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On the Curvature of Affine Homogeneous Convex Domains

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Introduction

Let Ω be a convex domain not containing any affine line in a real number space, and $G(\Omega)$ the group of all affine automorphisms of Ω. If the group $G(\Omega)$ acts on Ω transitively, then Ω is said to be homogeneous. For a convex domain Ω in the n-dimensional real number space, we denote by $D(\Omega)$ the tube domain over Ω in the n-dimensional complex number space. Then the tube domain $D(\Omega)$ is holomorphically equivalent to a bounded domain. Therefore, $D(\Omega)$ admits the Bergman metric. By restricting the Bergman metric of $D(\Omega)$ to Ω, we have a $G(\Omega)$-invariant Riemannian metric g_Ω on Ω, which is called the canonical metric of Ω ([13], [8]). Concerning the Bergman metrics of homogeneous bounded domains, D'Atri and Miatello ([2]) proved that a homogeneous bounded domain is symmetric if and only if the sectional curvature is non-positive. On the other hand, it is known that there exist non-symmetric homogeneous convex cones of non-positive sectional curvature in the canonical metric ([7]). The purpose of the present note is to determine homogeneous convex domains of non-positive sectional curvature among homogeneous convex domains of certain types or of low rank (Theorems 2.3, 3.3, 3.4, and 4.3).

The proofs for our results are carried out by calculations on T-algebras due to Vinberg ([13], [14]). The same notations and definitions as those in [6]–[10] will be employed.

§1. Preliminaries.

In this section, we fix notation and recall fundamental results on homogeneous convex domains and T-algebras mainly due to E. B. Vinberg ([13], [14]).

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1.1. Let $\mathfrak{A}=\sum_{1\leq i,j\leq r}\mathfrak{A}_{ij}$ be a T-algebra of rank r ($r>1$) provided with an involutive anti-automorphism *. General elements of the subspace $\mathfrak{A}_{ij}$ are denoted as a_{ij}, b_{ij}, $\cdots$, and also, general elements of $\mathfrak{A}$ are denoted like as matrices $a=(a_{ij})$, $b=(b_{ij})$, $\cdots$. We put

$$n_{ij}=\dim\mathfrak{A}_{ij} \quad\text{and}\quad n_i=1+\frac{1}{2}\sum_{k\neq i}n_{ki} \qquad (1\leqq i,\ j\leqq r)\ . \tag{1.1}$$

Let us define subsets T, $\Omega(\mathfrak{A})$, and X of $\mathfrak{A}$ by

$$\begin{aligned} &T=\{t=(t_{ij});\ t_{ii}>0\ (1\leqq i\leqq r-1),\ t_{rr}=1,\ t_{ij}=0\ (1\leqq j<i\leqq r)\}\ , \\ &\Omega(\mathfrak{A})=\{tt^*;\ t\in T\}\ , \quad\text{and}\quad X=\{x;\ x^*=x\}\ , \end{aligned} \tag{1.2}$$

respectively. Then it is known that $\Omega(\mathfrak{A})$ is a homogeneous convex domain in the affine subspace $\{x=(x_{ij})\in X;\ x_{rr}=1\}$ of X, and the set T becomes a connected triangular Lie subgroup of $G(\Omega(\mathfrak{A}))$ with respect to the multiplication in the T-algebra $\mathfrak{A}$, acting on $\Omega(\mathfrak{A})$ simply transitively in the following way:

$$(t,\ ss^*)\in T\times\Omega(\mathfrak{A})\longrightarrow(ts)(ts)^*\in\Omega(\mathfrak{A})\ .$$

The Lie algebra $\mathfrak{t}$ of T is given by

$$\mathfrak{t}=\{t=(t_{ij})\in\mathfrak{A};\ t_{rr}=0,\ t_{ij}=0\ \ (1\leqq j<i\leqq r)\}$$

with the bracket relation $[a, b]=ab-ba$. It is known that any homogeneous convex domain is affinely equivalent to one constructed in the above manner. If a homogeneous convex domain Ω is affinely equivalent to a domain $\Omega(\mathfrak{A})$ of the form (1.2), then the rank of the T-algebra $\mathfrak{A}$ is an affine invariant of Ω. Therefore, we define the rank of a domain Ω by the rank of $\mathfrak{A}$ minus one. It is easy to see that the rank of Ω is equal to the codimension of the commutator subalgebra $[\mathfrak{t}, \mathfrak{t}]$ in $\mathfrak{t}$. Throughout this note, we consider homogeneous convex domains of the form (1.2) given by T-algebras exclusively.

