Tokyo J. Math.
Vol. 9, No. 1, 1986

An Immersion of an n-dimensional Real Space Form into an n-dimensional Complex Space Form

Motoko KOTANI

Tokyo Metropolitan University
(Communicated by K. Ogiue)

Introduction

After the famous theorem of Hilbert "There exists no isometric immersion of a hyperbolic plane $H^2(-1)$ into a 3-dimensional Euclidean space." and his conjecture "There exists no isometric immersion of an n-dimensional hyperbolic space $H^n(-1)$ into a $(2n-1)$-dimensional Euclidean space." ([5]), we have studied the problem "Can an n-dimensional hyperbolic space $H^n(-1)$ be isometrically immersed in a Euclidean space R^N?" W. Henke ([4]) constructed an isometric immersion $H^n(-1)\to R^{4n-3}$. But few facts have been known beyond them.

In this paper, we get an example of a local immersion of $H^n(-1)$ into an n-dimensional complex Euclidean space C^n, as a totally real submanifold. Moreover we can determine the immersion of a real space form $M^n(c)$ into a complex space form $\tilde{M}^n(4\tilde{c})$ for $c<\tilde{c}$ as a totally real submanifold with a certain condition about a mean curvature vector (§1). This is a natural extention of the Ejiri's Theorem in [2] and contains an example of Vranceanu [6].

We remark that this immersion cannot be extended globally.

The author wishes to express her gratitude to Professors K. Ogiue and N. Ejiri for their valuable suggestions.

§1. Chen submanifolds.

Let M be a submanifold immersed in $\tilde{M}$. We denote by $\langle\ ,\ \rangle$ the Riemannian metrics on M and $\tilde{M}$. Let σ and h be the second fundamental form and the mean curvature vector of the immersion, respectively.

DEFINITION 1.1. A submanifold M immersed in $\tilde{M}$ is called a Chen submanifold if it satisfies the condition

Received January 10, 1985

$$\sum_{A,B} \langle\sigma(e_A, e_B), h\rangle\sigma(e_A, e_B) \tag{1.1}$$

is parallel to h, where $\{e_A\}$ is an orthonormal frame of M.

REMARK. Historically B. Y. Chen introduced and investigated an *A-submanifold* through the study of the Gauss map and showed that a pseudoumbilic submanifold is a trivial example ([1]) and it was then called a Chen submanifold and some more examples were given in [3].

LEMMA 1.2. *Let M be an n-dimensional totally real submanifold with constant sectional curvature c in an n-dimensional complex space form $\tilde{M}^n(4\tilde{c})$ with constant holomorphic sectional curvature $4\tilde{c}$. Then the following two conditions are equivalent.*

(i) *M is a Chen submanifold.*

(ii) *$\sigma(Jh, Jh)$ is parallel to h,*

where J is the complex structure of $\tilde{M}^n(4\tilde{c})$.

Before proving Lemma 1.2, we define a 3-symmetric tensor T by

$$T(X, Y, Z) = \langle\sigma(X, Y), JZ\rangle\ , \tag{1.2}$$

and we take an orthonormal basis $\{\varepsilon_A\}$ at each point x of M in such a way that

$$T(\varepsilon_1, \varepsilon_1, \varepsilon_1) = \max\{T(X, X, X);\ X \in T_xM,\ \|X\| = 1\}$$

and

$$T(\varepsilon_A, \varepsilon_A, \varepsilon_A) = \max_{X \in \cup_A}\{T(X, X, X)\}\ ,$$

where $\cup_A = \{X \in T_xM;\ \|X\| = 1,\ \langle X, \varepsilon_B\rangle = 0 \text{ for } B = 1, \cdots, A-1\}$.

