where $l = \max\{\nu \in S_1; \nu < j'\}$. Since $\{i, j, i', l\} \subset S_1$ and $\{l+1, \cdots, j'\} \subset S_2$, we have

$$d(i, j, i', j') \leqq |J(S_1)| + |J(S_2)| = |J| .$$

Similarly we have (20) in case $\{i, j, j'\} \subset S_1$ and $i' \in S_2$. Changing the roles of S_1 and S_2, we can prove (20) in every case. Thus the lemma follows.

LEMMA 6. *Every union J of intervals has a gathered union of intervals.*

PROOF. Let $J = I_0 \cup \cdots \cup I_m$ with $I_{i-1} < I_i$. On account of Lemma 2, we obtain that $I_i^* = (T_1c_i - p_i, T_1c_i + p_i)$ is a gathered interval of I_i for each i. Let $\{S_1, \cdots, S_l\}$ be a partition of $\{0, \cdots, m\}$ such that

(21) every $J(S_i)$ has a gathered interval $I^*(S_i)$.

Obviously $\{\{0\}, \cdots, \{m\}\}$ satisfies (21). We assume that $\{S_1, \cdots, S_l\}$ is a partition having the minimum length among partitions satisfying (21). We claim that $\{I^*(S_1), \cdots, I^*(S_l)\}$ is mutually disjoint. Suppose that $I^*(S_1) \cap I^*(S_2) \neq \emptyset$. Applying Lemma 5 to $J(S_1 \cup S_2)$, we observe that $J(S_1 \cup S_2)$ has a gathered interval. Hence $\{S_1 \cup S_2, S_3, \cdots, S_l\}$ is a partition satisfying (21). This is a contradiction. We see that $J^* = I^*(S_1) \cup \cdots \cup I^*(S_l)$ is a gathered union of intervals of J. In fact for any $k \in K$ and any $x \in \boldsymbol{R}$,

$$\int_J k(|x-y|)dy=\sum_{i=1}^{l}\int_{J(S_i)} k(|x-y|)dy$$
$$\leqq\sum_{i=1}^{l}\int_{I^*(S_i)} k(|T_1x-y|)dy=\int_{J^*} k(|T_1x-y|)dy\ .$$

Thus the lemma follows.

By a step function we mean a linear combination of characteristic functions of open rectangles in $\boldsymbol{R}^n$.

LEMMA 7. *If f is a nonnegative step function on $\boldsymbol{R}$, then there exists a nonnegative step function f^* on $\boldsymbol{R}$ such that $\|f^*\|_p\leqq\|f\|_p$ for all $p\geqq 1$ and*

$$\int k(|x-y|)f(y)dy\leqq\int k(|T_1x-y|)f^*(y)dy \tag{22}$$

for all $k\in K$ and all $x\in\boldsymbol{R}$.

PROOF. Let f be a nonnegative step function on $\boldsymbol{R}$. We can write $f=\sum_{i=1}^{m}\alpha_i\chi_{J_i}$ such that $J_1\supset\cdots\supset J_m$ and $\alpha_i>0$. By the aid of Lemma 6, every J_i has a gathered union J_i^* of intervals. Letting $f^*=\sum_{i=1}^{m}\alpha_i\chi_{J_i^*}$, we observe that f^* satisfies (22).

Now we prove $\|f^*\|_p\leqq\|f\|_p$ by induction on m. The case $m=1$ is obvious. We assume that

$$\left\|\sum_{i=1}^{m-1}\alpha_i\chi_{J_i^*}\right\|_p\leqq\left\|\sum_{i=1}^{m-1}\alpha_i\chi_{J_i}\right\|_p\ .$$

Noting that if $p\geqq 1$ and $v\geqq 0$, then $(u+v)^p-u^p$ is a nondecreasing function of $u\geqq 0$, we have

$$\|f^*\|_p^p=\int_{J_m^*}\left(\alpha_m+\sum_{i=1}^{m-1}\alpha_i\chi_{J_i^*}\right)^p dx+\int_{R\setminus J_m^*}\left(\sum_{i=1}^{m-1}\alpha_i\chi_{J_i^*}\right)^p dx$$
$$=\int_{J_m^*}\left\{\left(\alpha_m+\sum_{i=1}^{m-1}\alpha_i\chi_{J_i^*}\right)^p-\left(\sum_{i=1}^{m-1}\alpha_i\chi_{J_i^*}\right)^p\right\}dx+\left\|\sum_{i=1}^{m-1}\alpha_i\chi_{J_i^*}\right\|_p^p$$
$$\leqq\{(\alpha_1+\cdots+\alpha_m)^p-(\alpha_1+\cdots+\alpha_{m-1})^p\}|J_m|+\left\|\sum_{i=1}^{m-1}\alpha_i\chi_{J_i^*}\right\|_p^p\ .$$

On the other hand,

$$\|f\|_p^p=\{(\alpha_1+\cdots+\alpha_m)^p-(\alpha_1+\cdots+\alpha_{m-1})^p\}|J_m|+\left\|\sum_{i=1}^{m-1}\alpha_i\chi_{J_i}\right\|_p^p\ ,$$

so that $\|f^*\|_p\leqq\|f\|_p$.

