TOKYO J. MATH.
VOL. 10, NO. 1, 1987

Local Isometric Embedding of Two Dimensional Riemannian Manifolds into R^3 with Nonpositive Gaussian Curvature

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(Communicated by M. Obata)

Introduction

As it is well known, the problem of C^∞ local isometric embedding of a two dimensional Riemannian manifold into R^3 is a problem whether C^∞ functions $x(u, v)$, $y(u, v)$, $z(u, v)$ which satisfy

$$dx^2+dy^2+dz^2=Edu^2+2Fdudv+Gdv^2 \tag{0.1}$$

exist in a neighborhood of a point, say $(u, v)=0$, when the first fundamental form $Edu^2+2Fdudv+Gdv^2$ is given. The results already known are as follows. Let K be the Gaussian curvature of the two dimensional manifold, then the classical result is that the problem is affirmatively answered if $K\neq 0$ at $(u, v)=0$, and a recent interesting result due to Lin [4] is that it is also affirmative if $K=0$, $\operatorname{grad} K\neq 0$ at $(u, v)=0$. Now a natural question arises. Namely, is it affirmative when

$$K=\operatorname{grad} K=0 \quad \text{at} \quad (u, v)=0 \tag{0.2}$$

and one of the following conditions holds:

(i) $\operatorname{Hess} K(0, 0)>0$,
(ii) $\operatorname{Hess} K(0, 0)<0$,
(iii) $\operatorname{Hess} K(0, 0)$ has two eigenvalues with opposite signs?

Hereafter, for simplicity, we refer the case with conditions (0.2) and (i) (resp. (ii) and resp. (iii)) by (i) (resp. (ii) and resp. (iii)).

Then what we have obtained is the following.

THEOREM. *The problem of C^∞ local isometric embedding is also affirmative in the case* (ii). (Consult §4 for the related results.)

We note we can replace the equation (0.1) by a second order Monge-

Received September 12, 1986

Ampère equation (i.e., Darboux equation). Namely, reminding the fact that a two dimensional Riemannian manifold whose Gaussian curvature is zero is locally isometric to Euclidean space with its standard metric, it is enough to solve the following Darboux equation (0.3) for z under the condition $\nabla z(0, 0)=0$, which assures the Gaussian curvature of the metric $Edu^2+2Fdudv+Gdv^2-dz^2$ vanishes:

$$(0.3)\qquad \begin{aligned}(z_{11}-\Gamma^i_{11}z_i)(z_{22}-\Gamma^i_{22}z_i)-(z_{12}-\Gamma^i_{12}z_i)^2\\ =K(EG-F^2-Ez_2^2-Gz_1^2-2Fz_1z_2)\end{aligned}$$

where Γ^i_{jk} are the Christoffel symbols, z_i is the first derivative of z with respect to the i-th variable and z_{jk} is the second derivative with respect to the j-th and k-th variable by calling u the first variable and v the second variable.

This paper is organized as follows. In §1 we modify the equation (0.3) and construct an approximate solution for the modified equation. In order to get a solution of the modified equation by compensating the approximate solution, we set up the Nash implicit function theorem in §2. Then the subsequent sections are devoted to deriving the estimates related to the Nash implicit function theorem. The most hardest part is to derive the evolutional type energy inequality for the linearized equation which is an effectively hyperbolic equation. The method is to rewrite the linearized equation into a singular, hyperbolic, pseudodifferential system diagonal in its principal part by Morimoto [5] diagonalization process. This is done in §3, and we also derive the desired estimate for the linearized equation from this system and prove the other estimates. Finally, in §4 we remark about the other cases and problem which can be treated by similar methods.

§1. Modified Darboux equation and its approximate solution.

Following the usual notations used in the theory of partial differential equations, we hereafter rewrite the previous independent variables u, v by x, y (sometimes t, x) and the previous dependent variable z by u, respectively. The rest of the symbols are kept unchanged. We also use the convention of the Einstein summation rule and the convenient way of writing derivatives of u such as $u_{11}=u_{xx}$, $u_{12}=u_{xy}$, etc. In this notation, (0.3) becomes

$$(1.1)\qquad F(u)=g$$

where

$$(1.2)\qquad F(u)=(u_{11}-\Gamma^i_{11}u_i)(u_{22}-\Gamma^i_{22}u_i)-(u_{12}-\Gamma^i_{12}u_i)^2 + K(Eu_2^2+Gu_1^2-2Fu_1u_2)\ ,$$

$$(1.3)\qquad g=K(EG-F^2)\ .$$

From (ii), there exist $\delta_0>0$ and $d_0>0$ such that

$$(1.4)\qquad d_0^{-1}(x^2+y^2)\leqq -K(x, y)\leqq d_0(x^2+y^2)$$

on $x^2+y^2\leqq\delta_0^2$. Now take δ $(0<\delta<\delta_0)$ sufficiently small and $\gamma(\theta)\in C_0^\infty(\boldsymbol{R}^1)$ with the properties: $0\leqq\gamma\leqq 1$ and

$$(1.5)\qquad \gamma(\theta)=\begin{cases}1 & (|\theta|\leqq\delta/2)\\ 0 & (|\theta|\geqq\delta)\ .\end{cases}$$

Set

$$(1.6)\qquad \gamma_1(x, y)=\gamma((x^2+y^2)^{1/2})\ ,$$

$$(1.7)\qquad \gamma_2(x, y)=1-\gamma_1(x, y)\ .$$

Then, we modify the equation (1.1) to

$$(1.8)\qquad \tilde{F}(u)=\tilde{g}$$

where

$$(1.9)\qquad \tilde{F}(u)=\gamma_1\{(u_{11}-\Gamma^i_{11}u_i)(u_{22}-\Gamma^i_{22}u_i)-(u_{12}-\Gamma^i_{12}u_i)^2\} +\gamma_2(u_{11}-\Gamma^i_{11}u_i)-\gamma_2(u_{22}-\Gamma^i_{22}u_i)+\gamma_1K(Eu_2^2+Gu_1^2-2Fu_1u_2)\ ,$$

$$(1.10)\qquad \tilde{g}=\gamma_1 g-\gamma_2\ .$$

Note (1.1) and (1.10) are equivalent on $(x^2+y^2)^{1/2}\leqq\delta/2$.

