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# Small Deformations of Certain Compact Manifolds of Class $L$, II

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## Introduction

In this paper, we shall construct the complete and effectively parametrized complex analytic family of small deformations of a Blanchard manifold. Let $\boldsymbol{P}^3$ be the complex projective space of dimension 3 with the system of homogeneous coordinates $[z_0: z_1: z_2: z_3]$. We define a projective line in $\boldsymbol{P}^3$ by

$$l=\{[z_0: z_1: z_2: z_3] \in \boldsymbol{P}^3: z_2=z_3=0\}$$

and $Z$ by $\boldsymbol{P}^3-l$. Let $\alpha={}^t(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$, $\beta={}^t(\beta_1, \beta_2, \beta_3, \beta_4)$ be vectors in $\boldsymbol{C}^4$ such that $\det(\alpha\, \beta\, \bar{\alpha}\, \bar{\beta})\neq 0$. Then the matrices $A_i$ $(i=1, 2, 3, 4)$ defined by

$$A_i=\begin{pmatrix} \alpha_i & \beta_i \\ -\bar{\beta}_i & \bar{\alpha}_i \end{pmatrix} \in GL(2, \boldsymbol{C})$$

satisfy the condition $\det(\sum_{i=1}^{l} r_i A_i)\neq 0$ for any $(r_1, r_2, r_3, r_4) \in \boldsymbol{R}^4-\{(0, 0, 0, 0)\}$. We put

$$G_i=\begin{pmatrix} 1 & 0 & \multicolumn{2}{c}{\multirow{2}{*}{$A_i$}} \\ 0 & 1 & & \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and denote by $g_i$ the automorphism of $Z$ determined by $G_i$ for $i=1, 2, 3, 4$. Then it is easy to see that the group of automorphisms $\Gamma$ generated by $g_i$ $(i=1, 2, 3, 4)$ acts on $Z$ properly discontinuously and without fixed points.

DEFINITION 0.1. We define a *Blanchard manifold* $X$ by the quotient space of $Z$ by $\Gamma$.

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This manifold is thought out by Blanchard [2]. The construction here is due to M. Inoue.

DEFINITION 0.2. A complex 3-fold $M$ is called of *Class L* if there exists a holomorphic open embedding of $\{[z_0: z_1: z_2: z_3] \in \boldsymbol{P}^3; |z_0|^2+|z_1|^2< r(|z_2|^2+|z_3|^2)\}$ into $M$ for some $r>0$.

PROPOSITION 0.3. *A Blanchard manifold $X$ is of Class L.*

We verify this proposition later in §2.

REMARK. $X$ is a twistor space (see [4]).

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## §1. Cohomology groups.

In the following, we denote by $\mathcal{O}$ (resp. $\Theta$, resp. $\Omega^p$) the structure sheaf (resp. the tangent sheaf, resp. the sheaf of $p$-forms). First we have

PROPOSITION 1.1. $H^3(X, \Theta)=0$.

PROOF. By the result in [5], we have

$$H^0(M, (\Omega^1)^{\otimes m_1}\otimes(\Omega^2)^{\otimes m_2}\otimes(\Omega^3)^{\otimes m_3})=0$$

for any Class $L$ manifold $M$ and any non-negative integers $m_1$, $m_2$, $m_3$ which are not all zero. Using the Kodaira-Serre duality, we have

$$H^3(X, \Theta)\cong H^0(X, \Omega^1\otimes\Omega^3)=0 . \qquad \square$$

Let $V_i$ be domains in $\boldsymbol{P}^3$ on which $z_i\neq 0$ for $i=2, 3$. We take the system of local coordinates on $V_2$ (resp. $V_3$) as

$$u_2=z_0/z_2 , \quad v_2=z_1/z_2 , \quad w_2=z_3/z_2$$
$$(\text{resp. } u_3=z_0/z_3,\ v_3=z_1/z_3,\ w_3=z_2/z_3) .$$

LEMMA 1.2. *A vector field on $\boldsymbol{P}^3$ (resp. on $Z$) is expressed on $V_2$ by the following form.*

$$\begin{aligned}&(a_1+b_1u_2+c_1v_2+d_1w_2+eu_2^2+fu_2v_2+gu_2w_2)\frac{\partial}{\partial u_2}\\ &+(a_2+b_2u_2+c_2v_2+d_2w_2+eu_2v_2+fv_2^2+gv_2w_2)\frac{\partial}{\partial v_2}\\ &+(a_3+b_3u_2+c_3v_2+d_3w_2+eu_2w_2+fv_2w_2+gw_2^2)\frac{\partial}{\partial w_2}\end{aligned}$$

*where* $a_k$, $b_k$, $c_k$, $d_k$, $e$, $f$, $g$ *are complex numbers for* $k=1, 2, 3$. *Conversely any vector field on* $V_2$ *of the above form is extended to a vector field on* $\boldsymbol{P}^3$.

PROOF. Almost the same as the proof of Lemma 2.1 in [9]. □

PROPOSITION 1.3. $\dim H^0(X, \Theta)=4$.

PROOF. An element $\eta$ of $H^0(X, \Theta)$ is identified with a $g_{i*}$-invariant vector field on $Z$, i.e., $g_{i*}\eta=\eta$ for $i=1, 2, 3, 4$. We may assume that $\eta$ is a vector field as the one in Lemma 1.2. Then it is easy to check that