PROOF OF THEOREM 2. We may assume that every T_i $(2\leqq i\leqq n)$ is the identity, because the general case can be proved by iteration. First suppose that E is a compact set. In the same way as in [4; Lemma 2], we can prove

$$C_{k,p}(E)=\inf\left\{\|f\|_p^p;\ \int k(|x-y|)f(y)dy\geqq 1 \text{ on } E,\ f \text{ is a nonnegative step function}\right\}$$

by using [1; Lemma 2.2.1]. Given $\varepsilon>0$, we take a nonnegative step function on $\boldsymbol{R}^n$ such that $\int k(|x-y|)f(y)dy\geqq 1$ on E and $\|f\|_p^p<C_{k,p}(E)+\varepsilon$. We write $x=(x_1, x'')$, $f_{x''}(\cdot)=f((\cdot, x''))$ and $k_{x''}(t)=k((t^2+|x''|^2)^{1/2})$. Obviously $k_{x''}\in K$ for every $x''\in\boldsymbol{R}^{n-1}$. By the aid of Lemma 7 we can find nonnegative step functions $f^*_{y''}$ on $\boldsymbol{R}$ such that $\|f^*_{y''}\|_p^p\leqq\|f_{y''}\|_p^p$, $f^*((y_1, y''))=f^*_{y''}(y_1)$ is a step function on $\boldsymbol{R}^n$, and

$$\int k_{x''-y''}(|x_1-y_1|)f_{y''}(y_1)dy_1\leqq\int k_{x''-y''}(|T_1x_1-y_1|)f^*_{y''}(y_1)dy_1$$

for all $x_1\in\boldsymbol{R}$ and all x'', $y''\in\boldsymbol{R}^{n-1}$. Integrating the above quantities with respect to dy'', we have $\|f^*\|_p^p\leqq\|f\|_p^p$ and

$$\int k(|x-y|)f(y)dy\leqq\int k(|Tx-y|)f^*(y)dy$$

on $\boldsymbol{R}^n$; in particular, $\int k(|Tx-y|)f^*(y)dy\geqq 1$ on TE. Hence

$$C_{k,p}(TE)\leqq\|f^*\|_p^p\leqq\|f\|_p^p<C_{k,p}(E)+\varepsilon\ .$$

Since $\varepsilon>0$ is arbitrary, we have $C_{k,p}(TE)\leqq C_{k,p}(E)$.

Let V be an open set. Then there is a sequence of compact sets E_j such that $E_j\uparrow V$. On account of [4; Theorem 6], we have

$$C_{k,p}(TV)=\lim_{j\to\infty}C_{k,p}(TE_j)\leqq\lim_{j\to\infty}C_{k,p}(E_j)=C_{k,p}(V)\ .$$

Since $C_{k,p}$ is an outer capacity ([4; Theorem 1]), for any set E with $C_{k,p}(E)<\infty$ and $\varepsilon>0$, there is an open set V containing E and $C_{k,p}(V)<C_{k,p}(E)+\varepsilon$. Hence

$$C_{k,p}(TE)\leqq C_{k,p}(TV)\leqq C_{k,p}(V)<C_{k,p}(E)+\varepsilon\ ,$$

so that $C_{k,p}(TE)\leqq C_{k,p}(E)$. The proof is complete.

§2. Proof of Theorem 1.

We may assume that $1<p\leqq n/2$ and $2p-n<\beta<2p-1$, because if $p=1$, then Theorem 1 follows from Theorem A (iii), and if $\beta=2p-1$, then $\gamma=1$ and Theorem 1 is nothing but a known radial limit theorem of Green potentials (see e.g. [7; Theorem 6]). Note $n\geqq 3$. The following estimate of Green function for H is well known:

(23) $$A^{-1}x_n y_n|x-y|^{2-n}|x-\bar{y}|^{-2}\leqq G(x,y)\leqq Ax_n y_n|x-y|^{2-n}|x-\bar{y}|^{-2},$$

where $\bar{y}=(y_1,\cdots,y_{n-1},-y_n)$ for $y=(y_1,\cdots,y_n)$ and A is a positive constant depending only on the dimension n. We shall use the following notation:

$$I_j=\{x\in H;\ 2^{-j-1}\leqq|x|<2^{-j}\},$$

$$J_j=\{x\in H;\ 2^{-j-1}\leqq x_n<2^{-j}\},$$

$I_j^*=I_{j-1}\cup I_j\cup I_{j+1}$ and $J_j^*=J_{j-1}\cup J_j\cup J_{j+1}$. Unless otherwise specified, A will denote a positive constant depending only on n, p, β and γ, possibly changing from one occurrence to the next.

We take a constant c, $0<c<1/8$, satisfying $(2m)^{-1/\gamma}+4c\leqq\{(1-4c)/m\}^{1/\gamma}$. We observe that if $x\in R(\gamma,a,2m)$, $x_n<1/2$ and $y\in B(x,4cx_n)$, then

$$\begin{aligned}|y'-a'|&<|x'-a'|+4cx_n<\{x/(2m)\}^{1/\gamma}+4cx_n\\&\leqq\{(1-4c)x_n/m\}^{1/\gamma}<(y_n/m)^{1/\gamma},\end{aligned}$$

and $y_n<x_n+4cx_n<1$. Hence

(24) if $x\in R(\gamma,a,2m)$ and $x_n<1/2$, then $B(x,4cx_n)\subset R(\gamma,a,m)$.

We decompose u into the sum of v, w, and z so that $u(x)=v(x)+w(x)+z(x)$, where

$$v(x)=\int_{B(x,cx_n)}G(x,y)\lambda(y)dy,$$

$$w(x)=\int_{R(\gamma,a,m)\setminus B(x,cx_n)}G(x,y)\lambda(y)dy,$$

$$z(x)=\int_{H\setminus R(\gamma,a,m)}G(x,y)\lambda(y)dy.$$

We shall examine v, w and z seperately. Wu [10; Proposition 3] proved

LEMMA 8. *If*

(25) $$\int_{R(\gamma,a,m)}\lambda(y)^p y_n^{2p-n}dy<\infty,$$

then $w(x)\to 0$ *as* x *approaches* a *inside* $R(\gamma, a, m)$.