Next we proceed to construct an approximate solution for the equation (1.8).

LEMMA 1.1. *We can construct an approximate solution $u_0(x, y)\in C^\infty(\boldsymbol{R}^2)$ with the following property*:

$$(1.11)\qquad x^{-4}(\tilde{F}(u)-\tilde{g}),\ x^{-2}(u_0(x, y)-c_0y^2)\in B^\infty(\boldsymbol{R}^2)$$

where $c_0>0$ and, for any $S\subset\boldsymbol{R}^n$ $(n=1$ or $2)$, $B^\infty(S)$ is the set of C^∞ functions defined on S with bounded derivatives.

PROOF. If we seek $u_0(x, y)$ in the form

$$(1.12)\qquad u_0(x, y)=c_0y^2+\sum_{j=2}^5\phi_j(y)x^j\ ,\qquad \phi_j(y)\in B^\infty(\boldsymbol{R})\quad (2\leqq j\leqq 5)\ ,$$

we only have to check the first property $(1.11)_1$ of (1.12) by taking $\phi_j(y)$ $(2\leqq j\leqq 5)$ appropriately. In order to determine $\phi_j(y)$ $(2\leqq j\leqq 5)$, expand the coefficients and $\tilde{g}$ of (1.8) into Taylor series around $x=0$, and equate the coefficients of x^j. Then we get the recursion formula for ϕ_{j+2} $(0\leqq j\leqq 3)$. In this recursion formula, the coefficient of ϕ_{j+2} is

$$(j+2)(j+1)\{2\gamma_1(0, y)(c_0-y^2\Gamma^2_{22}(0, y))+\gamma_2(0, y)\} ,$$

which is bounded from below by $(j+2)(j+1)\min(c_0, 1)$ if we take $\delta>0$ small enough. Hence we can determine ϕ_{j+2} $(0\leqq j\leqq 3)$ so that (1.12) satisfies $(1.11)_1$. Q.E.D.

§ 2. Nash implicit function theorem.

Set

$$\Phi(u)=\tilde{F}(u+u_0)-\tilde{F}(u_0) , \tag{2.1}$$

$$f=\tilde{g}-\tilde{F}(u_0) , \tag{2.2}$$

and consider the equation

$$\Phi(u)=f . \tag{2.3}$$

Note that $u+u_0$ is a solution of (1.8) if u is a solution of (2.3).

For each m in the set $\boldsymbol{Z}_+$ of nonnegative integers, let E_m be the completion of $\{u(x, y)\in C^\infty([-T, T]\times \boldsymbol{R}^1_y);\ |u|_m<\infty\}$ with respect to the norm $|u|_m=\sup_{0<|x|\leqq T}\sum_{j=0}^{3}|x|^{-5+j}\|(\partial/\partial x)^j u(x, \cdot)\|_{m+2-j+m^*}$, and F_m be the completion of $\{u(x, y)\in C^\infty([-T, T]\times \boldsymbol{R}^1_y);\ \|u\|_m<\infty\}$ with respect to the norm $\|u\|_m=\sup_{0<|x|\leqq T}\sum_{j=0}^{1}|x|^{-3+j}\|(\partial/\partial x)^j u(x, \cdot)\|_{m+m^*-j}$. Here m^* is any fixed number, $m^*\geqq 2$, $\|u(x, y)\|_s$ denotes the Sobolev norm of order s in $\boldsymbol{R}^1_y$, and $T>0$ will be specified later. Then, E_m $(m\in \boldsymbol{R})$ and F_m $(m\in \boldsymbol{R})$ are the scales of Banach spaces with smoothing operators.

We also denote the closed unit ball in E_m by D_m, and for any $r>0$, $D'_m(r)$ denotes the ball of radius r centered at 0 in F_m, and set $E_\infty=\bigcap_m E_m$, $F_\infty=\bigcap_m F_m$.

NASH THEOREM (cf. [1]). *Let $\Phi: D_0\to F_0$ be a map with the following properties:*

(A) $\Phi(0)=0$.

(B) *For any $m\in \boldsymbol{Z}_+$, $\Phi: D_0\cap E_m\to F_m$ is twice Fréchet differentiable.*

(C) *For any $m\in \boldsymbol{Z}_+$, $u\in D_0\cap E_m$, v, $w\in E_m$,*

$$\|\Phi'(u)v\|_m\leqq C_m(|v|_m+|u|_m|v|_0) , \tag{2.4}$$

$$\|\Phi''(u)(v, w)\|_m \leqq C'_m(1+|u|_m)|v|_m|w|_m\ , \tag{2.5}$$

where C_m, C'_m *are independent of* u, v, w.

(D) *There exists* $\alpha \in Z_+$ *such that, for any* $\alpha \leqq m \in Z_+$, $u \in D_\alpha \cap E_\infty$, *there exists a right inverse* $Q(u)$ *of* $\Phi'(u)$ *which maps* F_m *into* $E_{m-\alpha}$ *and satisfies the estimate*:

$$|Q(u)h|_{m-\alpha} \leqq K_m(\|h\|_m + |u|_m\|h\|_\alpha) \qquad (m \in Z_+,\ m \geqq \alpha,\ u \in D_\alpha \cap E_\infty,\ h \in F_m)\ , \tag{2.6}$$

where K_m *is independent of* u, v.

Then, there exist $\rho>0$ *and a mapping* $\Psi: D'_{\eta\alpha}(\rho) \cap F_\infty \to D_\alpha \cap E_\infty$ *such that*

$$\Phi\cdot\Psi(f)=f \qquad (f \in D'_{\eta\alpha}(\rho) \cap F_\infty,\ \eta \geqq 11)\ . \tag{2.7}$$

REMARK. The validities of the estimates (2.3) and (2.4) follow from the well known inequality:

$$\|f\cdot g\|_m \leqq C_m(\|f\|_m\|g\|_1 + \|f\|_1\|g\|_m) \tag{2.8}$$

for any positive integer m and f, g in the m-th order Sobolev space $H^m(R^1)$ with its norm $\|\cdot\|_m$.