From the definition of $\{\varepsilon_A\}$ we get

$$T(\varepsilon_1, \varepsilon_1, \varepsilon_A) = 0 \quad \text{for} \quad A > 1\ , \tag{1.3}$$

$$T(\varepsilon_1, \varepsilon_1, \varepsilon_1) \geqq 2T(\varepsilon_1, \varepsilon_A, \varepsilon_A) \quad \text{for} \quad A > 1\ . \tag{1.4}$$

From (1.3) we can diagonalize $(T(\varepsilon_1, \varepsilon_A, \varepsilon_B))_{A,B}$ by an orthonormal basis $\{v_1 = \varepsilon_1, v_2, \cdots, v_n\}$ so that there exists $\beta_{1,A}$ such that

$$T(v_1, v_A, v_B) = \beta_{1,A}\delta_{A,B}\ .$$

But from Gauss equation we get

$$\beta_{1,A}^2 - \beta_{1,A}\beta_{1,1} - (\tilde{c} - c) = 0 \quad \text{for} \quad A > 1\ .$$

These, together with (1.4), imply that $\beta_{1,A}$ is independent of A. We define

$$\beta_1=\beta_{1,A}=T(v_1,\ v_A,\ v_A)\quad \text{for}\quad A>1\ ,$$
$$\alpha_1=\beta_{1,1}=T(v_1,\ v_1,\ v_1)\ .$$

Noting that an $(n-1,\ n-1)$ matrix $(T(v_1,\ v_A,\ v_B))$, $2\leqq A,\ B\leqq n$, is a scalar multiple of the identity, we can take $v_2=\varepsilon_2$. In the same way we diagonalize $(T(v_2,\ v_A,\ v_B))$, $2\leqq A,\ B\leqq n$, and denote its eigenvalues by

$$\alpha_2=T(v_2,\ v_2,\ v_2)\ ,$$
$$\beta_2=T(v_2,\ v_A,\ v_B)\delta_{A,B}\quad \text{for}\quad A,\ B>2\ .$$

Repeating this process, we see that there exist α_A and β_B such that

$$T(\varepsilon_A,\ \varepsilon_A,\ \varepsilon_A)=\alpha_A\ ,$$
$$T(\varepsilon_A,\ \varepsilon_B,\ \varepsilon_C)=\beta_A\delta_{BC}\ ,\quad \text{if}\quad A<B\ .$$

See [2] for detail.

PROOF OF LEMMA 1.2. Noting that

$$\sigma(\varepsilon_A,\ \varepsilon_B)=\beta_A J\varepsilon_B\quad \text{if}\quad A<B\ ,$$
$$\sigma(\varepsilon_A,\ \varepsilon_A)=\sum_{B=1}^{A-1}\beta_B J\varepsilon_B+\alpha_A J\varepsilon_A\ ,$$
$$nh=\sum_A \sigma(\varepsilon_A,\ \varepsilon_A)=\sum_A \{\alpha_A+(n-A)\beta_A\}J\varepsilon_A\ ,$$

we get

$$\langle\sigma(\varepsilon_A,\ \varepsilon_B),\ nh\rangle=\beta_A(\alpha_B+(n-B)\beta_B)\quad \text{if}\quad A<B\ ,$$

and

$$\langle\sigma(\varepsilon_A,\ \varepsilon_A),\ nh\rangle=\alpha_A(\alpha_A+(n-A)\beta_A)+\sum_{B=1}^{A-1}\beta_B\{\alpha_B+(n-B)\beta_B\}\ .$$

Then we have

$$\sum_{A,B}\langle\sigma(\varepsilon_A,\ \varepsilon_B),\ nh\rangle\sigma(\varepsilon_A,\ \varepsilon_B)=\sum_B\Big[\sum_{A<B}\beta_A\{\alpha_A+(n-A+2)\beta_A\}\{\alpha_B+(n-B)\beta_B\}$$
$$+\{\alpha_B^2+(n-B)\beta_B^2\}\{\alpha_B+(n-B)\beta_B\}+\sum_{B<C}(\alpha_C+(n-C)\beta_C\}^2\beta_B\Big]J\varepsilon_B\ .$$