$$\begin{aligned} g_{i*}\eta=&\{a_1-\alpha_i b_1+\bar{\beta}_i c_1+\alpha_i^2 e-\alpha_i\bar{\beta}_i f+\beta_i(a_3-\alpha_i b_3+\bar{\beta}_i c_3) \\ &+(b_1-2\alpha_i e+\bar{\beta}_i f+\beta_i b_3)u_2+(c_1-\alpha_i f+\beta_i c_3)v_2 \\ &+(d_1-\beta_i b_1-\bar{\alpha}_i c_1+\alpha_i\beta_i e+|\alpha_i|^2 f-\alpha_i g+\beta_i(d_3-\beta_i b_3-\bar{\alpha}_i c_3))w_2 \\ &+eu_2^2+fu_2v_2+(g-\beta_i e-\bar{\alpha}_i f)u_2w_2\}\frac{\partial}{\partial u_2} \\ &+\{a_2-\alpha_i b_2+\bar{\beta}_i c_2-\alpha_i\bar{\beta}_i e+\bar{\beta}_i^2 f+\bar{\alpha}_i(a_3-\alpha_i b_3+\bar{\beta}_i c_3) \\ &+(b_2+\bar{\beta}_i e+\bar{\alpha}_i b_3)u_2+(c_2-\alpha_i e+2\bar{\beta}_i f+\bar{\alpha}_i c_3)v_2 \\ &+(d_2-\beta_i b_2-\bar{\alpha}_i c_2-|\beta_i|^2 e-\bar{\alpha}_i\bar{\beta}_i f+\bar{\beta}_i g+\bar{\alpha}_i(d_3-\beta_i b_3-\bar{\alpha}_i c_3))w_2 \\ &+eu_2v_2+fv_2^2+(g-\beta_i e-\bar{\alpha}_i f)v_2w_2\}\frac{\partial}{\partial v_2} \\ &+\{a_3-\alpha_i b_3+\bar{\beta}_i c_3+b_3u_2+c_3v_2+(d_3-\beta_i b_3-\bar{\alpha}_i c_3-\alpha_i e+\bar{\beta}_i f)w_2 \\ &+eu_2w_2+fv_2w_2+(g-\beta_i e-\bar{\alpha}_i f)w_2^2\}\frac{\partial}{\partial w_2}\,. \end{aligned}$$

By the condition that $\det(\alpha\ \beta\ \bar{\alpha}\ \bar{\beta})\neq 0$, we have

$$a_3=b_k=c_k=d_3=e=f=g=0$$

for $k=1, 2, 3$, comparing the coefficients of $g_{i*}\eta$ with those of $\eta$. This proves that $\dim H^0(X, \Theta)=4$. □

PROPOSITION 1.4. $\dim H^2(X, \Theta)=5$.

To prove the above proposition, first we remark

PROPOSITION 1.5. *$X$ is a fibre space over $\boldsymbol{P}^1$ with the fibres of 2-dimensional complex tori.*

PROOF. Let $\tilde{p}$ be a projection of $Z$ to $\boldsymbol{P}^1$ defined by $\tilde{p}(z)=[z_2 : z_3]$.

Then we have $\tilde{p}\circ g_i(z)=\tilde{p}(z)$ for $i=1, 2, 3, 4$ and any $z\in Z$. Hence $\tilde{p}$ induces a projection $p$ of $X$ to $P^1$. The fibre $\tilde{p}^{-1}([z_2:z_3])$ is nothing other than $C^2$. Suppose $z_2\neq 0$. Then we take $u_2$, $v_2$ as a system of fibre coordinates. We see that the induced action of $g_i$ on the fibre $\tilde{p}^{-1}([z_2:z_3])$ is the one sending $(u_2, v_2)$ to $(\alpha_i+\beta_i w_2+u_2, -\bar{\beta}_i+\bar{\alpha}_i w_2+v_2)$ for $i=1, 2, 3, 4$. It is easy to see that the vectors ${}^t(\alpha_i+\beta_i w_2, -\bar{\beta}_i+\bar{\alpha}_i w_2)$ are linearly independent over $R$. In fact, suppose that we have $\sum_{i=1}^4 r_i\, {}^t(\alpha_i+\beta_i w_2, -\bar{\beta}_i+\bar{\alpha}_i w_2)=0$ with $r_i\in R$. The right-hand side of the above equation is equal to

$$\left(\sum_{i=1}^4 r_i\begin{pmatrix}\alpha_i & \beta_i\\ -\bar{\beta}_i & \bar{\alpha}_i\end{pmatrix}\right)\begin{pmatrix}1\\ w_2\end{pmatrix}.$$

Since

$$\det\left(\sum_{i=1}^4 r_i\begin{pmatrix}\alpha_i & \beta_i\\ -\bar{\beta}_i & \bar{\alpha}_i\end{pmatrix}\right)\neq 0$$

for any $(r_1, r_2, r_3, r_4)\in R^4-\{(0, 0, 0, 0)\}$, we have $(r_1, r_2, r_3, r_4)=0$. This verifies the linear independence of the four vectors over $R$. Hence the quotient space $p^{-1}([z_2:z_3])$ of the fibre $\tilde{p}^{-1}([z_2:z_3])$ is a 2-dimensional complex torus if $z_2\neq 0$. In the case $z_3\neq 0$, we can discuss in the same way. □

Applying the Kodaira-Serre duality and the Leray's spectral sequence, we have

$$H^2(X, \Theta)\cong H^1(X, \Omega^1\otimes\Omega^3)\cong E_3^{0,1}+E_2^{1,0}$$

where $E_2^{q,r}=H^q(P^1, R^r p_*(\Omega^1\otimes\Omega^3))$ and $E_3^{0,1}=\mathrm{Ker}(E_2^{0,1}\to E_2^{2,0})/\mathrm{Im}(E_2^{-1,2}\to E_2^{0,1})$. Since $E_2^{-1,2}=E_2^{2,0}=0$, we get

$$H^1(X, \Omega^1\otimes\Omega^3)\cong E_2^{0,1}+E_2^{1,0}.$$

**Lemma 1.6.** $\Omega_X^3\cong p^*((\Omega_{P^1}^1)^{\otimes 2})\cong p^*(\mathcal{O}_{P^1}(-4))$.

**Proof.** Let the system of local coordinates on $V_k=\{z_k\neq 0\}$ $(k=2, 3)$ as before. As we easily have

$$du_2=du_3/w_3-u_3dw_3/w_3^2,$$

$$dv_2=dv_3/w_3-u_3dw_3/w_3^2, \quad dw_2=-dw_3/w_3^2,$$

and

$$du_2\wedge dv_2\wedge dw_2=-(du_3\wedge dv_3\wedge dw_3)/w_3^4,$$

the lemma is obvious. □

LEMMA 1.7. $R^0p_*\Omega_X^1 \cong \Omega_{P^1}^1 \cong \mathcal{O}_{P^1}(-2)$.