She also stated that $z(x)\to 0$ as x approaches a under a certain condition on λ. Nevertheless her proof is not complete. The inequality on the 8-th line from the bottom in [10; p. 905] is not always valid. In fact, $x^j=(j^{-1}, 0, \cdots, 0, 10mj^{-\gamma})\in R(\gamma, 0, 5m)$ and $y^j=(j^{-1}, 0, \cdots, 0, mj^{-\gamma})\in H\backslash R(\gamma, 0, m)$, but $|x^j-y^j|/|y^{j\prime}|=9mj^{1-\gamma}$, which tends to 0 as $j\to\infty$ if $\gamma>1$. By using a different decomposition, Mizuta [8; Theorem 7] showed that $w+z$ has T_γ-limit zero at all points $a\in P$ apart from an exceptional set whose $\gamma(\beta+n-2p)$-dimensional Hausdorff measure zero. For reader's convenience we give

LEMMA 9. *If* $z\not\equiv\infty$ *and*

$$\lim_{t\to 0} t^{\gamma(2p-n-\beta)}\int_{B(a,t)\cap H}\lambda(y)^p y_n^\beta dy=0\ , \tag{26}$$

then $z(x)$ *has limit zero as* $x\to a$ *inside* $R(\gamma, a, 2m)$.

PROOF. We may assume $a=0$ and $\operatorname{supp}\lambda\subset B(0, 1/2)$. For $j\geqq 1$ we let

$$\varphi(j)=2^{j\gamma(\beta+n-2p)/p}\left(\int_{I_j\backslash R(\gamma,0,m)}\lambda(y)^p y_n^\beta dy\right)^{1/p}$$

We infer from (26) that

$$\lim_{j\to\infty}\varphi(j)=0\ . \tag{27}$$

Since $\beta<2p-1$, it follows that

$$\int_0^t s^{(p-\beta)/(p-1)}ds=At^{(2p-\beta-1)/(p-1)}\ , \tag{28}$$

so that

$$\begin{aligned}
&\int_{I_j\backslash R(\gamma,0,m)}\lambda(y)y_n dy \\
&\leqq\left\{\int_{I_j\backslash R(\gamma,0,m)}y_n^{(p-\beta)/(p-1)}dy\right\}^{(p-1)/p}2^{-j\gamma(\beta+n-2p)/p}\varphi(j) \\
&\leqq\left\{\int_{|y'|<2^{-j}}\int_0^{m2^{-\gamma j}}y_n^{(p-\beta)/(p-1)}dy_n dy'\right\}^{(p-1)/p}2^{-j\gamma(\beta+n-2p)/p}\varphi(j) \\
&\leqq A2^{-j\{(n-1)(p-1)/p+\gamma(2p-\beta-1)/p+\gamma(\beta+n-2p)/p\}}\varphi(j) \\
&=A2^{-j(n-1)(p-1+\gamma)/p}\varphi(j)\ .
\end{aligned} \tag{29}$$

If $x\in I_j$, $y\in I_{j-i}$ and $i\geqq 2$, then we have by (23)

$$G(x, y) \leqq A2^{n(j-i)}2^{-j}y_n .$$

Hence (29) leads to

$$\begin{aligned}\sup_{x \in I_j} \int_{I_{j-i} \setminus R(\gamma,0,m)} & G(x, y)\lambda(y)dy \\ & \leqq A2^{n(j-i)}2^{-j}2^{(i-j)(n-1)(p-1+\gamma)/p}\varphi(j-i) \\ & \leqq A2^{-i}2^{(i-j)(n-1)(\gamma-1)/p}\varphi(j-i) \leqq A2^{-i}\varphi(j-i) ,\end{aligned}$$

if $2 \leqq i \leqq j-1$. Therefore

$$\sup_{x \in I_j} \int_{\bigcup_{i=2}^{j-1} I_{j-i} \setminus R(\gamma,0,m)} G(x, y)\lambda(y)dy \leqq A \sum_{i=2}^{j-1} 2^{-i}\varphi(j-i) .$$

We infer from (27) that the last term tends to zero as $j \to \infty$. Similarly it follows from $G(x, y) \leqq A2^{(n-1)j}y_n$ for $x \in I_j$, $y \in I_{j+i}$ and $i \geqq 2$ that

$$\begin{aligned}\sup_{x \in I_j} \int_{\bigcup_{j=2}^{\infty} I_{j+i} \setminus R(\gamma,0,m)} & G(x, y)\lambda(y)dy \\ & \leqq A \sum_{i=2}^{\infty} 2^{(n-1)j}2^{-(i+j)(n-1)(p-1+\gamma)/p}\varphi(j+i) \\ & \leqq A \sum_{j=2}^{\infty} 2^{(1-n)i}\varphi(j+i) .\end{aligned}$$

The last term has limit zero as $j \to \infty$.