We define f_B by putting the right hand side of the above equation as

$$\sum_B f_B\{\alpha_B+(n-B)\beta_B\}J\varepsilon_B\ .$$

On the other hand, we get

$$\sigma(nJh, nJh)=\sum_B [\sum_{A<B} 2\beta_A\{\alpha_A+(n-A)\beta_A\}\{\alpha_B+(n-B)\beta_B\} + \alpha_B(\alpha_B+(n-B)\beta_B\}^2+\sum_{B<C}\{\alpha_C+(n-C)\beta_C\}^2\beta_B]J\varepsilon_B .$$

We define g_B by putting the right hand side of the above equation as

$$\sum_B g_B\{\alpha_B+(n-B)\beta_B\}J\varepsilon_B .$$

If we put $K_1=K=\tilde{c}-c>0$, $K_i=K_{i-1}+\beta^2_{i-1}$, then we get $K_i+\beta_i\alpha_i-\alpha^2_i=0$ from Gauss equation. Using this we note

$$\begin{aligned}(g_B-f_B)&\{\alpha_B+(n-B)\beta_B\}\\ &=[\sum_{A<B}\beta_A\{\alpha_A+(n-A-2)\beta_A\}+(n-B)(\alpha_B\beta_B-\beta^2_B)]\{\alpha_B+(n-B)\beta_B\}\\ &=[\sum_{A<B}\{-K_A+\beta^2_A+(n-A-2)\beta^2_A\}-K_B(n-B)]\{\alpha_B+(n-B)\beta_B\}\\ &=[\sum_{A<B}\{-K_A+(n-A-1)(K_{A+1}-K_A)\}-K_B(n-B)]\{\alpha_B+(n-B)\beta_B\}\\ &=[\sum_{A<B}\{(n-A-1)K_{A+1}-(n-A)K_A\}-K_A(n-B)]\{\alpha_B+(n-B)\beta_B\}\\ &=[(n-B)K_B-(n-1)K_1-K_B(n-B)]\{\alpha_B+(n-B)\beta_B\}\\ &=-(n-1)K_1\{\alpha_B+(n-B)\beta_B\} .\end{aligned}$$

Therefore we can easily see that the condition

"$\sum_B f_B\{\alpha_B+(n-B)\beta_B\}J\varepsilon_B$ *is parallel to* h"

is equivalent to the condition

"$\sum_B g_B\{\alpha_B+(n-B)\beta_B\}J\varepsilon_B$ *is parallel to* h ."

§2. Gauss equations.

Hereafter $(\tilde{M}, \langle\ ,\ \rangle, J)$ is an n-dimensional complex space form with constant holomorphic sectional curvature $4\tilde{c}$ and M is an n-dimensional totally real Chen submanifold with constant sectional curvature c. We may assume that M is not a minimal submanifold (cf. [2]).

LEMMA 2.1. *We can take a local field of orthonormal frames* $\{e_1, \cdots, e_n\}$ *for* M *so that the following relations hold for some numbers* λ_A, μ_1^+, μ_1^- *and some integer* $a\in\{2, \cdots, n\}$:

(2.1) $$\sigma(e_1, e_1)=\lambda_1 Je_1 ,$$

(2.2) $$\sigma(e_1, e_i)=\mu_1^+ Je_i ,$$

(2.3) $$\sigma(e_1, e_s)=\mu_1^- Je_s ,$$

(2.4) $$\sigma(e_i, e_i)=\mu_1^+ Je_1+\sum_{k=2}^{i-1}\mu_k Je_k+\lambda_i Je_i ,$$

(2.5) $$\sigma(e_i, e_j)=\mu_i Je_j \quad \text{for} \quad i<j ,$$

(2.6) $$\sigma(e_s, e_s)=\mu_1^- Je_1+\sum_{t=a+1}^{s-1}\mu_t Je_t+\lambda_s Je_s ,$$

(2.7) $$\sigma(e_s, e_t)=\mu_s Je_t \quad \text{for} \quad s<t ,$$

(2.8) $$\sigma(e_i, e_s)=0 .$$

Here we use the following indices conventions:

$$1\leqq A, B, \cdots\leqq n , \quad 1<i, j, \cdots\leqq a , \quad a<s, t, \cdots\leqq n .$$