PROOF. By the definition, $(R^0p_*\Omega_X^1)_z$, the stalk of $R^0p_*\Omega_X^1$ at $z=[z_2:z_3]\in \boldsymbol{P}^1$, is equal to $H^0(p^{-1}(z), \Omega_X^1)$. We assume that $z_2\neq 0$ and that a neighbourhood of $z$ does not contain the point $[1:0]\in \boldsymbol{P}^1$. Then an element $\phi(u_2, v_2, w_2)=f(u_2, v_2, w_2)du_2+g(u_2, v_2, w_2)dv_2+h(u_2, v_2, w_2)dw_2$ of $H^0(\tilde{p}^{-1}(z), \Omega_Z^1)$ induces an element of $H^0(p^{-1}(z), \Omega_X^1)$ if and only if $\phi(u_2, v_2, w_2)$ is $g_i^*$-invariant for $i=1, 2, 3, 4$. Since

$$\begin{aligned} g_i^*\phi = & f(\alpha_i+\beta_i w_2+u_2,\ -\bar{\beta}_i+\bar{\alpha}_i w_2+v_2,\ w_2)du_2 \\ & +g(\alpha_i+\beta_i w_2+u_2,\ -\bar{\beta}_i+\bar{\alpha}_i w_2+v_2,\ w_2)dv_2 \\ & +(\beta_i f(\alpha_i+\beta_i w_2+u_2,\ -\bar{\beta}_i+\bar{\alpha}_i w_2+v_2,\ w_2) \\ & +\bar{\alpha}_i g(\alpha_i+\beta_i w_2+u_2,\ -\bar{\beta}_i+\bar{\alpha}_i w_2+v_2,\ w_2) \\ & +h(\alpha_i+\beta_i w_2+u_2,\ -\bar{\beta}_i+\bar{\alpha}_i w_2+v_2,\ w_2))dw_2\ , \end{aligned}$$

$g_i^*\phi=\phi$ if and only if

$$f(\alpha_i+\beta_i w_2+u_2,\ -\bar{\beta}_i+\bar{\alpha}_i w_2+v_2,\ w_2)=f(u_2, v_2, w_2)\ ,$$

$$g(\alpha_i+\beta_i w_2+u_2,\ -\bar{\beta}_i+\bar{\alpha}_i w_2+v_2,\ w_2)=g(u_2, v_2, w_2)\ ,$$

$$\begin{aligned} & \beta_i f(\alpha_i+\beta_i w_2+u_2,\ -\bar{\beta}_i+\bar{\alpha}_i w_2+v_2,\ w_2) \\ & +\bar{\alpha}_i g(\alpha_i+\beta_i w_2+u_2,\ -\bar{\beta}_i+\bar{\alpha}_i w_2+v_2,\ w_2) \\ & +h(\alpha_i+\beta_i w_2+u_2,\ -\bar{\beta}_i+\bar{\alpha}_i w_2+v_2,\ w_2) \\ & =h(u_2, v_2, w_2)\ . \end{aligned}$$

The first two mean that $f$ and $g$ are independent of the variables $u_2$ and $v_2$, in other words, $f$ and $g$ are holomorphic functions of $w_2$. As for $h$, we differentiate the last equation by $u_2$ and get

$$\frac{\partial}{\partial u_2}h(\alpha_i+\beta_i w_2+u_2,\ -\bar{\beta}_i+\bar{\alpha}_i w_2+v_2,\ w_2)=\frac{\partial}{\partial u_2}h(u_2, v_2, w_2)\ .$$

Therefore $\partial h/\partial u_2$ is constant with respect to $u_2$ and $v_2$. Similarly $\partial h/\partial v_2$ is constant with respect to $u_2$ and $v_2$. Thus we have

$$h(u_2, v_2, w_2)=h_0(w_2)+h_{1,0}(w_2)u_2+h_{0,1}(w_2)v_2\ .$$

This equation, the relations among $f$, $g$ and $h$, and the assumption $\det(\alpha\,\beta\,\bar{\alpha}\,\bar{\beta})\neq 0$ imply that $f=g=h_{0,1}=h_{1,0}=0$. This concludes that $R^0p_*\Omega_X^1$ is isomorphic to $\Omega_{P^1}^1$ stalkwise. It is trivial that the isomorphism is extended globally. □

LEMMA 1.8. $\dim H^1(P^1, R^0p_*(\Omega^1\otimes\Omega^3))=5$.

PROOF. Since $R^0p_*(\Omega^1\otimes\Omega^3)\cong R^0p_*\Omega^1\otimes\mathcal{O}_{P^1}(-4)\cong\mathcal{O}_{P^1}(-2)\otimes\mathcal{O}_{P^1}(-4)\cong\mathcal{O}_{P^1}(-6)$, $H^1(P^1, R^0p_*(\Omega^1\otimes\Omega^3))$ is isomorphic to $H^0(P^1, \mathcal{O}_{P^1}(4))$, which is of dimension 5. □

Next we study $R^1p_*\Omega^1$.

LEMMA 1.9. *We have the following exact sequence of sheaves:*

$$0\longrightarrow p^*\Omega^1_{P^1}\longrightarrow\Omega^1_X\longrightarrow(p^*\mathcal{O}_{P^1}(-1))^{\oplus 2}\longrightarrow 0\ .$$

PROOF. Let $x$ be a point of $X\cap\{z_2\neq 0\}$. It is trivial that the homomorphism $\alpha_x$ of $(p^*\Omega^1_{P^1})_x$, the stalk of $p^*\Omega^1_{P^1}$ at $x$, into $\Omega^1_{X,x}$ defined by sending $f(u_2, v_2, w_2)\otimes dw_2$ to $f(u_2, v_2, w_2)dw_2$ is injective. It is also trivial that a homomorphism $\beta_x$ of $\Omega^1_{X,x}$ to $(p^*\mathcal{O}_{P^1}(-1))^{\oplus 2}_x$ defined by sending $fdu_2+gdv_2+hdw_2$ to $(f, g)$ is surjective. Moreover we easily have $\operatorname{Im}\alpha_x=\operatorname{Ker}\beta_x$. For $x\in X\cap\{z_3\neq 0\}$, the homomorphisms $\alpha_x$, $\beta_x$ are similarly defined. Then it is easy to see that $\alpha_x$ and $\beta_x$ are well-defined, in other words, defined independently of the choice of the local coordinates and we have the proposition. □