Since (A) and (B) clearly hold, we only have to prove (D).

§ 3. Evolutional type energy inequality for the linearized equation.

Hereafter, we take $\alpha \in Z_+$ appropriately large and $T>0$ appropriately small, and let $u \in D_\alpha \cap E_m$. Then, we can easily see the principal part $L_u(v)$ of $\Phi'(u)v$ is given by

$$L_u(v)=\{\gamma_1(w_{22}-\Gamma^i_{22}w_i)+\gamma_2\}v_{11}+\{\gamma_1(w_{11}-\Gamma^i_{11}w_i)-\gamma_2\}v_{22}-2\gamma_1(w_{12}-\Gamma^i_{12}w_i)v_{12} \tag{3.1}$$

where

$$w=u+u_0\ . \tag{3.2}$$

Here, from Lemma 1.1,

$$|\gamma_1(w_{22}-\Gamma^i_{22}w_i)+\gamma_2| \geqq A_0>0 \qquad (u \in D_\alpha \cap E_\infty)$$

for some constant A_0 if $\delta>0$ is small enough. Moreover, the discriminant $D(u)$ of (3.1) is

$$\begin{aligned} D(u) &= -\gamma_1\tilde{F}(w)+\gamma_2^2+\gamma_1^2K(Ew_2^2+Gw_1^2-2Fw_1w_2) \\ &= -\gamma_1\tilde{F}(w)+\gamma_2^2+\gamma_1^2O^3\ , \end{aligned}$$

where, for any $l \in Z_+$, O^l denotes a function $h \in C^\infty([-T, T] \times R_y^1)$ such that $h(x, y)/r^l$ is bounded as $r = (x^2+y^2)^{1/2} \to 0$. Since $\tilde{F}(w) = \gamma_1 K(EG-F^2) - \gamma_2 + O^3$ by (1.3), (1.10) and Lemma 1.1, we have

$$D(u) = -\gamma_1^2 K(EG-F^2) + \gamma_2^2 + \gamma_1\gamma_2 + \gamma_1 O^3 . \tag{3.3}$$

Hence taking $\delta > 0$ small enough, there exists a constant $d_1 > 0$ such that

$$d_1^{-1}(x^2+y^2) \leqq D(u) \leqq d_1 \min(x^2+y^2, 1) \qquad (x, y \in R) \tag{3.4}$$

for any $u \in D_\alpha \cap E_\infty$. Therefore the operator $\Phi'(u)$ is hyperbolic with respect to the time variable x. Moreover, $\Phi'(u)$ is an effectively hyperbolic operator (see [7] for the definition of effectively hyperbolic operators). To see this we quote the following remark due to Nishitani [7].

REMARK 3.1. Let P be a second order partial differential operator with principal part

$$P_2 = AD_t^2 + 2BD_tD_x + CD_x^2 , \tag{3.5}$$

where $D_t = (-1)^{1/2}\partial/\partial t$, $D_x = (-1)^{1/2}\partial/\partial x$ and $A(t, x)$, $B(t, x)$, $C(t, x) \in B^\infty([-T, T] \times R_x^1)$. Assume $|A(t, x)| \geqq A_0 > 0$ and $B^2 - AC \geqq 0$ on $[-T, T] \times R_x^1$ for some constant A_0, and $B^2 - AC = 0$ if and only if $(t, x) = 0$. Then the necessary and sufficient condition for the operator to be effectively hyperbolic is

$$((\partial/\partial t) + (B/A)(\partial/\partial x))^2 (B^2 - AC) > 0 \quad \text{at} \quad (t, x) = 0 . \tag{3.6}$$

Writing $L_u(v)$ in the form (3.5), B becomes

$$B = -\gamma_1\{(u+u_0)_{12} - \Gamma_{12}^i(u+u_0)_i\} = O^1 \tag{3.7}$$

and (3.4) yields $B^2 - AC = O^2$. Hence, the condition (3.6) reduces to

$$(\partial/\partial t)^2 (B^2 - AC) > 0 \quad \text{at} \quad (t, x) = 0 \tag{3.8}$$

in our case, which is valid because of (3.4).

Although the well-posedness of the effectively hyperbolic operator is well known and several types of energy inequalities have been derived, no one has mentioned about the evolutional type energy inequality which is crucial in our proof. (Prof. Nishitani told me that it is possible to derive our energy inequality from his method. However, to see this, it is necessary to rewrite the whole proof of [7].) So we will present a simple method of deriving an evolutional type energy inequality for our special type of effectively hyperbolic operator. We also note that this method is also effective for constructing a fundamental solution for a certain degenerate hyperbolic operator discussed in [8].

Our task is to prove the following lemma.

LEMMA 3.1. *For given* $s_0 \geqq m^*+2$, *there exist* $T>0$, $\alpha' \in \boldsymbol{Z}_+$, $M>0$ *such that, for any* $u \in D_{\alpha'} \cap E_\infty$, $|s| \leqq s_0$, $h \in C^0([-T, T], H^{s+M}(\boldsymbol{R}^1))$, *the Cauchy problem*:

$$\Phi'(u)v = h \tag{3.9}$$

with zero Cauchy data on $t=0$ *admits a unique solution* $v \in \bigcap_{j=0}^2 C^j([-T, T], H^{s-j}(\boldsymbol{R}^1))$ *which satisfies*

$$\|v(t, \cdot)\|_s + \|(\partial/\partial t)v(t, \cdot)\|_{s-1} \leqq C\left|\int_0^t \|h(t', \cdot)\|_{s+M} dt'\right| \quad (|t| \leqq T,\ |s| \leqq s_0)\ , \tag{3.10}$$

where $C^j([-T, T], H^s(\boldsymbol{R}^1))$ *denotes the set of* j *times continuously differentiable* $H^s(\boldsymbol{R}^1)$ *valued functions defined on* $[-T, T]$ *and* C *is a constant independent of* u. *Moreover, if* $h \in \bigcap_{j=0}^1 C^j([-T, T], H^{s+M-j}(\boldsymbol{R}^1))$, *then the above* v *belongs to* $\bigcap_{j=0}^3 C^j([-T, T], H^{s-j}(\boldsymbol{R}^1))$ *and satisfies*

$$|v|_m \leqq C\|h\|_{m+M+3} \quad (|s| \leqq s_0) \tag{3.11}$$

where $s = m+m^*+2$ *and* C *is a constant independent of* u. *Moreover, denoting this* v *by* $Q(u)h$, *the condition* (D) *of Nash Theorem holds for this* $Q(u)h$.