PROOF. There exists a non-minimal point. If we put $e_1=Jh/\|Jh\|$, then we get $T(e_1, e_1, e_A)=0$ for any $A>1$ from Lemma 1.2. Noting this, we take a local field of orthonormal frames $\{e_1, \cdots, e_n\}$ which diagonalizes $T(e_1, e_A, e_B))$. Then from Gauss equation we have

(2.9) $$K+\lambda_1 T(e_1, e_A, e_A)-\{T(e_1, e_A, e_A)\}^2=0 \quad \text{for any} \quad A>1 ,$$

Therefore the $(n-1, n-1)$-matrix $(T(e_1, e_A, e_B))_{A,B>1}$ has at most two distinct eigenvalues. Then we get

$$T(e_1, e_i, e_i)=\mu_1^+ \quad \text{and} \quad T(e_1, e_s, e_s)=\mu_1^- ,$$

for any $1<i\leqq a$ and any $a<s\leqq n$.

Again from Gauss equation, we have

$$\{T(e_1, e_i, e_i)-T(e_1, e_s, e_s)\}T(e_i, e_j, e_s)=0 .$$

Then we get $T(e_i, e_j, e_s)=0$ for any i and j. Similarly we have $T(e_s, e_t, e_i)=0$ for any i, s, t. Then as in Lemma 1.2, we take e_i in such a way that $T(e_i, e_i, e_i)=\max\{T(X, X, X); X\in TM, \langle X, e_1\rangle=0, \langle X, e_k\rangle=0, \langle X, e_s\rangle=0$ for any $1<k<i\leqq a<s.\}$, and take e_s in the similar way. For this basis we get (2.1)$\sim$(2.8). Q.E.D.

LEMMA 2.2. *If* $\{\lambda_A, \mu_1^+, \mu_1^-, \mu_B\}$ *are given as above, then*

$$\lambda_1=\mu_1^+ +\mu_1^- , \quad \mu_1^+\mu_1^-=-K ,$$
$$\mu_i=-\sqrt{K_2 a/(a-i+2)(a-i+1)}$$
$$=\sqrt{a(a-1)/(a-i+1)(a-i+2)}\mu_2 ,$$

$$\lambda_i+(a-i)\mu_i=0 ,$$
$$\mu_s=-\sqrt{K_{a+1}(n-a+1)/(n-s+2)(n-s+1)}$$
$$=\sqrt{(n-a)(n-a+1)/(n-s+2)(n-s+1)}\mu_{a+1} ,$$
$$\lambda_s+(n-s)\mu_s=0 .$$

PROOF. If we put $K_1=K$, $K_i=K_{i-1}+\mu_{i-1}^2$, then we get $K_i+\mu_i\lambda_i-\mu_i^2=0$ from Gauss equation. Since $\langle Je_i, h\rangle=0$, we see that $\lambda_i+(a-i)\mu_i=0$. Hence we have

$$\mu_i=-\sqrt{K_i/(a-i+1)} \quad \text{and} \quad K_{i+1}=K_i+\mu_i^2=(a-i+2)K_i/(a-n+1)$$

so that

$$K_i=K_2a/(a-i+2) \quad \text{and} \quad \mu_i=-\sqrt{K_2a/(a-i+2)(a-i+1)}$$
$$=\sqrt{a(a-1)/(a-i+2)(a-i+1)}\mu_2 .$$

We get μ_s in the same way. Q.E.D.

§ 3. Codazzi equations.

The Codazzi equation can be written as

$$(\nabla_{e_A}T)(e_B, e_C, e_D)=(\nabla_{e_B}T)(e_A, e_C, e_D) ,$$

where ∇ is the connection of M.