1.2. The unit element e of the group T is contained in the convex domain $\Omega(\mathfrak{A})$ and the tangent space of $\Omega(\mathfrak{A})$ at the point e can be naturally identified with the Lie algebra $\mathfrak{t}$. The canonical metric $\langle\ ,\ \rangle=g_{\Omega(\mathfrak{A})}$ at the point e is given by the formula

$$\langle a, b\rangle=\mathrm{Sp}((a+a^*)(b+b^*)) \qquad (a,\ b\in\mathfrak{t})\ ,$$

where $\mathrm{Sp}(a)=\sum_{1\leq i\leq r}n_i a_{ii}$; and $\mathfrak{t}$ is the orthogonal direct sum of the subspaces $\mathfrak{A}_{ii}$ ($1\leqq i\leqq r-1$) and $\mathfrak{A}_{ij}$ ($1\leqq i<j\leqq r$). Therefore, the connection

function α, and the curvature tensor R for the canonical metric can be represented in terms of the Lie algebra $\mathfrak{t}$ as follows

$$\alpha: \mathfrak{t}\times\mathfrak{t}\longrightarrow\mathfrak{t},$$

$$(1.3)\qquad 2\langle\alpha(a, b), c\rangle=\langle[c, a], b\rangle+\langle[c, b], a\rangle+\langle[a, b], c\rangle ;$$

$$R: \mathfrak{t}\times\mathfrak{t}\times\mathfrak{t}\longrightarrow\mathfrak{t},$$

$$R(a, b, c)=R(a, b)c=\alpha(a, \alpha(b, c))-\alpha(b, \alpha(a, c))-\alpha([a, b], c)$$

for all $a, b, c \in \mathfrak{t}$ ([4]). The sectional curvature K is given by the following formula:

$$K(a, b)=\langle R(a, b)b, a\rangle$$

for every orthonormal elements $a, b \in \mathfrak{t}$.

A homogeneous convex domain Ω is said to be of *metabelian* type if a transitive triangular Lie algebra $\mathfrak{t}$ of infinitesimal affine automorphisms of Ω is metabelian, that is, the commutator subalgebra $[\mathfrak{t}, \mathfrak{t}]$ is abelian. It shoud be remarked that for a homogeneous convex domain Ω, transitive connected triangular subgroups are conjugate in $G(\Omega)$ ([12]).

1.3. Let V be a homogeneous convex cone in $\boldsymbol{R}^n$ and F an $\boldsymbol{R}^n$-valued symmetric bilinear mapping defined on $\boldsymbol{R}^m\times\boldsymbol{R}^m$ satisfying the following conditions: (1) $F(y, y) \in \bar{V}$ (the topological closure of V); (2) $F(y, y)=0$ implies $y=0$; (3) the subgroup of $G(V)$ defined by $\{A \in G(V)$; there exists $B \in GL(\boldsymbol{R}^m)$ such that the equality $F(B(y), B(y))=A(F(y, y))$ holds for all $y \in \boldsymbol{R}^m\}$ acts on V transitively. Then the domain

$$\Omega(V, F)=\{(x, y) \in \boldsymbol{R}^n\times\boldsymbol{R}^m;\ x-F(y, y) \in V\}$$

is a homogeneous convex domain, which is called the *real Siegel domain* over the cone V and associated with the symmetric mapping F. It was proved that every homogeneous convex domain is affinely equivalent to a real Siegel domain ([12], [13]).

Let Ω_i be a real Siegel domain over a cone $V_i (i=1, 2)$. Then the tube domain $D(\Omega_i)$ is holomorphically equivalent to a Siegel domain over the cone V_i([10]). Therefore, if Ω_1 is affinely equivalent to Ω_2, then V_1 is linearly equivalent to V_2 ([3]). We call Ω a domain over a cone of square type (resp. dual square type) if Ω is affinely equivalent to a real Siegel domain over a cone of square type (resp. dual square type) (For the definition of cones of square type or of dual square type, see [15], [16].).

§ 2. Domains of metabelian type.

In this section, we calculate the sectional curvature of homogeneous convex domains of metabelian type.