REMARK 3.2. From Lemma 3.1, we can prove our Theorem in the following manner. Since the estimate (2.6) is a direct consequence of Lemma 3.1, the Nash implicit function theorem tells the existence of a solution $u+u_0$ of (1.1) in the neighborhood of the origin $(x, y)=0$ which is of C^3 class in x and C^∞ class in y and satisfies $u/x^3 = O^0$. Here, we have utilized the one extra factor x of (2.2) $f = \tilde{g} - \tilde{F}(u_0)$ to make its norm $\|f\|_{\eta\alpha}$ small so that $f \in D'_{\eta\alpha}(\rho) \cap F_\infty$. Then, reminding $u_0 = C_0 y^2 + x^2 O^0$, the coefficient $(u+u_0)_{yy} - \Gamma_{22}^i (u+u_0)_i$ of $(u+u_0)_{xx}$ in (1.1) is nonzero near the origin. Hence, using (1.1), $u+u_0$ is of C^∞ class in x, y near the origin. Thus Theorem is proved.

PROOF OF LEMMA 3.1. Whereas the proof of the estimate (3.10) is divided into several lemmas, the estimate (3.11) can be easily proved by assuming (3.10). So we do this first. Combining (3.1) and $v(t, x) = \int_0^t (\partial/\partial t')v(t', x)dt'$, we have

$$\|v(t, \cdot)\|_s \leqq C\left|\int_0^t \int_0^{t'} \|h(t'', \cdot)\|_{s+M+1} dt' dt''\right|\ , \tag{3.12}$$

where $C(s)$ is a general constant independent of u. Also, we have from (3.9) and (3.10),

$$\|(\partial/\partial t)^2 v(t,\cdot)\|_{s-2} \leqq C\left\{\|h(t,\cdot)\|_{s-2} + \left|\int_0^t \|h(t',\cdot)\|_{s+M} dt'\right|\right\}, \tag{3.13}$$

$$\|(\partial/\partial t)^3 v(t,\cdot)\|_{s-3} \leqq C\left\{\|(\partial/\partial t)h(t,\cdot)\|_{s-3} + \|h(t,\cdot)\|_{s-2} + \left|\int_0^t \|h(t',\cdot)\|_{s+M} dt'\right|\right\}. \tag{3.14}$$

Then, (3.11) easily follows from (3.12)~(3.14). Moreover the validity of (D) follows from the rearrangement argument given in [2; pp 739–740].

The rest of this section is devoted to proving (3.10).

DEFINITION 3.1. (i) Define the weight function $\nu(t, x, \xi)$ by

$$\nu(t, x, \xi) = \{t^2 + \rho(x)^2 + \langle\xi\rangle^{-1}\chi((t^2+\rho(x)^2)\langle\xi\rangle)\}^{1/2} \quad (x, \xi \in \boldsymbol{R}, |t| \leqq T), \tag{3.15}$$

$$\rho(x) = \{\gamma(x)x^2 + (1-\gamma(x))\}^{1/2}, \tag{3.16}$$

where $\langle\xi\rangle = (1+\xi^2)^{1/2}$, γ is the one already defined in §1 and $\chi \in C_0^\infty(\boldsymbol{R}^1)$ such that $0 \leqq \chi \leqq 1$, $\chi(\theta) = 1$ ($|\theta| \leqq 1$), $=0$ ($|\theta| \geqq 2$).

(ii) For $m, k \in \boldsymbol{R}$, $a(t, x, \xi) \in S^{\pm}[m, k]$ if the following properties hold.

(a) $a(t, x, \xi) \in C^0(\{0 \leqq \pm t \leqq \pm T\} \times \boldsymbol{R}_x^1 \times \boldsymbol{R}_\xi^1)$ and $a(t, x, \xi) \in C^\infty(\boldsymbol{R}_x^1 \times \boldsymbol{R}_\xi^1)$ for each fixed t ($0 \leqq \pm t \leqq \pm T$).

(b) For any $\alpha, \beta \in \boldsymbol{Z}_+$, there exists a constant $C_{\alpha,\beta} > 0$ such that

$$|D_\xi^\alpha D_x^\beta a(t, x, \xi)| \leqq C_{\alpha,\beta}\langle\xi\rangle^{m-\alpha}\nu^{k-\beta} \quad (x, \xi \in \boldsymbol{R}^1, 0 \leqq \pm t \leqq \pm T).$$

(iii) We also denote the class of operators with symbols in $S^{\pm}[m, k]$ by $S^{\pm}[m, k]$.

In order to simplify the notations, we will restrict our subsequent argument to the case $0 \leqq t \leqq T$ and simply write $S[m, k] = S^+[m, k]$.

We first desingularize the singularities of the characteristic roots of $\Phi'(u)$ by modifying its principal symbol. For this purpose, set

$$L_u(v) = AD_t^2 v + 2BD_t D_x v + CD_x^2 v, \tag{3.17}$$

$$b(t, x) = (B^2 - AC)/A^2, \tag{3.18}$$

$$T = \{b(t, x) + \langle\xi\rangle^{-1}\chi(b(t, x)\langle\xi\rangle)\}^{1/2}, \tag{3.19}$$

$$\lambda_j = -B\xi/A + (-1)^{j-1}\{\chi(T\xi) + (1-\chi(T\xi))|T\xi|\}, \tag{3.20}$$

$$\Lambda_j = D_t - \lambda_j(t, x, D_x) \quad (j = 1, 2) \tag{3.21}$$

and consider

(3.22) $$\tilde{P}_2 = \Lambda_1 \circ \Lambda_2$$

as the principal part of

(3.23) $$P \equiv A^{-1}\Phi'(u) .$$

LEMMA 3.2. $\lambda_j \in S[1, 1]$, $(\partial/\partial t)\lambda_j \in S[1, 0]$ $(j=1, 2)$.