By an easy but long computation we see that the connection ∇ and the connection $\tilde{\nabla}$ of $\tilde{M}$ satisfy the following relations with respect to a local field of orthonormal frames $\{e_A\}$ given in § 2:

$$\tilde{\nabla}_{e_1}e_1=\lambda_1Je_1 , \tag{3.1}$$

$$\tilde{\nabla}_{e_1}e_i=\mu_1^+Je_i , \tag{3.2}$$

$$\tilde{\nabla}_{e_1}e_s=\mu_1^-Je_s , \tag{3.3}$$

$$\tilde{\nabla}_{e_i}e_1=-b_1^+e_i+\mu_1^+Je_i \quad \text{for some} \quad b_1^+ , \tag{3.4}$$

$$\tilde{\nabla}_{e_i}e_i=b_1^+e_1+\mu_1^+Je_1+\sum_{k<i}\mu_kJe_k+\lambda_iJe_i , \tag{3.5}$$

$$\tilde{\nabla}_{e_i}e_j=\begin{cases}\mu_iJe_j & \text{if} \quad i<j \\ \mu_jJe_i & \text{if} \quad i>j ,\end{cases} \tag{3.6}$$

$$\tilde{\nabla}_{e_i}e_s=0 , \tag{3.7}$$

$$\tilde{\nabla}_{e_s}e_1=-b_1^-e_s+\mu_1^-Je_s \quad \text{for some} \quad b_1^- , \tag{3.8}$$

$$\tilde{\nabla}_{e_s} e_i = 0 , \tag{3.9}$$

$$\tilde{\nabla}_{e_s} e_s = b_1^- e_1 + \mu_1^- J e_1 + \sum_{t<s} \mu_t J e_t + \lambda_s J e_s , \tag{3.10}$$

$$\tilde{\nabla}_{e_s} e_t = \begin{cases} \mu_s J e_t & \text{if} \quad s<t \\ \mu_t J e_s & \text{if} \quad s>t , \end{cases} \tag{3.11}$$

$$\tilde{\nabla}_{e_1} \mu_1^+ = b_1^+ (2\mu_1^+ - \lambda_1^+) , \quad \nabla_{e_i} \mu_1^+ = 0 , \quad \nabla_{e_s} \mu_1^+ = 0 , \tag{3.12}$$

$$\tilde{\nabla}_{e_1} \mu_1^- = b_1^- (2\mu_1^- - \lambda_1^-) , \quad \nabla_{e_i} \mu_1^- = 0 , \quad \nabla_{e_s} \mu_1^- = 0 , \tag{3.13}$$

$$\tilde{\nabla}_{e_1} \mu_i = b_1^+ \mu_i , \quad \tilde{\nabla}_{e_j} \mu_i = 0 , \quad \tilde{\nabla}_{e_s} \mu_i = 0 , \tag{3.14}$$

$$\tilde{\nabla}_{e_1} \mu_s = b_1^- \mu_s , \quad \tilde{\nabla}_{e_i} \mu_s = 0 , \quad \tilde{\nabla}_{e_t} \mu_s = 0 , \tag{3.15}$$

$$b_1^+ \mu_1^- = b_1^- \mu_1^+ , \tag{3.16}$$

§ 4. Construction of an immersion.

We will construct an immersion of an n-dimensional real space form $M = M^n(c)$ into an n-dimensional complex space form $\tilde{M} = \tilde{M}^n(4\tilde{c})$ as a totally real Chen submanifold.

Before constructing such an immersion, we will determine the above b_1^+, b_1^- by using the condition that M has constant sectional curvature c.

LEMMA 4.1. $(b_1^+)^2 = (b_1^-)^2 = -c$ *if* $n>2$.

$$e_1 b_1 = b_1^2 + c \quad \textit{if} \quad n=2 .$$

Moreover we get $c=0$ *or* $a=n$.