LEMMA 1.10. *$R^1p_*\mathcal{O}$ is a rank two vector bundle over $P^1$.*

PROOF. Let $z\in P^1$ and $\mathfrak{m}_z$ the maximal ideal in $\mathcal{O}_{P^1,z}$. Then it is obvious that for any $z\in P^1$, the ringed space $(p^{-1}(z), \mathcal{O}_X/p^*\mathfrak{m}_z\mathcal{O}_X)$ is isomorphic to a 2-dimensional complex torus $T_z=(T_z, \mathcal{O}_T)$. Hence the mapping

$$R^ip_*\mathcal{O}_z/\mathfrak{m}_z(R^ip_*\mathcal{O}_z)\longrightarrow H^i(p^{-1}(z), \mathcal{O}_X/p^*\mathfrak{m}_z\mathcal{O}_X)=R^ip_*(\mathcal{O}_x/p^*\mathfrak{m}_z\mathcal{O}_X)$$

is bijective for any $z\in P^1$ and $i=0, 1$. Applying the statement in [1] p. 151, we see that $R^1p_*\mathcal{O}$ is locally free and

$$\dim R^1p_*(\mathcal{O}_x/p^*\mathfrak{m}_z\mathcal{O}_X)=\dim H^1(p^{-1}(z), \mathcal{O}_X/p^*\mathfrak{m}_z\mathcal{O}_X)=2.$$ □

LEMMA 1.11. $R^1p_*\mathcal{O}\cong\mathcal{O}^2_{P^1}(1)$.

PROOF. We define $C$-valued $C^\infty$ functions $q_1$, $q_2$ of $Z$ by

$$q_1=(z_0\bar{z}_2+\bar{z}_1z_3)/(|z_2|^2+|z_3|^2)\ ,$$

$$q_2=(-z_0\bar{z}_3+\bar{z}_1z_2)/(|z_2|^2+|z_3|^2)\ .$$

Then an easy calculation shows that $q_1\circ g_i=q_1+\alpha_i$ and $q_2\circ g_i=q_2+\beta_i$ for $i=1, 2, 3, 4$. Hence $\bar\partial q_1$, $\bar\partial\bar{q}_1$, $\bar\partial q_2$, $\bar\partial\bar{q}_2$ induce $C^\infty$ $\bar\partial$-closed (0, 1)-forms on $X$.

Since $R^1p_*\mathcal{O}$ is locally free of rank 2, $R^1p_*\mathcal{O}_{V_2}$ (resp. $R^1p_*\mathcal{O}_{V_3}$) is isomorphic to $\mathcal{O}_{V_2}^2$ (resp. $\mathcal{O}_{V_3}^2$) where $V_l=\{[z_2:z_3]\in \boldsymbol{P}^1;\, z_l\neq 0\}$ for $l=2, 3$. On the other hand, by Dolbeault's theorem, $\Gamma(V_1, R^1p_*\mathcal{O})$ is isomorphic to $\Gamma(p^{-1}(V_1), \bar\partial\mathscr{A}^{0,0})/\bar\partial\Gamma(p^{-1}(V_1), \mathscr{A}^{0,0})$. We show that the classes $[\bar\partial q_2]$ and $[\bar\partial\bar q_1]$ (resp. $[\bar\partial q_1]$, $[\bar\partial\bar q_2]$) form a basis of $\Gamma(p^{-1}(V_2), \bar\partial\mathscr{A}^{0,0})/\bar\partial\Gamma(p^{-1}(V_2), \mathscr{A}^{0,0})$ (resp. $\Gamma(p^{-1}(V_3), \bar\partial\mathscr{A}^{0,0})/\bar\partial\Gamma(p^{-1}(V_3), \mathscr{A}^{0,0})$) as a $\Gamma(V_2, \mathcal{O}_{p^{-1}(V_2)})$-module (resp. $\Gamma(V_3, \mathcal{O}_{p^{-1}(V_3)})$-module). For that, it suffices to show that $[\bar\partial q_2]$ and $[\bar\partial\bar q_1]$ (resp. $[\bar\partial q_1]$ and $[\bar\partial\bar q_2]$) form a basis at each stalk

$$\Gamma(p^{-1}(z), \bar\partial\mathscr{A}^{0,0})/\bar\partial\Gamma(p^{-1}(z), \mathscr{A}^{0,0}) ,$$

which we denote by $M_z$. Since $M_z$ is a module over a local ring $\mathcal{O}_z$ with the maximal ideal $\mathfrak{m}_z$, it is sufficient to prove that the images $\pi_z[\bar\partial q_2]$ and $\pi_z[\bar\partial\bar q_1]$ (resp. $\pi_z[\bar\partial q_1]$ and $\pi_z[\bar\partial\bar q_2]$) in $M_z/\mathfrak{m}_zM_z$ form a basis in it for every $z\in V_2$ (resp. $z\in V_3$), where $\pi_z$ is the projection of $M_z$ onto $M_z/\mathfrak{m}_zM_z$. We apply the following sublemma to show that.

SUBLEMMA ([8], (5.1)). *Let $A$ be a local ring with the maximal ideal $\mathfrak{m}$ and $M$ a finite $A$-module. Let $n$ be the dimension of $M/\mathfrak{m}M$ over $A/\mathfrak{m}$. Take a basis of $M/\mathfrak{m}M$ over $A/\mathfrak{m}$, say $\{u_1, u_2, \cdots, u_n\}$ and choose inverse images $\tilde u_i$ of $u_i$ in $M$ for each $i$. Then $\{\tilde u_1, \tilde u_2, \cdots, \tilde u_n\}$ is a basis of $M$. Conversely any basis of $M$ is obtained in the above way.*