We first prove the following

LEMMA 2.1. *A homogeneous convex domain $\Omega=\Omega(\mathfrak{A})$ is of metabelian type if and only if the condition $n_{ij}n_{jk}=0$ holds for every triple (i, j, k) of indices $1\leqq i<j<k\leqq r$.*

PROOF. The commutator subalgebra of $\mathfrak{t}$ is given by

$$[\mathfrak{t}, \mathfrak{t}]=\sum_{i<j}\mathfrak{A}_{ij},$$

and the identities

$$[x_{ij}, x_{jk}]=x_{ij}x_{jk} \quad \text{and} \quad \|x_{ij}x_{jk}\|^2=(1/2n_j)\|x_{ij}\|^2\|x_{jk}\|^2 \tag{2.1}$$

hold for all $x_{ij}\in\mathfrak{A}_{ij}$, $x_{jk}\in\mathfrak{A}_{jk}$ $(i<j<k)$ (cf. [6], [8]). Therefore, the lemma follows from these identities. Q.E.D.

We now define subspaces $\mathfrak{A}_0$ and $\mathfrak{n}$ of $\mathfrak{A}$ by

$$\mathfrak{A}_0=\sum_{1\leq i\leq r-1}\mathfrak{A}_{ii} \quad \text{and} \quad \mathfrak{n}=\sum_{1\leq i<j\leq r}\mathfrak{A}_{ij},$$

respectively. Then $\mathfrak{n}=[\mathfrak{t}, \mathfrak{t}]$, and $\mathfrak{A}_0$ is the orthogonal complement of $\mathfrak{n}$ in $\mathfrak{t}$ with respect to the canonical metric. Let us put

$$e_i=(1/2\sqrt{n_i})e_{ii} \qquad (1\leqq i\leqq r-1),$$

where e_{ii} is the unit element of the subalgebra $\mathfrak{A}_{ii}$. Then $\{e_i\}$ is an orthonormal basis of the subspace $\mathfrak{A}_0$. We define linear endomorphisms A_i of $\mathfrak{n}$ by $A_i(x)=[e_i, x]$ for every $x\in\mathfrak{n}$. Then we have

$$A_i(x)=(1/2\sqrt{n_i})(\sum_{i<j}x_{ij}-\sum_{j<i}x_{ji}) \tag{2.2}$$

for every $x\in\mathfrak{n}$. (cf. [6], [8]). On the other hand, from (1.3) and (2.2) it follows that the condition

$$\begin{aligned}2\langle\alpha(x, e_i), y\rangle&=\langle[y, x], e_i\rangle+\langle[y, e_i], x\rangle+\langle[x, e_i], y\rangle\\&=-2\langle A_i(x), y\rangle\end{aligned}$$

holds for all $x, y\in\mathfrak{n}$. Therefore we have the following

LEMMA 2.2. *The linear endomorphisms A_i are self-adjoint with respect to the canonical metric and satisfy the conditions*

$$A_i(x)=-\alpha(x, e_i) \qquad (1\leqq i\leqq r-1)$$

for every $x\in\mathfrak{n}$.

By using the above lemmas, we have the following

THEOREM 2.3. *If a homogeneous convex domain* Ω *is of metabelian type, then the sectional curvature of* Ω *is non-positive.*

PROOF. We can assume that the dimension of Ω is larger than one and Ω is of the form $\Omega(\mathfrak{A})$ by a T-algebra $\mathfrak{A}$. Since the Lie algebra $\mathfrak{t}$ is metabelian, the identity

$$\alpha(x, y)=\sum_{1\leqq i\leqq r-1}\langle A_i(x), y\rangle e_i$$

is satisfied for $x, y\in\mathfrak{n}$. Thus, by Lemma 2.2, we have

$$R(x, y)z=\sum_{1\leqq i\leqq r-1}(\langle A_i(x), z\rangle A_i(y)-\langle A_i(y), z\rangle A_i(x))$$

for all $x, y, z\in\mathfrak{n}$. From this it follows that

$$K(x, y)=\sum_{1\leqq i\leqq r-1}(\langle A_i(x), y\rangle^2-\langle A_i(x), x\rangle\langle A_i(y), y\rangle)$$

for all $x, y\in\mathfrak{n}$. By calculations using Lemma 2.1 and the condition (2.2), we have the following formula:

$$\begin{aligned}&\langle A_i(x), y\rangle^2-\langle A_i(x), x\rangle\langle A_i(y), y\rangle\\ &\quad=(1/4n_i)\sum_{i<j}(\langle x_{ij}, y_{ij}\rangle^2-\|x_{ij}\|^2\|y_{ij}\|^2)+(1/4n_i)\sum_{j<i}(\langle x_{ji}, y_{ji}\rangle^2-\|x_{ji}\|^2\|y_{ji}\|^2)\\ &\quad\quad+(1/4n_i)\sum_{i<j<k}(2\langle x_{ij}, y_{ij}\rangle\langle x_{ik}, y_{ik}\rangle-\|x_{ij}\|^2\|y_{ik}\|^2-\|x_{ik}\|^2\|\|y_{ij}\|^2)\\ &\quad\quad+(1/4n_i)\sum_{s<t<i}(2\langle x_{si}, y_{si}\rangle\langle x_{ti}, y_{ti}\rangle-\|x_{si}\|^2\|y_{ti}\|^2-\|x_{ti}\|^2\|y_{si}\|^2)\end{aligned}$$

for all $x, y\in\mathfrak{n}$. Hence, $K(x, y)\leqq 0$ for all $x, y\in\mathfrak{n}$. On the other hand, it is easy to see that the sectional curvature K is non-positive if and only if $K(x, y)$ is non-positive for all $x, y\in\mathfrak{n}$ (cf. [1, Proposition 4.3]). Q.E.D.

§ 3. Domains of low rank.

In this section, we determine all homogeneous convex domains of non-positive curvature of rank two or of rank three.

3.1. We first generalize a result in [7] as follows:

LEMMA 3.1. *If the sectional curvature of a homogeneous convex domain* $\Omega(\mathfrak{A})$ *is non-positive, then the conditions*

$$n_i \leqq n_j \quad and \quad n_k \leqq n_j$$

hold for every triple (i, j, k) *of indices* $1 \leqq i < j < k \leqq r-1$ *satisfying* $n_{ij} n_{jk} \neq 0$.

The above lemma can be proved in the same way as in [7, Theorem 3.4]. So we may omit the proof.

LEMMA 3.2. *If the sectional curvature of a homogeneous convex domain* $\Omega(\mathfrak{A})$ *is non-positive, then the condition* $n_{ij} n_{jr} = 0$ *holds for every pair* (i, j) *of indices* $1 \leqq i < j \leqq r-1$.

PROOF. By (1.3), we have

$$R(x_{ir}, x_{jr}) x_{jr} = -\alpha(x_{jr}, \alpha(x_{ir}, x_{jr})) = -\frac{1}{2}\alpha(x_{jr}, x_{ir} x_{jr}^{*})$$
$$= \frac{1}{4}(x_{ir} x_{jr}^{*}) x_{jr}$$

for all $x_{ir} \in \mathfrak{A}_{ir}$ and $x_{jr} \in \mathfrak{A}_{jr}$ (cf. [7], [8]). Hence, we have

$$K(x_{ir}, x_{jr}) = \frac{1}{4}\langle (x_{ir} x_{jr}^{*}) x_{jr}, x_{ir} \rangle = \frac{1}{4}\| x_{ir} x_{jr}^{*} \|^2 .$$

From this, it follows that $x_{ir} x_{jr}^{*} = 0$ and $\langle x_{ij} x_{jr}, x_{ir} \rangle = 0$ for every $x_{ij} \in \mathfrak{A}_{ij}$. Putting $x_{ir} = x_{ij} x_{jr}$, we have $n_{ij} n_{jr} = 0$ by (2.1). Q.E.D.

By the above lemma, we have the following

THEOREM 3.3. *Let* Ω *be a homogeneous convex domain over a cone of metabelian type. Then the sectional curvature of* Ω *is non-positive if and only if* Ω *is of metabelian type.*

PROOF. There exists a T-algebra $\mathfrak{A}$ of rank r satisfying the conditions $\Omega = \Omega(\mathfrak{A})$ and $n_{ij} n_{jk} = 0$ $(1 \leqq i < j < k \leqq r-1)$. By Lemma 3.2, we see that the condition of Lemma 2.1 holds for $\mathfrak{A}$. Q.E.D.