Now let c be the constant taken at the beginning of this section. We observe from (24) that if $x \in R(\gamma, 0, 2m)$, $x_n < 1/2$, $y \in H \setminus R(\gamma, 0, m)$ and $|y'-x'| < cx_n$, then $|x_n - y_n| \geqq 3cx_n$. Put $S_x = \{y \in I_j^* \setminus R(\gamma, 0, m);\ |y'-x'| < cx_n\}$ and $S'_x = I_j^* \setminus (R(\gamma, 0, m) \cup S_x)$ for $x \in I_j \cap R(\gamma, 0, 2m)$. It follows from (23) that if $x \in R(\gamma, 0, 2m)$ and $y \in S_x$, then $G(x, y) \leqq Ax_n^{1-n}y_n$. Hence we have by (28)

$$\begin{aligned}\sup_{x \in I_j \cap R(\gamma,0,2m)} & \int_{S_x} G(x, y)\lambda(y)dy \\ & \leqq A\{\varphi(j-1)+\varphi(j)+\varphi(j+1)\}2^{j\gamma(2p-\beta-n)/p} \\ & \quad \times \sup_{x \in I_j \cap R(\gamma,0,2m)} x_n^{1-n}\left(\int_{S_x} y_n^{(p-\beta)/(p-1)}dy\right)^{1/p'} \\ & \leqq A\{\varphi(j-1)+\varphi(j)+\varphi(j+1)\}2^{j\gamma(2p-\beta-n)/p} \\ & \quad \times \sup_{x \in I_j \cap R(\gamma,0,2m)} x_n^{1-n}\left\{\int_{|y'-x'|<cx_n} dy' \int_0^{m2^{-\gamma(j-1)}} y_n^{(p-\beta)/(p-1)}dy_n\right\}^{1/p'} \\ & = A\{\varphi(j-1)+\varphi(j)+\varphi(j+1)\} ,\end{aligned}$$

where $p' = p/(p-1)$. Using $G(x, y) \leqq Ax_ny_n|x'-y'|^{-n}$, we have

$$\sup_{x\in I_j\cap R(\gamma,0,2m)} \int_{S'_x} G(x, y)\lambda(y)dy$$
$$\leqq A\{\varphi(j-1)+\varphi(j)+\varphi(j+1)\}2^{j\gamma(2p-\beta-n)/p}$$
$$\times \sup_{x\in I_j\cap R(\gamma,0,2m)} x_n\left\{\int_{|y'-x'|\geqq cx_n} |x'-y'|^{-np'}dy'\int_0^{m2^{-\gamma(j-1)}} y_n^{(p-\beta)/(p-1)}dy_n\right\}^{1/p'}$$
$$\leqq A\{\varphi(j-1)+\varphi(j)+\varphi(j+1)\} .$$

Therefore the lemma follows from (27).

REMARK 1. The assumption (26) can be replaced by

$$(26') \qquad \int_{B(a,t)\cap H} \lambda(y)^p y_n^{2p-n}dy<\infty \qquad \text{for some} \quad t>0 .$$

In fact, letting $\beta=2p-n$ and

$$\varphi(j)=\left(\int_{I_j\setminus R(\gamma,0,m)} \lambda(y)^p y_n^{2p-n}dy\right)^{1/p} ,$$

we observe that (27) and (28) hold, and that the same argument as above is applicable.

We refer a proof of the following covering lemma to [2; Lemma 3.2]. Let $C(x, r)$ be the closed ball with center at x and radius r.

LEMMA 10. *Suppose that a set $E\subset \boldsymbol{R}^n$ is covered by closed balls $C(x, r(x))$ such that $x\in E$ and $\sup_{x\in E} r(x)<\infty$. Then there exists a covering $\{C(x_j, r(x_j))\}_j$ of E whose multiplicity is not larger than a certain constant N_0 depending only on the dimension n.*

Let $V=\{a\in P$; (25) does not hold for some $m>0\}$ and $V'=\{a\in P$; (26) does not hold for some $m>0\}$. On account of Lemmas 8 and 9, if $a\in P\setminus(V\cup V')$, then

$$(30) \qquad \lim_{x\to a,\, x\in R(\gamma,a,m)} \{w(x)+z(x)\}=0 \qquad \text{for} \quad m>0 .$$

Hence it is natural to ask if we can take $V\cup V'$ as V_γ in Theorem 1. We shall show that this is true. First we see

LEMMA 11. *Let V and V' be as above and let λ satisfy* (1). *Then the $(\beta-2p+n)\gamma$-dimensional Hausdorff measure of $V\cup V'$ is zero.*

PROOF. Suppose that λ satisfies (1). Wu [10; Proposition 6] proved that V has $(\beta-2p+n)\gamma$-dimensional Hausdorff measure zero. Note $V'=\bigcup_{\varepsilon>0} V'_\varepsilon$ with

$$V'_\varepsilon=\left\{a \in P;\ \limsup_{t\to 0} t^{\gamma(2p-\beta-n)} \int_{B(a,t)\cap H} \lambda(y)^p y_n^\beta dy > \varepsilon\right\} .$$

It is sufficient to prove that the $(\beta-2p+n)\gamma$-dimensional Hausdorff measure of V'_ε is zero. On account of (1), there is $\rho>0$ such that

$$\int_{\{y;0<y_n\leqq\rho\}} \lambda(y)^p y_m^\beta dy < \eta .$$

For each $a \in V'_\varepsilon$ find $r(a)$, $0<r(a)\leqq\rho$, such that

$$r(a)^{\gamma(2p-\beta-n)} \int_{C(a,r(a))\cap H} \lambda(y)^p y_n^\beta dy > \varepsilon .$$

By Lemma 10 we can choose $\{a_j\}_j \subset V'_\varepsilon$ such that

$$V'_\varepsilon \subset \bigcup_j C(a_j, r(a_j)) ,$$

$$\varepsilon^{-1} \int_{C(a_j,r(a_j))\cap H} \lambda(y)^p y_n^\beta dy > r(a_j)^{\gamma(\beta-2p+n)} ,$$

the multiplicity of $\{C(a_j, r(a_j))\}_j \leqq N_0$.