PROOF. If we note ν, T can estimate each other and the same is true for ν, $\langle\xi\rangle^{1/2}$ on supp $\chi(b\langle\xi\rangle)$, it is not hard to prove $(\partial/\partial t)^j T \in S[0, 1-j]$ $(j=0, 1)$. Next write λ_j in the form:

(3.24) $$\lambda_j = -B\xi/A + \tilde{\lambda}_j(T\xi) ,$$

where

(3.25) $$\lambda_j(\eta) = (-1)^{j-1}\{\chi(\eta) + (1-\chi(\eta))|\eta|\}$$

belongs to the Hörmander class $S^1_{1,0}$. Then the assertion easily follows from $(\partial/\partial t)^j T \in S[0, 1-j]$ $(j=0, 1)$ and

(3.26) $$\langle T\xi\rangle^{-1} \leqq C\langle\xi\rangle^{-1}\nu^{-1}$$

for some constant C. Q.E.D.

LEMMA 3.3. *For $j \neq k$ $(1 \leqq j, k \leqq 2)$, there exist a_{jk}, $b_{jk} \in S[-1, -1]$ such that $a_{jk} \circ (\lambda_j - \lambda_k) = I + b_{jk}$.*

PROOF. This can be easily proved by noting (3.24), (3.26) and $|\tilde{\lambda}_j(\eta) - \tilde{\lambda}_k(\eta)| \geqq C\langle\eta\rangle$ for some constant $C>0$. Q.E.D.

LEMMA 3.4. *For $j \neq k$ $(1 \leqq j, k \leqq 2)$,*

(3.27) $$[\Lambda_j, \Lambda_k] = p_{jk} \circ \Lambda_j + q_{jk} \circ \Lambda_k + r_{jk}$$

for some p_{jk}, q_{jk}, $r_{jk} \in S[0, -1]$.

PROOF. From (3.21),

(3.28) $$\begin{aligned}[\Lambda_j, \Lambda_k] &= (D_t\lambda_j - D_t\lambda_k) + [\lambda_j, \lambda_k] \\ &\equiv A + B .\end{aligned}$$

Here, by applying the usual product formula of pseudodifferential operators, we have $A \in S[1, 0]$, $B \in S[1, 1]$. From Lemma 3.3, $I = a_{jk} \circ (\Lambda_j - \Lambda_k) - b_{jk}$ for some a_{jk}, $b_{jk} \in S[-1, -1]$. Hence

(3.29) $$\begin{cases} A = A \circ a_{jk} \circ (\Lambda_j - \Lambda_k) - A \circ b_{jk} , \\ B = B \circ a_{jk} \circ (\Lambda_j - \Lambda_k) - B \circ b_{jk} . \end{cases}$$

From (3.28) and (3.29),

$$[\Lambda_j, \Lambda_k]=p_{jk}\circ\Lambda_j+q_{jk}\circ\Lambda_k+r_{jk} \tag{3.30}$$

where

$$\begin{cases} p_{jk}=A\circ a_{jk}+B\circ a_{jk}\in S[0, -1] , \\ q_{jk}=-A\circ a_{jk}-B\circ a_{jk}\in S[0, -1] , \\ r_{jk}=-A\circ b_{jk}-B\circ b_{jk}\in S[0, -1] . \end{cases} \tag{3.31}$$

Q.E.D.

LEMMA 3.5. *Let $k\in \boldsymbol{R}$ and $c\in S[1, k]$. Then,*

$$c=c_{11}\Lambda_1+c_{12}\Lambda_2+c_0 \tag{3.32}$$

for some $c_{11}, c_{12}, c_0\in S[0, k-1]$,

PROOF. This easily follows from Lemma 3.3. Q.E.D.

LEMMA 3.6. *If we denote the principal part of P by P_2, we have*

$$P_2-\tilde{P}_2\in S[0, -2] . \tag{3.33}$$

PROOF. By definitions,

$$P_2=D_t^2+2(B/A)D_tD_x+(C/A)D_x^2 , \tag{3.34}$$

$$\begin{aligned}\tilde{P}_2&=\{D_t^2-(\lambda_1+\lambda_2)D_t+\lambda_1\lambda_2\}+\{\lambda_1\circ\lambda_2-D_t\lambda_2\}\\ &\equiv\tilde{P}_{21}+\tilde{P}_{22} .\end{aligned} \tag{3.35}$$

Here, we have

$$-(\lambda_1+\lambda_2)=2B\xi/A \tag{3.36}$$

from (3.20), and

$$\tilde{P}_{22}\in S[0, 1]\subset S[0, -2] \tag{3.37}$$

from the product formula of pseudodifferential operators. Moreover, since $|T\xi|\geqq 2$ for $|\xi|^{1/2}\geqq 2$,

$$\lambda_1\lambda_2=C|\xi|^2/A-\langle\xi\rangle^{-1}\chi(b\langle\xi\rangle)|\xi|^2 \quad (|\xi|^{1/2}\geqq 2) \tag{3.38}$$

follows from (3.18)~(3.20). Reminding ν and $\langle\xi\rangle^{-1/2}$ are equivalent on $\operatorname{supp}\chi(b\langle\xi\rangle)$,

$$\langle\xi\rangle^{-1}\chi(b\langle\xi\rangle)|\xi|^2\in S[0, -2] . \tag{3.39}$$

Hence $P_2-\tilde{P}_{21}\in S[0, -2]$ which immediately entails the assertion. Q.E.D.

LEMMA 3.7.

$$R \equiv P - P_2 = \sum_{j=1}^{2} \gamma'_{1j} \Lambda_j + \gamma'_0 \tag{3.40}$$

for some γ'_{1j}, $\gamma'_0 \in S[0, 0] \subset S[0, -1]$ $(j=1, 2)$.