By easy calculations, we have

$$\bar\partial q_2=\frac{d\bar v_2}{1+|w_2|^2}-\frac{u_2+\bar v_2w_2}{(1+|w_2|^2)^2}d\bar w_2 ,$$

$$\bar\partial\bar q_1=\frac{d\bar u_2}{1+|w_2|^2}+\frac{v_2-\bar u_2w_2}{(1+|w_2|^2)^2}d\bar w_2 ,$$

$$\bar\partial q_1=\frac{d\bar v_3}{1+|w_3|^2}+\frac{u_3-\bar v_3w_3}{(1+|w_3|^2)^2}d\bar w_3 ,$$

$$\bar\partial\bar q_2=-\frac{d\bar u_3}{1+|w_3|^2}+\frac{v_3+\bar u_3w_3}{(1+|w_3|^2)^2}d\bar w_3 ,$$

and in $M_z/\mathfrak{m}_zM_z$,

$$\pi_z[\bar\partial q_2]=\left[\frac{d\bar v_2}{1+|w_2|^2}\right], \quad \pi_z[\bar\partial\bar q_1]=\left[\frac{d\bar u_2}{1+|w_2|^2}\right],$$

$$\pi_z[\bar\partial q_1]=\left[\frac{d\bar v_3}{1+|w_3|^2}\right], \quad \pi_z[\bar\partial\bar q_2]=-\left[\frac{d\bar u_3}{1+|w_3|^2}\right].$$

In the last two lines, $w_2$ and $w_3$ are values, not coordinates. These asserts that the classes $[\bar\partial q_2]$ and $[\bar\partial \bar q_1]$ (resp. $[\bar\partial q_1]$ and $[\bar\partial \bar q_2]$) form a basis of $\Gamma(p^{-1}(V_2), \bar\partial\mathscr{A}^{0,0})/\bar\partial\Gamma(p^{-1}(V_2), \mathscr{A}^{0,0})$ (resp. $\Gamma(p^{-1}(V_3), \bar\partial\mathscr{A}^{0,0})/\bar\partial\Gamma(p^{-1}(V_3), \mathscr{A}^{0,0})$). Hence we can now define an isomorphism $\alpha_2$ (resp. $\alpha_3$) of $\Gamma(p^{-1}(V_2), \bar\partial\mathscr{A}^{0,0})/\bar\partial\Gamma(p^{-1}(V_2), \mathscr{A}^{0,0})$ (resp. $\Gamma(p^{-1}(V_3), \bar\partial\mathscr{A}^{0,0})/\bar\partial\Gamma(p^{-1}(V_3), \mathscr{A}^{0,0})$) to $\mathscr{O}_{V_2}$ (resp. $\mathscr{O}_{V_3}$) by sending $f_2[\bar\partial q_2]+g_2[\bar\partial\bar q_1]$ to $(f_2, g_2)$ (resp. $f_3[\bar\partial q_1]+g_3[\bar\partial\bar q_2]$ to $(f_3, g_3)$). Since

$$[\bar\partial q_2]=w_3[\bar\partial q_1]\ ,\quad [\bar\partial\bar q_1]=w_3[\bar\partial\bar q_2]$$

on $p^{-1}(V_2)\cap p^{-1}(V_3)$, $\alpha_2$ and $\alpha_3$ give an isomorphism of $R^1p_*\mathscr{O}$ onto $\mathscr{O}^2_{P^1}(1)$. □

LEMMA 1.12. *We have the following exact sequence:*

$$\mathscr{O}^2_{P^1}(-5)\longrightarrow R^1p_*(\Omega^1\otimes\Omega^3)\longrightarrow\mathscr{O}^4_{P^1}(-4)\ .$$

PROOF. Tensoring $\Omega^3_X\cong p^*\mathscr{O}_{P^1}(-4)$ with the exact sequence in Lemma 1.9, we get

$$\begin{aligned}0\longrightarrow p^*\Omega^1_{P^1}\otimes p^*\mathscr{O}_{P^1}(-4)\longrightarrow\Omega^1_X\otimes\Omega^3_X\\ \longrightarrow(p^*\mathscr{O}_{P^1}(-1))^{\oplus 2}\otimes p^*\mathscr{O}_{P^1}(-4)\longrightarrow 0\ ,\end{aligned}$$

which is exact. From this, we obtain the exact sequence

$$\longrightarrow R^1p_*(p^*\mathscr{O}_{P^1}(-6))\longrightarrow R^1p_*(\Omega^1_X\otimes\Omega^3_X)\longrightarrow R^1p_*(p^*\mathscr{O}_{P^1}(-5))^2\longrightarrow\ .$$

Since $R^1p_*(p_*\mathscr{O}_{P^1}(-6))\cong\mathscr{O}_{P^1}(-6)\otimes R^1p_*\mathscr{O}_X\cong\mathscr{O}^2_{P^1}(-5)$ and $R^1p_*(p^*\mathscr{O}_{P^1}(-5))^{\oplus 2}\cong(\mathscr{O}_{P^1}(-5)\otimes R^1p_*\mathscr{O})^{\oplus 2}\cong\mathscr{O}^4_{P^1}(-4)$, we have the conclusion. □

LEMMA 1.13. $H^0(\boldsymbol{P}^1, R^1p_*(\Omega^1\otimes\Omega^3))=0$.

PROOF. Let $\mathscr{F}$ (resp. $\mathscr{G}$) be the kernel (resp. image) of the first (resp. second) arrow in Lemma 1.12. Then we obtain the cohomology exact sequence:

$$0\longrightarrow H^0(\boldsymbol{P}^1, \mathscr{F})\longrightarrow H^0(\boldsymbol{P}^1, R^1p_*(\Omega^1\otimes\Omega^3))\longrightarrow H^0(\boldsymbol{P}^1, \mathscr{G})\ .$$

Since $\mathscr{F}$ (resp. $\mathscr{G}$) is a subsheaf of $\mathscr{O}^2_{P^1}(-5)$ (resp. $\mathscr{O}^4_{P^1}(-4)$) and $H^0(\boldsymbol{P}^1, \mathscr{O}^2_{P^1}(-5))=H^0(\boldsymbol{P}^1, \mathscr{O}^4_{P^1}(-4))=0$, we have

$$H^0(\boldsymbol{P}^1, \mathscr{F})=H^0(\boldsymbol{P}^1, \mathscr{G})=0\ .$$ □

By Lemma 1.8 and Lemma 1.13, we obtain Proposition 1.4. As for $H^1(X, \Theta)$, we have

PROPOSITION 1.14. $\dim H^1(X, \Theta)=9$.