Since $0<r(a_j)\leqq\rho$,

$$\sum_j r(a_j)^{\gamma(\beta-2p+n)} \leqq N_0\varepsilon^{-1} \int_{\{y;0<y_n\leqq\rho\}} \lambda(y)^p y_n^\beta dy < N_0\varepsilon^{-1}\eta .$$

From the arbitrariness of η we obtain that V'_ε has $(\beta-2p+n)\gamma$-dimensional Hausdorff measure zero. The lemma follows.

Next we show that if $a \in P\setminus(V\cup V')$, then there is a certain exceptional set F such that

$$\lim_{x\to a, x\in R(\gamma,a,m)\setminus F} u(x)=0 \qquad \text{for} \quad m>0 .$$

To make it precise, we introduce

DEFINITION 1 (cf. [8; §1]). A set $F\subset H$ is called $(2, p)$-thin on P if

$$\sum_{j=1}^{\infty} B_{2,p}(S_j(F\cap J_j)) < \infty ,$$

where $S_j x=2^j x$. A function f on H is said to have $(2, p)$-fine T_γ-limit l at a if there is a set F $(2, p)$-thin on P for which

$$\lim_{x\to a, x\in R(\gamma,a,m)\setminus F} f(x)=l \qquad \text{for} \quad m>0 .$$

REMARK 2. Let $2p>n$. It is known that there is a positive constant κ such that $B_{2,p}(E)\geqq\kappa$ for $E\neq\varnothing$ ([4; Theorem 20]). Hence if F is $(2, p)$-thin on P, then $\bar{F}\cap P=\varnothing$; the $(2, p)$-fine T_γ-limit and T_γ-limit are equivalent.

LEMMA 12 (cf. [8; Theorem 7]). *Let V and V' be as in Lemma 11. If $a\in P\backslash V$, then v has $(2, p)$-fine T_γ-limit zero at a. Moreover if $a\in P\backslash(V\cup V')$, then u has $(2, p)$-fine T_γ-limit zero at a.*

PROOF. Let $a\in P\backslash V$ and $m>0$. Without loss of generality we may assume that $a=0$. Let c be the constant taken at the beginning of this section. We observe from (24) that if $x\in J_j\cap R(\gamma, 0, 2m)$, then

$$\text{(31)}\qquad \bigcup_{y\in C(x,2^{-1-j}c)\cap J_j} B(y, cy_n)\subset B(x, 2^{-1-j}c+2^{-j}c)$$
$$\subset B(x, 2^{1-j}c)\subset J_j^*\cap B(x, 4cx_n)\subset J_j^*\cap R(\gamma, 0, m)\ .$$

By Lemma 10 we can choose $\{x_i^{(j)}\}_{i=1}^{N(j)}\subset J_j\cap R(\gamma, 0, 2m)$ such that

$$\text{(32)}\qquad J_j\cap R(\gamma, 0, 2m)\subset \cup_{i=1}^{N(j)} C(x_i^{(j)}, 2^{-1-j}c)$$
$$\subset \cup_{i=1}^{N(j)} B(x_i^{(j)}, 2^{1-j}c)\subset J_j^*\cap R(\gamma, 0, m)\ ,$$

and the multiplicity of $\{C(x_i^{(j)}, 2^{-1-j}c)\}_{i=1}^{N(j)}$ is not greater than N_0. Hence the multiplicity of $\{B(x_i^{(j)}, 2^{1-j}c)\}_{i=1}^{N(j)}$ is not greater than a constant depending only on the dimension n.

Let $F_{m,t}=\{x\in R(\gamma, 0, 2m);\ v(x)\geqq t\}$ for $t>0$. Since $G(x, y)\leqq|x-y|^{2-n}$, we have from (31)

$$F_{m,t}\cap C(x_i^{(j)}, 2^{-1-j}c)\cap J_j$$
$$\subset\left\{x\in C(x_i^{(j)}, 2^{-1-j}c)\cap J_j;\ \int_{B(x_i^{(j)},2^{1-j}c)}|x-y|^{2-n}\lambda(y)dy\geqq t\right\}\ ,$$

so that

$$R_{2,p}(F_{m,t}\cap C(x_i^{(j)}, 2^{-1-j}c)\cap J_j;\ B(x_i^{(j)}, 2^{1-j}c))$$
$$\leqq t^{-p}\int_{B(x_i^{(j)},2^{1-j}c)}\lambda(y)^p dy\ ,$$

where $R_{2,p}(E; U)$ denotes the Riesz capacity relative to an open set U defined by $R_{2,p}(E; U)=\inf\left\{\|f\|_p^p;\ \int k_2(|x-y|)f(y)dy\geqq 1 \text{ on } E,\ f \text{ is a non-negative measurable function vanishing on } R^n\backslash U\right\}$ with $k_2(t)=t^{2-n}$ (see [7; p. 116]). Since the multiplicity of the covering $\{B(x_i^{(j)}, 2^{1-j}c)\}_{i=1}^{N(j)}$ is independent of j, it follows from (32) that

$$\sum_{i=1}^{N(j)} R_{2,p}(F_{m,t}\cap C(x_i^{(j)}, 2^{-1-j}c)\cap J_j; B(x_i^{(j)}, 2^{1-j}c))$$
$$\leqq At^{-p}\int_{J_j^*\cap R(\gamma,0,m)} \lambda(y)^p dy .$$