PROOF. From (1.9) and (2.1), we have

$$\begin{aligned} AR(v) \equiv \gamma_1[& -\{(u+u_0)_{22} - \Gamma^i_{22}(u+u_0)_i\}\Gamma^i_{11} v_i \\ & -\{(u+u_0)_{11} - \Gamma^i_{11}(u+u_0)_i\}\Gamma^i_{22} v_i \\ & +2\{(u+u_0)_{12} - \Gamma^i_{12}(u+u_0)_i\}\Gamma^i_{12} v_i] \\ & -\gamma_2 \Gamma^i_{11} v_i + \gamma_2 \Gamma^i_{22} v_i \\ & +2\gamma_1 K\{E(u+u_0)_2 v_2 + G(u+u_0)_1 v_1 \\ & \qquad - F(u+u_0)_2 v_1 - F(u+u_0)_1 v_2\} . \end{aligned} \tag{3.41}$$

Then, writing R in the form

$$R = q_1(t, x) D_t + q_2(t, x) D_x , \tag{3.42}$$

we have

$$q_1, \ q_2 \in S[0, 0] . \tag{3.43}$$

Applying Lemma 3.5 to $q_2 D_x$ and $q_1 \lambda_1 \in S[1, 1]$ of $q_1 D_t = q_1 \Lambda_1 + q_1 \lambda_1$,

$$q_1 D_t = \sum_{j=1}^{2} h_{1j} \Lambda_j + h_0 , \tag{3.44}$$

$$q_2 D_x = \sum_{j=1}^{2} k_{1j} \Lambda_j + k_0 \tag{3.45}$$

for some h_{1j}, $h_0 \in S[0, 0]$ $(j=1, 2)$, k_{1j}, $k_0 \in S[0, -1]$ $(j=1, 2)$. Now the assertion is almost clear from (3.44), (3.45). Q.E.D.

Summing up what we have proved in Lemmas 3.2~3.7, we have the following.

PROPOSITION 3.1. λ_j $(j=1, 2)$ *defined* by (3.20) *are real valued functions and satisfy* $\lambda_j \in S[1, 1]$, $(\partial/\partial t)\lambda_j \in S[1, 0]$, *and* Λ_j $(j=1, 2)$ *defined by* (3.21) *satisfy* (3.27) $[\Lambda_1, \Lambda_2] = p_{12}\Lambda_1 + q_{12}\Lambda_2 + r_{12}$, *where* p_{12}, $q_{12} \in S[0, -1]$, $r_{12} \in S[0, -2]$. *Also, the operator* P *defined by* (3.23) *has the expression*:

$$P = \Lambda_1 \circ \Lambda_2 + \sum_{j=1}^{2} \gamma_{1j} \Lambda_j + \gamma_0 , \tag{3.46}$$

where $\gamma_{1j} \in S[0, -1]$ $(j=1, 2)$, $\gamma_0 \in S[0, -2]$.

REMARK 3.3. (i) An operator P with the above properties is a variant of ν-involutively hyperbolic operator introduced by Kumano-go [3].

(ii) The above procedure of rewriting the operator P into the form (3.46) is a slight modification of the method due to Yamamoto [8].

Next we rewrite (3.46) into an equivalent system by Morimoto's diagonalization process (cf. [5]). Namely, set

$$v_0=\nu^{-1}v\ ,\quad v_1=\Lambda_1 v\ ,\quad v_2=\Lambda_2 v\ , \tag{3.47}$$

$$V={}^t[v_0,\ v_1,\ v_2]\ , \tag{3.48}$$

where the symbol t denotes the transposition. Then, we have the following lemma.

LEMMA 3.8. *The equation* (3.46) *is equivalent to*

$$LV=H\ , \tag{3.49}$$

where

$$L=D_tI+D(t)+B(t)\nu(t)^{-1}\ , \tag{3.50}$$

$$D(t)=-\begin{bmatrix}\nu^{-1}\lambda_1\nu, & 0, & 0\\ 0, & \lambda_2, & 0\\ 0, & 0, & \lambda_2\end{bmatrix}\in S[1,1]\ , \tag{3.51}$$

$$B(t)\in S[0,0]\ ,\quad H=A^{-1}\,{}^t[0,\ h,\ h] \tag{3.52}$$

and I is the identity matrix.

PROOF. From Proposition 3.1,

$$D_tv_0=(\nu^{-1}\lambda_1\nu)v_0+\{(D_t\nu^{-1})\nu\}v_0+\nu^{-1}v_1\ , \tag{3.53}$$

$$\begin{aligned}D_tv_1&=(\Lambda_1\circ\Lambda_2-[\Lambda_1,\ \Lambda_2])v+\lambda_2v_1\\ &=\lambda_2v_1-\{(\gamma_0+r_{12})\nu\}v_0-(\gamma_{11}+p_{12})v_1-(\gamma_{12}+q_{12})v_2+A^{-1}h\ ,\end{aligned} \tag{3.54}$$

$$\begin{aligned}D_tv_2&=\Lambda_1\circ\Lambda_2v+\lambda_1v_2\\ &=\lambda_1v_2-(\gamma_0\nu)v_0-\gamma_{11}v_1-\gamma_{12}v_2+A^{-1}h\ .\end{aligned} \tag{3.55}$$

These equations immediately entail the assertion. Q.E.D.

LEMMA 3.9. *For given $s_0>0$, there exist $T>0$, $\alpha''\in Z_+$, $M'>0$ such that, for any $u\in D_{\alpha''}\cap E_\infty$, s ($|s|\leqq s_0$), $H\in C^0([-T,\ T],\ H^{s+M'}(R^1))$, the*

Cauchy problem:

$$LV=H \tag{3.56}$$

with zero Cauchy data on $t=0$ *admits a unique solution* $V\in\bigcap_{j=0}^{1}C^{j}([-T,T], H^{s-j}(\boldsymbol{R}^1))$ *which satisfies*

$$\|V(t,\cdot)\|_s\leqq C'(s)\left|\int_0^t\|H(t',\cdot)\|_{s+M'}dt'\right|\qquad(|t|\leqq T,\ |s|\leqq s_0)\ , \tag{3.57}$$

where C' *is a constant independent of* u.

