TOKYO J. MATH.
VOL. 9, NO. 2, 1986

Vanishing of Certain 1-form Attached to a Configuration

Kazuhiko AOMOTO

Nagoya University
(Communicated by T. Saito)

This is a remark to my paper [1], "Configurations and invariant Gauss-Manin connections of integrals". In the sequel we use the terminologies in [1].

Consider the integral

$$\hat{\varphi}(\phi)=\int \hat{U}(\lambda)dx_1\wedge\cdots\wedge dx_n \tag{1}$$

for $\hat{U}(\lambda)=\hat{f}_0^{\lambda_0}\hat{f}_1^{\lambda_1}\cdots\hat{f}_m^{\lambda_m}$, $\lambda_0,\cdots,\lambda_m\in\boldsymbol{C}$ where $\hat{f}_0$ and $\hat{f}_j$ denote functions $1-x_1^2-\cdots-x_n^2$, $\sqrt{-1}\sum_1^n u_{j,\nu}x_\nu+u_{j,0}$ respectively. We put $a_{0,0}=1$, $a_{j,k}=\sum_0^n u_{j,\nu}u_{k,\nu}$ and $a_{j,0}=u_{j,0}$ for $1\leqq j,\ k\leqq m$. $u_{j,\nu}$ are normalized such that $a_{j,j}=1$ for all j. For the symmetric configuration matrix $A=((a_{j,k}))_{0\leqq j,k\leqq m}$ we denote by $A\begin{pmatrix} i_1, & \cdots, & i_p \\ j_1, & \cdots, & j_p\end{pmatrix}$ the subdeterminant of the $i_1,\cdots,i_p$ th lines and the $j_1,\cdots j_p$ th columns. A sequence of 1-forms $\theta\begin{pmatrix} & \phi & \\ i_1, & \cdots, & i_p\end{pmatrix}$ for $1\leqq p\leqq n+1$ are defined in an inductive way:

$$\theta\begin{pmatrix}\phi\\ i\end{pmatrix}=da_{0,i} \tag{2_1}$$

$$\theta\begin{pmatrix}\phi\\ j,k\end{pmatrix}=da_{j,k}+\frac{A\begin{pmatrix}0, & k\\ k, & j\end{pmatrix}}{A(0,k)}da_{0,k}+\frac{A\begin{pmatrix}0, & j\\ j, & k\end{pmatrix}}{A(0,j)}da_{0,j} \tag{2_2}$$

$$\theta\begin{pmatrix} & \phi & \\ i_1, & \cdots, & i_p\end{pmatrix}=\sum_1^p(-1)^\nu\theta\begin{pmatrix} & & \phi & & \\ i_1, & \cdots, & i_\nu, & \cdots, & i_p\end{pmatrix}\cdot\frac{A\begin{pmatrix}0, & i_1, & \cdots, & \hat{i}_\nu, & \cdots i_p\\ & i_1, & \cdots, & i_p & \end{pmatrix}}{A(0,i_1,\cdots,\hat{i}_\nu,\cdots i_p)}, \tag{2_p}$$

for $p\geqq 3$.

where $\cdots,\hat{i}_\nu\cdots$ denotes the deletion of the index i_ν. (There are misprints in (3, 9), (3, 10), (3, 11) and (3, 12), [1] which should be corrected as above.)

Received October 30, 1985

Then we have

LEMMA. *For arbitrary* $n+1$ *indices* $i_1, \cdots, i_{n+1}$, *the form* $\theta\begin{pmatrix} & \phi & \\ i_1, & \cdots, & i_{n+1}\end{pmatrix}$ *always vanishes.*

PROOF. First we notice that $\theta\begin{pmatrix} & \phi & \\ i_1, & \cdots, & i_{n+1}\end{pmatrix}$ depends only on the functions $f_{i_1}, \cdots, f_{i_{n+1}}$ from its definition. So we have only to prove the Lemma in case of $m=n+1$ and $i_1=1, \cdots, i_{n+1}=n+1$. We specialize the coefficients $u_{j,\nu}$ such that f_j are all real and that the domain $\Delta: f_1 \geqq 0, \cdots, f_{n+1} \geqq 0$ is not empty and is contained in the domain: $f_0<0$. If we take λ to be -1 in the equation (E, III_0), [1], then the latter is simplified as follows:

$$0=\lambda_1 \cdots \lambda_{n+1}\theta\begin{pmatrix} & \phi & \\ 1, 2, & \cdots, & n+1\end{pmatrix}\int_\Delta \hat{U}(\lambda)\cdot\frac{dx_1\wedge\cdots\wedge dx_n}{f_1\cdots f_{n+1}}. \tag{3}$$

Suppose λ_j are all positive, then the integral part in the right hand side is positive. Hence the form $\theta\begin{pmatrix} & \phi & \\ i_1, & \cdots, & i_{n+1}\end{pmatrix}$ must vanish identically as an analytic continuation. Q.E.D.

As a consequence of it, *the formula* (E, III_0) *in Proposition* 3, 4, [1], *turns out to be*

$$d\hat{\varphi}(\phi)=\sum_1^n \frac{1}{s!}\sum_{i_1,\dots,i_s}\frac{\lambda_{i_1}\cdots\lambda_{i_s}}{(\mu_0+1)\cdots(\mu_0+s-1)}\theta\begin{pmatrix} & \phi & \\ i_1, & \cdots, & i_s\end{pmatrix} \times\frac{A(i_1,\cdots,i_s)}{A(0,i_1,\cdots,i_s)}\hat{\varphi}_*(i_1,\cdots,i_s) \tag{4}$$

for $\mu_0=-2\lambda_0-n-1-\sum_1^m\lambda_j$.

REMARK. By a direct computation $\theta\begin{pmatrix} \phi \\ j, k\end{pmatrix}$ is expressed in the following manner:

$$\frac{A\begin{pmatrix}0, j\\0, k\end{pmatrix}\theta\begin{pmatrix}\phi\\j, k\end{pmatrix}}{A(0,j,k)}=\frac{1}{2}\{-d\log A(0,j,k)+d\log A(0,j)+d\log A(0,k)\}, \tag{5}$$

so that the Lemma is trivial for $n=1$, because $A(0, j, k)$ identically vanishes. However, generally it seems rather difficult to prove in a purely algebraic way the vanishing of the forms $\theta\begin{pmatrix} & \phi & \\ i_1, & \cdots, & i_{n+1}\end{pmatrix}$.

References

[1] K. AOMOTO, Configurations and invariant Gauss-Manin connections of integrals I, Tokyo J. Math., **5** (1982), 249-287.

Present Address:
MATHEMATICAL INSTITUTE
FACULTY OF SCIENCES
NAGOYA UNIVERSITY
CHIKUSA-KU, NAGOYA, 464