On account of (25), we have

$$\sum_{j=1}^{\infty} 2^{j(n-2p)} \sum_{i=1}^{N(j)} R_{2,p}(F_{m,t}\cap C(x_i^{(j)}, 2^{-1-j}c)\cap J_j; B(x_i^{(j)}, 2^{1-j}c))<\infty .$$

It is well known that $R_{2,p}(rE; rU)=r^{n-2p}R_{2,p}(E; U)$ (see [7; p. 116]), so that

$$\sum_{j=1}^{\infty}\sum_{i=1}^{N(j)} R_{2,p}(S_j(F_{m,t}\cap C(x_i^{(j)}, 2^{-1-j}c)\cap J_j); S_j(B(x_i^{(j)}, 2^{1-j}c)))<\infty .$$

Since the radius of $S_j(B(x_i^{(j)}, 2^{1-j}c))$ is $2c$, independent of j and i, there is a positive constant A independent of j and i such that $g_2(|x-y|)\geqq A|x-y|^{2-n}$ for $x\in S_j(C(x_i^{(j)}, 2^{-1-j}c))$ and $y\in S_j(B(x_i^{(j)}, 2^{1-j}c))$, where g_2 is the Bessel kernel as was defined in the introduction. From the definition of L^p-capacities it follows that

$$B_{2,p}(S_j(F_{m,t}\cap C(x_i^{(j)}, 2^{-1-j}c)\cap J_j))$$
$$\leqq AR_{2,p}(S_j(F_{m,t}\cap C(x_i^{(j)}, 2^{-1-j}c)\cap J_j); S_j(B(x_i^{(j)}, 2^{1-j}c))) ,$$

and hence

$$\sum_{j=1}^{\infty}\sum_{i=1}^{N(j)} B_{2,p}(S_j(F_{m,t}\cap C(x_i^{(j)}, 2^{-1-j}c)\cap J_j))<\infty .$$

Since $B_{2,p}$ is countably subadditive, we have

$$\sum_{j=1}^{\infty} B_{2,p}(S_j(F_{m,t}\cap J_j))<\infty ,$$

so that $F_{m,t}$ is $(2, p)$-thin on P.

For positive integers k and l we find $j(k, l)$ such that

$$\sum_{j=j(k,l)}^{\infty} B_{2,p}(S_j(F_{1/k,1/l}\cap J_j))<2^{-k-l} .$$

Let

$$F=\bigcup_{k=1}^{\infty}\bigcup_{l=1}^{\infty}\bigcup_{j=j(k,l)}^{\infty} F_{1/k,1/l}\cap J_j .$$

Since $B_{2,p}$ is countably subadditive,

$$\sum_{j=1}^{\infty} B_{2,p}(S_j(F\cap J_j))\leqq\sum_{j=1}^{\infty}\sum_{j(k,l)\leqq j} B_{2,p}(S_j(F_{1/k,1/l}\cap J_j))$$
$$\leqq\sum_{k=1}^{\infty}\sum_{l=1}^{\infty}\sum_{j=j(k,l)}^{\infty} B_{2,p}(S_j(F_{1/k,1/l}\cap J_j))$$
$$\leqq\sum_{k=1}^{\infty}\sum_{l=1}^{\infty} 2^{-k-l}<\infty ,$$

so that F is $(2, p)$-thin on P. From the construction of F it is easy to see that

$$\lim_{x\to 0,\, x\in R(\gamma,0,m)\setminus F} v(x)=0 \qquad \text{for} \quad m>0 .$$

Hence v has $(2, p)$-fine T_γ-limit zero at 0. The second assertion immediately follows from Lemmas 8 and 9. The proof is complete.

REMARK 3. Let $2p>n$. It follows from Remark 2 that if $a\in P\setminus(V\cup V')$, then u has T_γ-limit zero.

We consider a relation between $(2, p)$-fine T_γ-limits and limits along curves $\Gamma(\gamma, a, b)$. We have

LEMMA 13. *Let $1\leqq p\leqq n/2$. If f has $(2, p)$-fine T_γ-limit l at a, then $\{b\in Q;\ f(x)$ does not approach l as $x\to a$ along $\Gamma(\gamma, a, b)\}$ has $B_{2,p}$-capacity zero.*

PROOF. We may assume $a=0$. Let F be $(2, p)$-thin on P and

$$\lim_{x\to 0,\, x\in R(\gamma,0,m)\setminus F} f(x)=l \qquad \text{for} \quad m>0 .$$

It is sufficient to show

$$E'=\{b\in Q;\ \Gamma(\gamma, 0, b)\subset R(\gamma, 0, 2m),\ f(x) \text{ does not approach } l \text{ as } x\to a \text{ along } \Gamma(\gamma, 0, b)\}$$

has $B_{2,p}$-capacity zero for each $m>0$.