To prove the proposition, first we apply the Riemann-Roch formula:

$$\sum_{i=0}^{3}(-1)^i \dim H^i(X, \Theta)=\frac{1}{2}c_1^3+\frac{19}{24}c_1c_2-\frac{1}{2}c_3 .$$

From the results we have already got, the above equation reduces to the next:

$$9-\dim H^1(X, \Theta)=\frac{1}{2}c_1^3+\frac{19}{24}c_1c_2-\frac{1}{2}c_3 .$$

It is known that $X$ is diffeomorphic to $T^4\times S^2$ where $T^4$ is a real 4-dimensional torus so the following lemma is proved easily.

LEMMA 1.15. $c_3=0$.

LEMMA 1.16. $c_1^3=0$.

PROOF. Obvious by Lemma 1.6. □

Lastly we study $c_1c_2$.

LEMMA 1.17. $c_1c_2=0$.

Again applying the Riemann-Roch formula, we have

$$c_1c_2=24\sum_{i=0}^{3}(-1)^i \dim H^i(X, \mathcal{O}) .$$

Since $X$ is a compact Class $L$ manifold, we have $H^0(X, \mathcal{O})=\boldsymbol{C}$, $H^3(X, \mathcal{O})\cong H^0(X, \Omega^3)=0$. As for $H^1(X, \mathcal{O})$, the Leray spectral sequence shows that $H^1(X, \mathcal{O})\cong H^0(\boldsymbol{P}^1, R^1p_*\mathcal{O})+H^1(\boldsymbol{P}^1, R^0p_*\mathcal{O})$. Since $R^1p_*\mathcal{O}\cong\mathcal{O}_{P^1}^2(1)$ and $R^0p_*\mathcal{O}\cong\mathcal{O}_{P^1}$ as we have already seen, we get

$$\dim H^1(X, \mathcal{O})=\dim H^0(\boldsymbol{P}^1, \mathcal{O}_{P^1}^2(1))=4 .$$

As for $H^2(X, \mathcal{O})$, again by the Leray spectral sequence, we have

$$\begin{aligned} H^2(X, \mathcal{O}) &\cong H^1(\boldsymbol{P}^1, R^1p_*\mathcal{O})+H^0(\boldsymbol{P}^1, R^2p_*\mathcal{O}) \\ &\cong H^1(\boldsymbol{P}^1, \mathcal{O}_{P^1}^2(1))+H^0(\boldsymbol{P}^1, R^2p_*\mathcal{O}) \\ &\cong H^0(\boldsymbol{P}^1, R^2p_*\mathcal{O}) . \end{aligned}$$

LEMMA 1.18. $R^2p_*\mathcal{O}\cong\mathcal{O}_{P^1}(2)$.

PROOF. This lemma is proved in almost the same way as Lemma 1.11. First remark that $R^2p_*\mathcal{O}$ is a line bundle which is proved in almost the same way as Lemma 1.10. Secondly we remark that the class $[\bar{\partial}q_2\wedge\bar{\partial}\bar{q}_1]$ (resp. $[\bar{\partial}q_1\wedge\bar{\partial}\bar{q}_2]$) forms a basis of $\Gamma(p^{-1}(V_2), \bar{\partial}\mathscr{A}^{0,1})/\bar{\partial}\Gamma(p^{-1}(V_2), \mathscr{A}^{0,1})$ (resp. $\Gamma(p^{-1}(V_3), \bar{\partial}\mathscr{A}^{0,1})/\bar{\partial}\Gamma(p^{-1}(V_3), \mathscr{A}^{0,1})$) over $\Gamma(V_2, \mathscr{O}_{V_2})$ (resp. $\Gamma(V_3, \mathscr{O}_{V_3})$). This is proved by applying the sublemma in the proof of Lemma 1.11. Thirdly we define a mapping $\beta_2$ (resp. $\beta_3$) of $\Gamma(p^{-1}(V_2), \bar{\partial}\mathscr{A}^{0,1})/\bar{\partial}\Gamma(p^{-1}(V_2), \mathscr{A}^{0,1})$ (resp. $\Gamma(p^{-1}(V_3), \bar{\partial}\mathscr{A}^{0,1})/\bar{\partial}\Gamma(p^{-1}(V_3), \mathscr{A}^{0,1})$) to $\Gamma(V_2, \mathscr{O}_{V_2})$ (resp. $\Gamma(V_3, \mathscr{O}_{V_3})$) by sending $f_2[\bar{\partial}q_2\wedge\bar{\partial}\bar{q}_1]$ (resp. $f_3[\bar{\partial}q_1\wedge\bar{\partial}\bar{q}_2]$) to $f_2$ (resp. $f_3$), which is isomorphic. Lastly since

$$[\bar{\partial}q_2\wedge\bar{\partial}\bar{q}_1]=w_3^2[\bar{\partial}q_1\wedge\bar{\partial}\bar{q}_2]$$

on $p^{-1}(V_2)\cap p^{-1}(V_3)$, $\{\beta_2, \beta_3\}$ forms an isomorphism of $R^2p_*\mathscr{O}$ onto $\mathscr{O}_{P^1}(2)$. □

By Lemma 1.18, we see

$$\dim H^2(X, \mathscr{O})=\dim H^0(\boldsymbol{P}^1, \mathscr{O}_{P^1}(2))=3\,.$$

Therefore we obtain Lemma 1.17 and Proposition 1.14.

## §2. Small deformations of a Blanchard manifold.

Let $\delta$ be a sufficiently small positive number. We define a polydisc $B$ in $\boldsymbol{C}^9$ by

$$B=\{t=(t_1, t_2, \cdots, t_9)\in\boldsymbol{C}^9;\ |t_k|<\delta,\ k=1, 2, \cdots, 9\}\,.$$

We can assume that

$$(*)\qquad \det\begin{pmatrix}\alpha_1 & \beta_1 & \bar{\alpha}_1\\ \alpha_2 & \beta_2 & \bar{\alpha}_2\\ \alpha_3 & \beta_3 & \bar{\alpha}_3\end{pmatrix}\neq 0\,,\quad \alpha_1\neq 0$$

without loss of generality since $\det(\alpha\,\beta\,\bar{\alpha}\,\bar{\beta})\neq 0$. We define $A_i(t)\in GL(2, \boldsymbol{C})$ for $i=1, 2, 3, 4$ and $t\in B$ by

$$A_1(t)=\begin{pmatrix}\alpha_1+t_1 & \beta_1\\ -\bar{\beta}_1 & \bar{\alpha}_1\end{pmatrix},\quad A_2(t)=\begin{pmatrix}\alpha_2+t_2 & \beta_2\\ -\bar{\beta}_2 & \bar{\alpha}_2+t_3\end{pmatrix},$$

$$A_3(t)=\begin{pmatrix}\alpha_3+t_4 & \beta_3\\ -\bar{\beta}_3 & \bar{\alpha}_3+t_5\end{pmatrix},\quad A_4(t)=\begin{pmatrix}\alpha_4+t_6 & \beta_4+t_7\\ -\bar{\beta}_4+t_8 & \bar{\alpha}_4+t_9\end{pmatrix}.$$