PROOF. From the constancy of the sectional curvature of M, we get

$$c = \langle \nabla_{e_1} \nabla_{e_i} e_i - \nabla_{e_i} \nabla_{e_1} e_i - \nabla_{[e_1, e_i]} e_i , e_1 \rangle = e_1 b_1^+ - (b_1^+)^2 ,$$
$$c = \langle \nabla_{e_i} \nabla_{e_j} e_j - \nabla_{e_j} \nabla_{e_i} e_j - \nabla_{[e_i, e_j]} e_j , e_i \rangle = -(b_1^+)^2 ,$$
$$c = \langle \nabla_{e_i} \nabla_{e_s} e_s - \nabla_{e_s} \nabla_{e_i} e_s - \nabla_{[e_i, e_s]} e_s , e_i \rangle = -b_1^- b_1^+ .$$

We also get

$$c = e_1 b_1^- - (b_1^-)^2 ,$$
$$c = -(b_1^-)^2 .$$

From these we get $b_1^+ = b_1^- = \sqrt{-c}$. If $c \neq 0$, we have $\mu_1^+ = \mu_1^-$ since $b_1^+ \mu_1^- = b_1^- \mu_1^+$. This is a contradiction. Thus we get $c=0$ or $a=n$. Q.E.D.

Now we are in a position to construct an immersion. We may consider the following two cases.

Case 1: $a \neq n$. We get $c=0$ and $b_1^{\pm}=0$ from Lemma 4.1, so that $K=\tilde{c}-c=\tilde{c}>0$, $\nabla_{e_A} e_B=0$ and $e_A \mu_B=0$. Then, noting $\{e_A\}$ is eigenvectors and $\{\mu_B\}$ is an eigenvalues of second fundamental form, M must be a flat parallel submanifold in an n-dimensional complex projective space, so that M must be a flat torus immersed in a standard way. Conversely, a flat torus which is a Chen submanifold is a minimal one.

Therefore, if $a \neq n$, M is a minimal flat torus.

Case 2: $a=n$. Now we will construct M which is a totally real Chen submanifold with constant sectional curvature c and also show that such a submanifold must be obtained in such a way. Moreover we will show that such an M cannot be complete.

2–1: $n=2$. From (3.1) we see that an integral curve γ of e_1 in C^2 satisfies

$$\begin{cases} \dot{\gamma}=e_1 , \\ \dot{e}_1=\lambda_1 J e_1 \ (\text{consequently } (Je_1)'=-\lambda_1 e_1) . \end{cases} \tag{4.1}$$

We note that γ is a curve in a 1-dimensional holomorphic plane with arc length parameter t. Along this curve we get a solution of the differential equations in Lemma 4.1 and (3.12) given by

$$b_1=\sqrt{c}\tan\{\sqrt{c}(t-t_0)\} , \tag{4.2}$$

$$\mu_1^2+K=\cup\{\cos\sqrt{c}(t-t_0)\}^2 . \tag{4.3}$$

On the other hand, since $e_2 b_1/2b_1=e_2 e_1 b_1=e_1 e_2 b_1-[e_1, e_2]b_1=0$, b_1 and μ_1 is constant along an integral curve of e_2 in C^2 which is given by

$$\begin{cases} \dot{\Gamma}=e_2 , \\ \dot{e}_2=b_1 e_1+\mu_1 J e_1=R\varepsilon \quad \text{where} \quad R^2=b_1^2+\mu_1^2 , \\ \dot{\varepsilon}=-\sqrt{b_1^2+\mu_1^2}\, e_2 . \end{cases} \tag{4.4}$$

Then we construct M as follows. Hereafter μ_1 and b_1 are given by (4.2) and (4.3) respectively. On a plane curve in a 1-dimensional holomorphic plane defined by (4.1), we define the 4-dimensional vector space spanned by $\{e_1=\dot{\gamma}, Je_1, e_2, Je_2\}$, where e_2 is given by the equation $(de_2/dt)=\mu_1 Je_2$. Then we get a surface M by attaching to each point on γ a circle whose center is given by $\tau(t)+b_1 e_1+\mu_1 J e_1$ and whose tangent vector is e_2. We verify that this construction gives in fact a surface with the condition in Theorem which will be stated at the end of this section. Conversely, it is clear that M with the condition in Theorem must be given as above.