Let $T_jx=(2^{-j(1-1/\gamma)}x', x_n)$, $Ux=(x_n^{-1/\gamma}x', x_n)$ and $\pi x=(x', 1)$. Since $x\in \Gamma(\gamma, 0, b)$ if and only if $x'=x_n^{1/\gamma}b'$, we have

$$F\cap\Gamma(\gamma, 0, b)\cap J_j\neq\emptyset\Leftrightarrow\pi UT_jS_j(F\cap\Gamma(\gamma, 0, b)\cap J_j)=\{b\} .$$

Hence $E'\subset\limsup_{j\to\infty}\pi UT_jS_j(F\cap J_j)$. We observe that π and T_j are affine contraction mappings and U is a bi-Lipschitz mapping on $R(\gamma, 0, m)\cap J_0$. Hence Corollary 2 and [6; Lemma 3] yield

$$B_{2,p}(\pi UT_jS_j(F\cap J_j))\leqq AB_{2,p}(S_j(F\cap J_j)) ,$$

so that

$$\sum_{j=1}^{\infty} B_{2,p}(\pi UT_jS_j(F\cap J_j))<\infty ,$$

which implies that

$$B_{2,p}(\limsup_{j\to\infty} \pi UT_jS_j(F\cap J_j))=B_{2,p}(E')=0 .$$

The proof is complete.

PROOF OF THEOREM 1 AND COROLLARY 1. Combining Lemma 11, 12 and 13, we obtain the theorem. Suppose that λ satisfies (1) with $\beta\leqq 2p-n$. Then

$$\int_{B(a,t)\cap H} \lambda(y)^p y_n^{2p-n} dy<\infty \qquad \text{for} \quad a\in P,\ 0<t<1 ,$$

and in particular (25) for all $\gamma\geqq 1$ holds at any point a on P. Hence by Lemma 8 and Remark 1 we have (30) for any $a\in P$ and $\gamma\geqq 1$. This with Lemmas 12, 13 and Remark 3 yields the corollary.

§3. Remarks.

We shall show that the size of each of V_γ and E_a in Theorem 1 is best possible. We give

PROPOSITION 1. *Let* $p\geqq 1$, $2p-n<\beta\leqq 2p-1$ *and* $1\leqq\gamma\leqq(n-1)/(\beta-2p+n)$. *Suppose that* $V\subset P$ *and* $(\beta-2p+n)\gamma$*-dimensional Hausdorff measure of* V *is zero. Then there exists a nonnegative measurable function* λ *on* H *satisfying* (1) *such that* $G(x, \lambda)$ *fails to have* $(2, p)$*-fine* T_γ*-limit zero at any* $\xi\in V$.

In order to prove the proposition, we give a necessary condition for a set to be $(2, p)$-thin on P. Let $\delta(x)=\mathrm{dist}(x, P)$.

LEMMA 14. *Let* $F\subset H$. *If there are* l, $0<l<1/4$, *and* $\{x^i\}_i\subset F$ *such that* $B(x^i, l\delta(x^i))\subset F$ *and* $\liminf_{i\to\infty}\delta(x^i)=0$, *then* F *is not* $(2, p)$*-thin on* P.

PROOF. Let $j(i)$ be the integer such that $2^{-j(i)-1}\leqq\delta(x^i)<2^{-j(i)}$. Taking a subsequence of $\{x^i\}$, if necessary, we may assume that $\lim_{i\to\infty} j(i)=\infty$. Since $0<l<1/4$ and $B(x^i, l\delta(x^i))\subset F$, it follows that $F\cap J_{j(i)}$ includes a ball with radius $l2^{-j(i)-2}$, so that $S_{j(i)}(F\cap J_{j(i)})$ includes a ball with radius $l/4$. Hence $B_{2,p}(S_{j(i)}(F\cap J_{j(i)}))\geqq A$ with A independent of i, so that F is not $(2, p)$-thin on P.

PROOF OF PROPOSITION 1. Let $m>0$. We shall find a nonnegative measurable function λ satisfying (1) such that $G(\cdot, \lambda)\not\equiv\infty$ and $\{x\in R(\gamma, \xi, m);\ G(x, \lambda)\geqq 1\}$ is not $(2, p)$-thin on P for every $\xi\in V$. Obviously $G(\cdot, \lambda)$ fails to have $(2, p)$-fine T_γ-limit zero at $\xi\in V$.

Take a constant $k>m$. We find a constant l, $0<l<1/4$, such that

$$m(kl+1)^\gamma\leqq(1-l)k\ . \tag{33}$$

Let $i\geqq 2$ and $ki^{-\gamma}\leqq 1$. Since the $(\beta-2p+n)\gamma$-dimensional Hausdorff measure of V is zero, there are $\xi^{ij}=(\xi_1^{ij}, \cdots, \xi_{n-1}^{ij}, 0)\in P$ and r_{ij}, $0<r_{ij}<1/i$, such that

$$V\subset\bigcup_j B(\xi^{ij}, r_{ij})\ ,\qquad \sum_j r_{ij}^{(\beta-2p+n)\gamma}<2^{-i}\ . \tag{34}$$

Put $x^{ij}=(\xi_1^{ij}, \cdots, \xi_{n-1}^{ij}, kr_{ij}^\gamma)$, $\lambda_{ij}(x)=\delta(x^{ij})^{-2}$ on $B(x^{ij}, 2l\delta(x^{ij}))$ and $\lambda_{ij}(x)=0$ elsewhere. We obtain from (23) that if $x\in B(x_i^j, l\delta(x^{ij}))$ and $y\in B(x^{ij}, 2l\delta(x^{ij}))$, then $G(x, y)\geqq A\delta(x^{ij})^{2-n}$, so that

$$G(x, \lambda_{ij})\geqq A\delta(x^{ij})^{2-n}\,\|\lambda_{ij}\|_1=A\quad\text{on}\quad B(x^{ij}, l\delta(x^{ij}))\ ,$$

where A is independent of i and j. Hence we can find a constant A_0 such that $\lambda_i=A_0\sup_j\lambda_{ij}$ satisfies

$$G(x, \lambda_i)\geqq 1\quad\text{on}\quad\bigcup_j B(x^{ij}, l\delta(x^{ij}))\ .$$