PROOF. As before we only illustrate the case $0\leqq t\leqq T$. Suppose $V\in\bigcap_{j=0}^{1}C^{j}([0,T], H^{s-j}(\boldsymbol{R}^1))$, $V|_{t=0}=0$, then we can estimate V in terms of $H=LV$ as follows. Namely, take $\theta>0$, $\sigma>0$ appropriately large and estimate $(d/dt)\|W(t)\|_s^2$, where

$$W(t)\equiv W(t,\cdot)=e^{-\theta t}\tilde{\nu}(t)^{-\sigma}V(t,\cdot)\ , \tag{3.58}$$

$$\tilde{\nu}(t)=t+\langle\xi\rangle^{-1/2}\ . \tag{3.59}$$

Using (3.56),

$$\begin{aligned}2\|W(t)\|_s(d/dt)\|W(t)\|_s&=(d/dt)\|W(t)\|_s^2=2\,\mathrm{Re}((d/dt)W(t),W(t))_s\\&=-2\,\mathrm{Re}((\theta+\sigma\tilde{\nu}(t)^{-1})W(t),W(t))_s+2\,\mathrm{Re}(e^{-\theta t}\tilde{\nu}(t)^{-\sigma}(d/dt)V(t),W(t))_s\\&=-2\theta\|W(t)\|_s^2-2\sigma\|\tilde{\nu}(t)^{-1/2}W(t)\|_s^2+2\,\mathrm{Im}(e^{-\theta t}\tilde{\nu}(t)^{-\sigma}(D(t)\\&\qquad+B(t)\nu(t)^{-1})V(t),W(t))_s+2\,\mathrm{Im}(e^{-\theta t}\tilde{\nu}(t)^{-\sigma}H(t),W(t))_s\\&\leqq-2\theta\|W(t)\|_s^2-2\sigma\|\tilde{\nu}(t)^{-1/2}W(t)\|_s^2+2\|e^{-\theta t}\tilde{\nu}(t)^{-\sigma}H(t)\|_s\|W(t)\|_s+2Z\ ,\end{aligned} \tag{3.60}$$

where

$$Z=\mathrm{Im}(e^{-\theta t}\tilde{\nu}(t)^{-\sigma}(D(t)+B(t)\nu(t)^{-1})V(t),W(t))_s \tag{3.61}$$

and $(\ ,\ )_s$ denotes the inner product of the Hilbert space $H^s(\boldsymbol{R}^1)$. Since

$$[\tilde{\nu}(t)^{-\sigma},D(t)]=\tilde{D}(t)\tilde{\nu}(t)^{-\sigma}\ , \tag{3.62}$$

$$[\tilde{\nu}(t)^{-\sigma},B(t)\nu(t)^{-1}]=\tilde{B}(t)\tilde{\nu}(t)^{-\sigma} \tag{3.63}$$

for some $\tilde{D}(t)\in S[0,0]$, $\tilde{B}(t)\in S[-1,2]\subset S[0,0]$, and

$$Z\leqq C_1(s)\|W(t)\|_s^2+\mathrm{Im}((D(t)+B(t)\nu(t)^{-1})W(t),W(t))_s \tag{3.64}$$

for some constant $C_1(s)>0$. If Λ^s denotes the pseudodifferential operator with symbol $(1+\xi^2)^{s/2}$,

$$\text{(3.65)}\quad \operatorname{Im}(D(t)W(t),\, W(t))_s = -(-1)^{1/2}/2((D(t)-D(t)^*)\Lambda^s W(t),\, \Lambda^s W(t))_0 + \operatorname{Im}([\Lambda^s,\, D(t)]W(t),\, \Lambda^s W(t))_0\,.$$

Here we note $D(t)\in S[1,\,1]$ is a real valued symbol and $[\Lambda^s,\, D(t)]\in S[s,\,0]$. Hence

$$\text{(3.66)}\quad \operatorname{Im}(D(t)W(t),\, W(t))_s \leqq C_2(s)\|W(t)\|_s^2$$

with some constant $C_2(s)>0$. As for $\operatorname{Im}(B(t)\nu(t)^{-1}W(t),\, W(t))_s$,

$$\text{(3.67)}\quad \operatorname{Im}(B(t)\nu(t)^{-1}W(t),\, W(t))_s = \operatorname{Im}(\{\tilde{\nu}(t)^{1/2}\Lambda^s B(t)\nu(t)^{-1}\tilde{\nu}(t)^{1/2}\Lambda^{-s}\}\Lambda^s\tilde{\nu}(t)^{-1/2}W(t),\, \Lambda^s\tilde{\nu}(t)^{-1/2}W(t))_0\,,$$

$$\text{(3.68)}\quad \tilde{\nu}(t)^{1/2}\Lambda^s B(t)\nu(t)^{-1}\tilde{\nu}(t)^{1/2}\Lambda^{-s} - \tilde{\nu}(t)^{1/2}B(t)\nu(t)^{-1}\tilde{\nu}(t)^{1/2}\in S[-1,\,-1]$$

and $\tilde{\nu}(t)^{1/2}B(t)\nu(t)^{-1}\tilde{\nu}(t)^{1/2}\in S[0,\,0]$. Thus, for some constant $C_3(s)>0$ and $C_4>0$,

$$\text{(3.69)}\quad \operatorname{Im}(B(t)\nu(t)^{-1}W(t),\, W(t))_s \leqq C_3(s)\|W(t)\|_s^2 + C_4\|\tilde{\nu}(t)^{-1/2}W(t)\|_s^2\,.$$

From (3.60), (3.66), (3.69),

$$\text{(3.70)}\quad \|W(t)\|_s(d/dt)\|W(t)\|_s \leqq \Big(\sum_{j=1}^{3} C_j(s)-\theta\Big)\|W(t)\|_s^2 + (C_4-\sigma)\|\tilde{\nu}(t)^{-1/2}W(t)\|_s^2 + \|W(t)\|_s\|e^{-\theta t}\tilde{\nu}(t)^{-\sigma}H(t)\|_s\,.$$