REMARK. Since $b_1 \to \infty$ as $t \to t_0$, M cannot be extended globally.

We prepare one more fact. Let $\tilde{M}^n$ be a complex space form and S^{2n-1} be a geodesic hypersphere in $\tilde{M}$. Then we get the fibration $S^{2n-1} \to P^{n-1}$, whose fibre is defined by S^1 action on S^{2n-1}. (This fibration is called the Hopf fibration when $\tilde{M}=C^n$.) An easy computation shows that P^{n-1} becomes an $(n-1)$-dimensional complex projective space whose structure is induced from the contact structure of S^{2n-1}. Moreover let L^d be a d-dimensional totally real submanifold of P^{n-1}. Then there is a unique horizontal lift $\tilde{L}^d$ of L^d in S^{2n-1} (cf. N. Ejiri [7]).

2-2: $n \neq 2$. In this case, $b_1=\sqrt{-c}$ and $\mu_1^2+K=Ue^{\sqrt{-c}t}$.

Let M be a totally real Chen submanifold with constant sectional curvature c and $\{e_A\}$ be a local field of orthonormal frames as above. From §3 we see that $\{e_2, \cdots, e_n\}$ deines a completely integrable distribution. Let $\tilde{L}$ be a leaf of $\{e_2, \cdots, e_n\}$. Then its connection D, 2nd fundanental form σ' and the mean curvature vector h' in $\tilde{M}$ are given by

$$D_{e_i}e_j=0 , \tag{4.5}$$

$$\sigma'(e_i, e_j)=\sigma(e_i, e_j)+\delta_{i,j}b_1e_1 , \tag{4.6}$$

$$h'=b_1e_1+\mu_1Je_1 . \tag{4.7}$$

We consider a mapping $F: M \to \tilde{M}$ given by $F(x)=\exp_x(h'/\|h'\|^2)$. Then $F_*e_A=0$ so that $F(\tilde{L})$ is a point and $\|h'\|$ is constant on $\tilde{L}$. Thus $\tilde{L}$ is contained in a geodesic hypersphere as a minimal submanifold. $\pi(\tilde{L})=L$ is a totally real flat parallel submanifold.

Conversely, let $\tilde{L}^{n-1}$ be a lift of a parallel flat submanifold L^{n-1} in P^{n-1}. We define e_1 such that the position vector of $\tilde{L}$ in a geodesic hypersphere with radius 1 in $\tilde{M}(4\tilde{c})$ is equal to $b_1e_1+\mu_1Je_1$. Through each point of $\tilde{L}$ we define an integral curve $\gamma_p(t)$ by (4.1).

LEMMA 4.2. *$\cup_{t\in I}\cup_{p\in\tilde{L}}\gamma(t)$ is a totally real Chen submanifold with constant sectional curvature c in $\tilde{M}$ for an interval I of R.*

PROOF. We will deal with the case $\tilde{M}=C^n$. Other cases are quite similar. Since $\gamma_p(t)$ is contained in a 1-dimensional holomorphic plane, $\dot{\gamma}_p(t)$ is written as

$$\dot{\gamma}_p(t)=\cos\theta(t)v_1+\sin\theta(t)Jv_1=e_1(t) ,$$

where $\dot{\theta}(t)=\mu_1(t)$ and $v_1=e_1(p)$. Then $\gamma_p(t)$ is written as

$$\gamma_p(t)=\Phi_1(t)v_1+\Phi_2(t)Jv_1+C ,$$

where $\Phi_1(t)=\int_0^t \cos\theta(s)ds$, $\Phi_2(t)=\int_0^t \sin\theta(s)ds$ and C is independent of t. Thus we have

$$\begin{aligned}\gamma_p(t)_* e_i(p) &= \tilde{\nabla}_{e_i(p)}\gamma_p(t)\Phi_1(t)\tilde{\nabla}_{e_i}e_1(p)+\Phi_2(t)J\tilde{\nabla}_{e_i}e_1(p)+e_iC \\ &= \Phi_1(t)(-b_1e_i+\mu_1Je_i)-\Phi_2(t)(\mu_1e_i+b_1Je_i)+e_iC \\ &= -(b_1\Phi_1+\mu_1\Phi_2)e_i+(\Phi_1\mu_i-b_1\Phi_2)Je_i+e_iC\ .\end{aligned}$$