We define $G_i(t)\in PGL(4, \boldsymbol{C})$ for $i=1, 2, 3, 4$ and $t\in B$ by

$$G_i(t)=\begin{pmatrix} 1 & 0 & \multicolumn{2}{c}{\multirow{2}{*}{$A_i(t)$}} \\ 0 & 1 & & \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and denote the corresponding automorphisms of $Z$ by $g_i(t)$.

We construct a complex manifold $\mathfrak{X}$ as follows. Let $\tilde{g}_i$ be automorphisms of $Z\times B$ defined by $\tilde{g}_i(z, t)=(g_i(t)(z), t)$. We denote by $\tilde{\Gamma}$ the group of automorphisms of $Z\times B$ generated by $\tilde{g}_i$. Then it is easy to see that $\tilde{\Gamma}$ acts on $Z\times B$ properly discontinuously and has no fixed points. Therefore the quotient space $\mathfrak{X}$ of $Z\times B$ by $\tilde{\Gamma}$ is a complex manifold. We define a projection $\varpi$ of $X$ to $B$ by sending a point represented by $(z, t)$ to $t$. We have the following theorem.

THEOREM. *$(\mathfrak{X}, B, \varpi)$ is the complete and effectively parametrized complex family of small deformations of $X=\varpi^{-1}(0)$.*

Before proving the theorem, we define an open covering of $X$ as follows. We put $J=\{-1, 0, 1\}$. We define real 3-dimensional submanifolds $W_{(j,i)}$ by

$$W_{(j,i)}=\Big\{[s(\alpha_i z_2^i+\beta_i z_3^i): s(-\bar{\beta}_i z_2^i+\bar{\alpha}_i z_3^i): z_2^i: z_3^i]\in Z;$$
$$-\frac{1}{4}+\frac{j}{3}<s<\frac{1}{4}+\frac{j+1}{3}\Big\}$$

where $j\in J$ and $i\in\{1, 2, 3, 4\}$. For $z^i=[z_0^i:z_1^i:z_2^i:z_3^i]\in W_{(j,i)}$ with $[z_2^i:z_3^i]=[z_2:z_3]$ for all $i\in\{1, 2, 3, 4\}$, we define

$$\sum_{i=1}^4 z^i=\Big[\sum_{i=1}^4 z_0^i:\sum_{i=1}^4 z_1^i:z_2:z_3\Big].$$

Put

$$V_{(j_1,j_2,j_3,j_4)}=\Big\{\sum_{i=1}^4 z^i;\ [z_2:z_3]\in \boldsymbol{P}^1,\ z^i\in W_{(j,i)}\Big\}$$

for $(j_1, j_2, j_3, j_4)\in J^4$, which are open subsets in $Z$. For $j=(j_1, j_2, j_3, j_4)$, $j'=(j'_1, j'_2, j'_3, j'_4)\in J^4$, we define

$$|j-j'|=\max_{1\le i\le 4}|j_i-j'_i|.$$

Then the following is easy.

$$|j-j'|=0 \quad \text{if and only if} \quad V_j=V_{j'},$$

$$|j-j'|=1 \quad \text{if and only if} \quad V_j \neq V_{j'} \quad \text{and} \quad V_j \cap V_{j'} \neq \emptyset\ ,$$

$$|j-j'|=2 \quad \text{if and only if} \quad V_j \cap V_{j'} = \emptyset\ .$$

Note that $z \in V_j$ and $z' \in V_{j'}$, $|j-j'|=2$, represent the same point in $X$ if and only if

$$z' = g_1^{[(j_1-j_1')/2]} \circ g_2^{[(j_2-j_2')/2]} \circ g_3^{[(j_3-j_3')/2]} \circ g_4^{[(j_4-j_4')/2]}(z)$$

where [ ] denotes the Gauss symbol. In the following, we denote $p(V_j)$ by $V_j$ for simplicity. Then $\mathscr{V}=\{V_j\}_{j\in J^4}$ forms an open covering of $X$.

REMARK. $V_{(0,0,0,0)}$ contains $U_r$ with $r$ sufficiently small. Since $p(V_{(0,0,0,0)})\subset X$ is biholomorphic to $V_{(0,0,0,0)}$, $X$ is a Class $L$ manifold. We remark also that all $V_{(j_1,j_2,j_3,j_4)}$ are biholomorphic to $V_{(0,0,0,0)}$.

PROOF OF THEOREM. First we decompose the Kodaira-Spencer map into three maps: $\theta$ of $T_0(B)$ to $Z^1(\mathscr{V}, \Theta)$ and $q$, the quotient map of $Z^1(\mathscr{V}, \Theta)$ to $H^1(\mathscr{V}, \Theta)$, and the inclusion map $i$ of $H^1(\mathscr{V}, \Theta)$ to $H^1(X, \Theta)$. Suppose that we have a relation

$$\sum_{k=1}^{9} \gamma_k q \cdot \theta\left(\frac{\partial}{\partial t_k}\right)=0$$

where $\gamma_k$ are complex numbers. It is enough to show that all $\gamma_k$ vanish. The above equation is equivalent to the following equation

$$(**) \qquad \sum_{k=1}^{9} \gamma_k \theta\left(\frac{\partial}{\partial t_k}\right)=\delta\eta$$

with $\eta \in C^0(\mathscr{V}, \Theta)$. Since the support of $\theta(\partial/\partial t_k)$ does not contain $V_j \cap V_{j'}$ with $|j-j'|\leq 1$, $\eta$ is an element of $H^0(Z, \Theta)$. Let $\theta_k^i$ (resp. $\eta^i$) denote the value of $\theta(\partial/\partial t_k)$ (resp. $\delta\eta$) on $V_j \cap V_{j'}$ with $|j_i-j_i'|=2$ and $j_\iota=j_\iota'$ for $\iota\neq i$. Here $\eta^i$ is equal to the value of $g_{i*}\eta-\eta$ on $V_j\cap V_{j'}$ and $g_{i*}\eta$ is already calculated in the proof of Proposition 1.3. Then it is easy to see that