Now take θ, σ large enough so that

$$\text{(3.71)}\quad \theta \geqq \sum_{j=1}^{3} C_j(s)\,,$$

$$\text{(3.72)}\quad \sigma \geqq C_4\,.$$

Then, from (3.70),

$$\text{(3.73)}\quad (T+1)^{-\sigma}e^{-\theta T}\|V(t)\|_s \leqq \|W(t)\|_s \leqq \int_0^t \|e^{-\theta t'}\tilde{\nu}(t')^{-\sigma}H(t')\|_s dt' \leqq C_5(s)\int_0^t \|H(t')\|_{s+\sigma/2}dt' \quad (0\leqq t\leqq T)$$

for some constant $C_5(s)>0$. Therefore, there exists a constant $C_6(T)>0$ such that, for any s ($|s|\leqq s_0$),

$$\text{(3.74)}\quad \|V(t)\|_s \leqq C_6(s,\, T)\int_0^t \|H(t')\|_{s+\sigma/2}dt' \quad (0\leqq t\leqq T)\,.$$

We remark here $C_6(T)$ does not depend on $u\in D_{\alpha''}\cap E_\infty$. This is because that, for any s ($|s|\leqq s_0$), the operator norm of a zero-th order pseudodifferential operator on $H^s(R^1)$ depends only on the bound of the derivatives of its symbol up to finite order (say k). Hence, we must take α'' large enough so that

$$\alpha''+m^*\geqq k+1 . \tag{3.75}$$

To complete the proof, we note that L^* has the same form as L. Then, we can prove the assertion by applying the usual duality argument (cf. [6] p. 166). Q.E.D.

Finally, we remark that Lemma 3.1 follows from Lemma 3.9. The proof is as follows. From Lemma 3.9,

$$\|v_0(t, \cdot)\|_s,\ \|v_1(t, \cdot)\|_s\leqq C'\left|\int_0^t \|h(t', \cdot)\|_{s+M'}dt'\right| \quad (0\leqq t\leqq T,\ |s|\leqq s_0) \tag{3.76}$$

for some constant $C'>0$. Reminding (3.21) and (3.47),

$$\begin{aligned}
&\|v(t, \cdot)\|_s=\|\nu v_0(t, \cdot)\|_s\leqq C''\left|\int_0^t \|h(t', \cdot)\|_{s+M'}dt'\right| ,\\
&\|D_t v(t, \cdot)\|_{s-1}=\|v_1(t, \cdot)+\lambda_1 v(t, \cdot)\|_{s-1}\leqq \|v_1(t, \cdot)\|_{s-1}+C_3\|v(t, \cdot)\|_s\\
&\quad\leqq C'\left|\int_0^t \|h(t', \cdot)\|_{s+M'-1}dt'\right|+C'C_3\left|\int_0^t \|h(t', \cdot)\|_{s+M'}dt'\right|\\
&\quad\leqq C_4\left|\int_0^t \|h(t', \cdot)\|_{s+M'}dt'\right| \quad (|t|\leqq T,\ |s|\leqq s_0)
\end{aligned}$$

for some constants C'', $C_4>0$. Thus, we have proved our claim.

§4. Some extension of our result and the related problem.

The problem of the local existence of a nonparametric surface with prescribed Gaussian curvature K is to find a C^∞ function $f(x, y)$ defined in a neighborhood of the origin such that

$$(f_{xx}f_{yy}-f_{xy}^2)/(1+|\nabla f|^2)^2=K(x, y) . \tag{4.1}$$

Let us call this "problem II" and the previous one "problem I". Suppose K has the form:

$$K=LM^l , \tag{4.2}$$

near the origin, where l is a positive integer and L, M are C^∞ functions defined in a neighborhood of the origin such that

$$L<0 ,\quad M=\operatorname{grad} M=0 ,\quad \operatorname{Hess} M>0 \tag{4.3}$$

at $(x, y)=(0, 0)$. Then, we have the following.

THEOREM. *The problems I, II are affirmative for any positive integer.*

PROOF. Although we have to alter the definition of the symbol class and the modification of the principal part of $\Phi'(u)$, the proof is essentially the same as before. So we omit the proof and only point out the necessary alterations and remarks. As for the definition of norms, we have to replace the number 5 (resp. 3) in the definition $|u|_m$ (resp. $\|u\|_m$) by $2l+3$ (resp. $2l+1$). As for the definition of the symbol class, change the definition of $\nu(t, x, \xi)$ to

$$\nu(t, x, \xi)=\{(t^2+\rho(x)^2)^l+\langle\xi\rangle^{-2l\omega}\chi((t^2+\rho(x)^2)^l\langle\xi\rangle^{2l\omega})\}^{1/(2l)}, \tag{4.4}$$

where

$$\omega=1/(l+1). \tag{4.5}$$

As for the modification of the principal part of $\Phi'(u)$, change the definitions of $b(t, x)$ and $T(t, x, \xi)$ to

$$b(t, x)=(B^2-AC)/A^2 \tag{4.6}$$

and

$$T=\{b(t, x)^l+\langle\xi\rangle^{-2l\omega}\chi(b(t, x)^l\langle\xi\rangle^{2l\omega})\}^{1/(2l)}. \tag{4.7}$$

We remark that, as the order $2l$ of the degeneracy of K becomes higher, we need the so called Levi condition for the linearized operator $\Phi'(u)$. Namely, for any second order classical pseudodifferential operator

$$P=\sum_{j=0}^{2} p_j(t, x, D_x)D_t^{2-j}, \quad p_j\sim\sum_{k=0}^{\infty} p_{jk}, \ \operatorname{ord} p_{jk}=j-k$$

with the same principal symbol as that of $\Phi'(u)$, the condition is that $p_{21}\in S[1, l-1]$. Since $\langle\xi\rangle^{-1}$ is dominated by ν^{l+1} and we can choose an approximate solution $u_0(t, x)=c_0t^2+\sum_{j\geq 2}\Phi_j(x)t^j$ such that $\Phi_j(v)=O(x^{2l+2-j})$, the Levi condition holds for both $\Phi'(u)$ and its rearrangement, if we express the Darboux equation in terms of the geodesic parallel coordinate.

Q.E.D.

REMARK. By a similar method we can also prove a similar result for the both problems when K vanishes on a hypersurface provided that some of the Christoffel symbols vanish up to certain order on this surface for the problem I.

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