Noting that e_2C is independent of t, we get

$$e_iC=\gamma_p(0)_*e_i(p)-(\Phi_1(0)b_1+\Phi_2(0)\mu_1)e_i(p)+(\Phi_1(0)\mu_1-\Phi_2(0)b_1)Je_i(p)=e_i(p)\ .$$

Then we get

$$\gamma_p(t)_*e_i=(1-b_1\Phi_1-\mu_1\Phi_2)e_i+(\mu_1\Phi_1-b_1\Phi_2)Je_i\ .$$

Since $T_{\gamma_p(t)}M$ is spanned by $e_1(t)$, $\gamma_p(t)_*e_2, \cdots, \gamma_p(t)_*e_n$, we easily see that M is totally real. Checking that $e_i(t)=\gamma_{p*}e_i/\|\gamma_{p*}e_i\|$ satisfy the differential equations in § 3, we prove M is a Chen submanifold with constant curvature c. We put $f(t)=\|\gamma_{p*}e_i\|^2=(1-b_1\Phi_1-\mu_1\Phi_2)^2+(\mu_1\Phi_1-b_1\Phi_2)^2$.

Now we state our Theorem.

THEOREM. *If M is an n-dimensional totally real Chen submanifold with constant sectional curvature c isometrically immersed in an n-dimensional complex space form $\tilde{M}(4\tilde{c})$, where $\tilde{c}>c$. Then*

(i) *If M is minimal, then M is a totally geodesic submanifold or a flat torus (Ejiri* [2]*).*

(ii) *Unless M is minimal, then $M=(I\times\tilde{L}^{n-1}, dt^2+f(t)g)$, where I is an interval of R and $(\tilde{L}, g)$ is the following submanifold in $\tilde{M}(4\tilde{c})$:*

$$\begin{array}{ccccc}\tilde{L}^{n-1} & \subset & S^{2n-1} & \subset & \tilde{M}(4\tilde{c})\ , \\ \downarrow & & \downarrow & & \\ L^{n-1} & \subset & P^{n-1} & & \end{array}$$

where S^{2n-1} is a geodesic hypersphere in $\tilde{M}(4\tilde{c})$ and $\tilde{L}$ is a horizontal lift of a minimal flat torus L in P^{n-1}. The immersion is given as above.

References

[1] B. Y. CHEN, Pseudo-umbilical submanifolds of a Riemannian manifold of constant curvature II, J. Math. Soc. Japan, **25** (1973), 105–114.

[2] N. EJIRI, Totally real minimal immersions of n-dimensional real space forms into n-dimensional complex space forms, Proc. Amer. Math. Soc., **84** (1982), 243–246.

[3] L. GHEYSENS, P. VERHEYEN and L. VERSTRAELEN, Characterization and examples of Chen submanifolds, J. Geom., **20** (1983), 47–62.

[4] W. HENKE, Isometrische Immersionen des n-Dim. Hyperbolishen Raumes H^n in E^{4n-3}, Manuscripta Math., **34** (1981), 265-278.

[5] D. HILBERT, Uber Flachen von Konstanten Gausschen Krummeng, Trans. Amer. Math. Soc., **2** (1901), 87-99.

[6] G. VRANCEANU, Surfaces de rotation dans E^4, Rev. Roumaine Math. Pures Appl., **2** (1977), 857-862.

[7] N. EJIRI, A generalization of minimal cones, Trans. Amer. Math. Soc., **276** (1983), 347-360.

Present Address:
DEPARTMENT OF MATHEMATICS
FACULTY OF SCIENCES
TOKYO METROPOLITAN UNIVERSITY
FUKAZAWA, SETAGAYA-KU, TOKYO 158