$$\theta\left(\frac{\partial}{\partial t_k}\right)(V_j\cap V_{j'})=\sum_{i=1}^{4}\left[\frac{j_i-j_i'}{2}\right]\theta_k^i$$

and that

$$\delta\eta(V_j\cap V_{j'})=((g_1^{[(j_1-j_1')/2]}\circ \cdots \circ g_4^{[(j_4-j_4')/2]})_*\eta-\eta)(V_j\cap V_{j'})$$
$$=\sum_{i=1}^{4}\left[\frac{j-j'}{2}\right]\eta^i\ .$$

Hence it suffices to consider the equation (**) only on $V_j\cap V_{j'}$ with

$|j_\iota - j'_\iota| = 2$ for only one $i$ and $j_\iota = j'_\iota$ for $\iota \neq i$. So (**) reduces to the equations below:

$$\gamma_1 \frac{\partial}{\partial u_2} = \eta^1 \qquad \text{on} \quad V_{(0,j_2,j_3,j_4)} \cap V_{(2,j_2,j_3,j_4)} ,$$

$$\gamma_2 \frac{\partial}{\partial u_2} + \gamma_3 w_2 \frac{\partial}{\partial v_2} = \eta^2 \qquad \text{on} \quad V_{(j_1,0,j_3,j_4)} \cap V_{(j_1,2,j_3,j_4)} ,$$

$$\gamma_4 \frac{\partial}{\partial u_2} + \gamma_5 w_2 \frac{\partial}{\partial v_2} = \eta^3 \qquad \text{on} \quad V_{(j_1,j_2,0,j_4)} \cap V_{(j_1,j_2,2,j_4)} ,$$

$$\gamma_6 \frac{\partial}{\partial u_2} + \gamma_7 w_2 \frac{\partial}{\partial u_2} + \gamma_8 \frac{\partial}{\partial v_2} + \gamma_9 w_2 \frac{\partial}{\partial v_2} = \eta^4 \quad \text{on} \quad V_{(j_1,j_2,j_3,0)} \cap V_{(j_1,j_2,j_3,2)} .$$

In the above,

$$\begin{aligned}
\eta^i = \{ & -\alpha_i b_1 + \bar{\beta}_i c_1 + \alpha_i^2 e - \alpha_i \bar{\beta}_i f + \beta_i (a_3 - \alpha_i b_3 + \bar{\beta}_i c_3) \\
& + (-2\alpha_i e + \bar{\beta}_i f + \beta_i b_3) u_2 + (-\alpha_i f + \beta_i c_3) v_2 \\
& + (-\beta_i b_1 - \bar{\alpha}_i c_1 + \alpha_i \beta_i e + |\alpha_i|^2 f - \alpha_i g + \beta_i (d_3 - \beta_i b_3 - \bar{\alpha}_i c_3)) w_2 \\
& + (-\beta_i e - \bar{\alpha}_i f) u_2 w_2 \} \frac{\partial}{\partial u_2} \\
& + \{ -\alpha_i b_2 + \bar{\beta}_i c_2 - \alpha_i \bar{\beta}_i e + \bar{\beta}_i^2 f + \bar{\alpha}_i (a_3 - \alpha_i b_3 + \bar{\beta}_i c_3) \\
& + (\bar{\beta}_i e + \bar{\alpha}_i b_3) u_2 + (-\alpha_i e + 2\bar{\beta}_i f + \bar{\alpha}_i c_3) v_2 \\
& + (-\beta_i b_2 - \bar{\alpha}_i c_2 - |\beta_i|^2 e - \bar{\alpha}_i \bar{\beta}_i f + \bar{\beta}_i g + \bar{\alpha}_i (d_3 - \beta_i b_3 - \bar{\alpha}_i c_3)) w_2 \\
& + (-\beta_i e - \bar{\alpha}_i f) v_2 w_2 \} \frac{\partial}{\partial v_2} \\
& + \{ -\alpha_i b_3 + \bar{\beta}_i c_3 + (-\beta_i b_3 - \bar{\alpha}_i c_3 - \alpha_i e + \bar{\beta}_i f) w_2 \\
& + (-\beta_i e - \bar{\alpha}_i f) w_2^2 \} \frac{\partial}{\partial w_2} .
\end{aligned}$$

Comparing the coefficients of $w_2 \partial/\partial w_2$ of both sides of the above four equations, we have

$$(\alpha\, \beta\, \bar{\alpha}\, \bar{\beta})^t (e, b_3, c_3, -f) = 0 .$$

Hence $b_3 = c_3 = e = f = 0$. Next we compare the coefficients of $w_2 \partial/\partial u_2$ and $\partial/\partial v_2$ of the three equations except the fourth and we have

$$\begin{pmatrix} \alpha_1 & \beta_1 & \bar{\alpha}_1 \\ \alpha_2 & \beta_2 & \bar{\alpha}_2 \\ \alpha_3 & \beta_3 & \bar{\alpha}_3 \end{pmatrix} \begin{pmatrix} g \\ b_1 - d_3 \\ c_1 \end{pmatrix} = 0 ,$$

$$\begin{pmatrix} \bar{\alpha}_1 & \bar{\beta}_1 & \alpha_1 \\ \bar{\alpha}_2 & \bar{\beta}_2 & \alpha_2 \\ \bar{\alpha}_3 & \bar{\beta}_3 & \alpha_3 \end{pmatrix} \begin{pmatrix} a_3 \\ c_2 \\ -b_2 \end{pmatrix} = 0 .$$

By the assumption (*), $a_3 = b_2 = c_1 = c_2 = g = b_1 - d_3 = 0$. Comparing the coefficients of $w_2\partial/\partial v_2$ of the first equation, we have $d_3 = 0$ since $\alpha_1 \neq 0$, and we also get $b_1 = 0$. Therefore the right-hand side of the four equations vanish and we have $\gamma_k = 0$ for all $k$. □

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