3.2. Let $\Omega = \Omega(\mathfrak{A})$ be an irreducible homogeneous convex domain. If the rank of Ω is one, then Ω is affinely equivalent either to the half-line or to an elementary domain. The sectional curvature of an elementary domain is a negative constant ([8]). If the rank of Ω is two and the sectional curvature is non-positive, then by Lemma 3.2, the condition $n_{23} = 0$ holds. Therefore, by Lemma 2.1, Ω is of metabelian type. In the case where $n_{13} = 0$, Ω is the $n_{12} + 2$ dimensional circular cone, which is self-dual. If the rank of Ω is three and the sectional curvature is non-positive, then by Lemma 3.2, either Ω is of metabelian type or Ω satisfies the conditions $n_{24} = n_{34} = 0$ and $n_{12} n_{23} \neq 0$. Therefore, in the last case, we have

$$2n_1=2+n_{12}+n_{13}+n_{14}\ ,\quad 2n_2=2+n_{12}+n_{23}\ ,\quad 2n_3=2+n_{13}+n_{23}\ .$$

By (2.1), we can see that $n_{12}\leqq n_{13}$ and $n_{23}\leqq n_{13}$. Hence, by Lemma 3.1, we have $n_{14}=0$ and $n_{12}=n_{23}=n_{13}$. This condition implies that Ω is a self-dual cone (cf. [14]). Furthermore, it was proved in [11] that every reducible homogeneous convex domain is decomposed uniquely into the direct product of irreducible homogeneous convex domains. Therefore, summing up results stated above, we have a generalization of [7, Theorem 4.6] as follows:

THEOREM 3.4. *Let Ω be a homogeneous convex domain whose rank is less than four. Then the sectional curvature of Ω is non-positive if and only if every irreducible component of Ω is affinely equivalent either to a homogeneous self-dual cone or to a homogeneous convex domain of metabelian type.*

§4. Domains over cones of square type or of dual square type.

In this section, we determine all homogeneous convex domains of non-positive sectional curvature among homogeneous convex domains over cones of square type or of dual square type.

4.1. Let Ω be a homogeneous convex domain of rank $r-1$ ($r\geqq 3$) over a cone of square type. Then there exists a T-algebra $\mathfrak{A}$ of rank r satisfying the conditions $\Omega=\Omega(\mathfrak{A})$ and

$$n_{1i}=n_{2i}=\cdots=n_{i-1,i}\qquad(=m_i>0) \tag{4.1}$$

for $2\leqq i\leqq r-1$ (cf. [10, pp. 213 and 214], [15]).

LEMMA 4.1. *Let Ω be a homogeneous convex domain over a cone of square type and of rank $r-1$ ($r\geqq 4$). Then the sectional curvature of Ω is non-positive if and only if Ω is affinely equivalent to a homogeneous self-dual cone.*

PROOF. By (4.1) and (1.1), we have

$$2n_i=2+(i-1)m_i+m_{i+1}+\cdots+m_{r-1}+n_{ir}\qquad(1\leqq i\leqq r-1)\ ,$$

and by (2.1),

$$m_2\leqq m_3\leqq\cdots\leqq m_{r-1}\ .$$

Let us suppose that the sectional curvature of Ω is non-positive. Then by Lemma 3.2, we have

$$n_{2r}=n_{3r}=\cdots=n_{r-1,r}=0 .$$

Moreover, by Lemma 3.1, we have the equalities

$$m_2=m_3=\cdots=m_{r-1} \quad \text{and} \quad n_{1r}=0 .$$

Hence, Ω is a self-dual cone. Since the sectional curvature of every self-dual cone is non-positive ([5]), the converse assertion holds. Q.E.D.

4.2. Let Ω be a homogeneous convex domain over a cone of dual square type. Then there exists a T-algebra $\mathfrak{A}$ such that the conditions $\Omega=\Omega(\mathfrak{A})$ and

$$n_{i,i+1}=n_{i,i+2}=\cdots=n_{i,r-1} \qquad (>0)$$

are satisfied for $1\leqq i\leqq r-2$ (cf. [10, pp. 213 and 214], [16]). The following lemma can be proved similarly as Lemma 4.1. So, we may omit the proof.

LEMMA 4.2. *Let Ω be a homogeneous convex domain over a cone of dual square type and of rank $r-1$ ($r\geqq 4$). Then the sectional curvature of Ω is non-positive if and only if Ω is affinely equivalent to a homogeneous self-dual cone.*

4.3. Combining Lemmas 4.1 and 4.2 with Theorem 3.4, we have the main theorem of the present note as follows:

THEOREM 4.3. *Let Ω be a homogeneous convex domain over a cone of square type or of dual square type. Then the sectional curvature of Ω is non-positive if and only if Ω is affinely equivalent to one of the following: a homogeneous self-dual cone; an elementary domain (i.e., the hyperbolic space form); a non-symmetric homogeneous convex domain of metabelian type and of rank two.*

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