Since $\delta(x^{ij})=kr_{ij}^\gamma$, it follows from (34) that

$$\int\lambda_i(y)^p\delta(y)^\beta dy\leqq A_0^p\sum_j\int\lambda_{ij}(y)^p\delta(y)^\beta dy\leqq A\sum_j\delta(x^{ij})^{\beta-2p+n}<A2^{-i}\ .$$

Noting $n-1\geqq\beta-2p+n$ and $\delta(x^{ij})=kr_{ij}^\gamma<ki^{-\gamma}\leqq 1$, we have

$$\int\lambda_i(y)\delta(y)dy\leqq A_0\sum_j\int\lambda_{ij}(y)\delta(y)dy\leqq A\sum_j\delta(x^{ij})^{n-1}<A2^{-i}\ .$$

Let $\lambda=\sum_{i=2}^\infty\lambda_i$. Then

$$G(x, \lambda)\geqq 1\quad\text{on}\quad\bigcup_{i,j}B(x^{ij}, l\delta(x^{ij})) \tag{35}$$

and λ satisfies (1). Since $\int\lambda(y)\delta(y)dy<\infty$, $G(\cdot, \lambda)\not\equiv\infty$.

Let $\xi\in V$. We shall show that $F=\{x\in R(\gamma, \xi, m);\ G(x, \lambda)\geqq 1\}$ is not $(2, p)$-thin on P. Let $i\geqq 2$ and $ki^{-\gamma}\leqq 1$. From (34) we find ξ^{ij} and r_{ij} such that $\xi\in B(\xi^{ij}, r_{ij})$ and $0<r_{ij}<1/i$. For simplicity we put $\xi(i)=\xi^{ij}$, $r(i)=r_{ij}$ and $x(i)=x^{ij}=(\xi_1^{ij}, \cdots, \xi_{n-1}^{ij}, kr_{ij}^\gamma)$. Take $x\in B(x(i), l\delta(x(i)))$ and observe that

$$|x'-\xi'|\leqq|x-x(i)|+|\xi(i)-\xi|<l\delta(x(i))+r(i)$$
$$=(lkr(i)^{\gamma-1}+1)r(i)\leqq(lk+1)r(i) ,$$

and that

$$(1-l)kr(i)^{\gamma}=(1-l)\delta(x(i))<\delta(x)<(1+l)kr(i)^{\gamma}<(1+l)ki^{-\gamma} .$$

On account of (33), we have

$$m|x'-\xi'|^{\gamma}<m(lk+1)^{\gamma}r(i)^{\gamma}\leqq(1-l)kr(i)^{\gamma}<\delta(x)<(1+l)ki^{-\gamma} ,$$

so that

$$B(x(i), l\delta(x(i)))\subset R(\gamma, \xi, m) \qquad \text{for } i, (1+l)ki^{-\gamma}<1 .$$

Hence (35) leads to $B(x(i), l\delta(x(i)))\subset F$ if i is large. Since $\lim_{i\to\infty}\delta(x(i))=0$, we infer from Lemma 14 that F is not $(2, p)$-thin on P. The proof is complete.

Let us see that the size of E_a is best possible.

PROPOSITION 2. *Let $1\leqq p\leqq n/2$ and $\gamma\geqq 1$. If $E\subset Q$ and $B_{2,p}(E)=0$, then there is a nonnegative measurable function λ on H such that $G(\cdot, \lambda)$ has $(2, p)$-fine T_{γ}-limit zero at 0, but*

$$\limsup_{x\to 0, x\in\Gamma(\gamma,0,b)} G(x, \lambda)=\infty \tag{36}$$

for any $b\in E$.

PROOF. Let $E_j=\{(2^{-j/\gamma}x', 2^{-j}); x\in E\}$. Note that $B_{2,p}(E_j)=0$. We recall that the Bessel kernel $g_2(t)$ is comparable to t^{2-n} as $t\to 0$. On account of (23), we find a nonnegative measurable function λ_j such that

$$\operatorname{supp}\lambda_j\subset J_{j-1}\cup J_j\cup J_{j+1} ,$$

$$\int\lambda_j(y)^p dy<\infty ,$$

$$G(x, \lambda_j)=\infty \quad \text{on} \quad E_j ,$$

$$G(x, \lambda_j)\not\equiv\infty .$$

Let $\eta\leqq 2p-n$. We choose positive numbers α_j such that the function λ defined by $\sum_j \alpha_j\lambda_j$ satisfies $\int_H \lambda(y)^p y_n^{\eta} dy<\infty$ and $G(x, \lambda)\not\equiv\infty$. It is clear that (25) for any $m>0$ and (26') hold at any $a\in P$, so that Lemma 8, 12 and Remark 1 yield that $G(\cdot, \lambda)$ has $(2, p)$-fine T_γ-limit zero at any $a \in P$. If $b \in E$, then $b^j = (2^{-j/\gamma} b', 2^{-j}) \in \Gamma(\gamma, 0, b) \cap \{x \in H; \delta(x) = 2^{-j}\} \subset E_j$, so that $G(b^j, \lambda) = \infty$. Hence (36) holds.

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Present Address:
DEPARTMENT OF MATHEMATICS
FACULTY OF SCIENCE
GAKUSHUIN UNIVERSITY
MEJIRO, TOSHIMA-KU, TOKYO 171