TOKYO J. MATH.
VOL. 10, NO. 1, 1987

Isogenies between Algebraic Surfaces with Geometric Genus One

David R. MORRISON*

Princeton University

(Communicated by K. Ogiue)

The classical theory of the Albanese variety provides a geometric interpretation of first cohomology groups of complex projective varieties in the following way: a variety X and its Albanese variety $\mathrm{Alb}(X)$ have isomorphic first cohomology groups, and there is a mapping $X \to \mathrm{Alb}(X)$ inducing this isomorphism, the formation of which is functorial for maps between complex varieties.

This note develops the beginnings of an analogous theory for second cohomology groups of algebraic surfaces with geometric genus one. Specifically, we show that to any such surface is associated a so-called K3 surface, whose transcendental second cohomology is isomorphic to that of the original surface. This isomorphism is not in general given by a mapping from the surface to its associated K3 surface, but one can hope (and the Hodge conjecture would imply) that it is induced by a correspondence between the two surfaces. We use the term *isogeny* to denote a correspondence giving such an isomorphism. We have elsewhere [5, 7] given constructions of isogenies between algebraic surfaces with geometric genus one and their associated K3 surfaces in several particular cases.

If we require that the isomorphism between second transcendental cohomology groups preserve both the intersection pairings and the integral Hodge structures, then most algebraic surfaces with geometric genus one have a unique associated K3 surface (see Theorem 1 for the precise statement). However, even when the associated K3 surface is not unique, a theorem of Mukai [8] guarantees the existence of an isogeny between any pair of associated K3 surfaces; hence, any algebraic surface with geometric genus one has a unique isogeny class of associated $K3$ surfaces.

The "functoriality property" of our construction should be the follow-

Received December 16, 1985

* Research supported by a National Science Foundation Postdoctoral Fellowship.

ing: if X_1 and X_2 are algebraic surfaces with geometric genus one and Y_1 and Y_2 are the associated K3 surfaces, then every isogeny between X_1 and X_2 is induced by an isogeny between Y_1 and Y_2. We prove this in section 3 under the additional hypothesis that there exist isogenies between X_1 and Y_1 and between X_2 and Y_2; the general case would be a consequence of the Hodge conjecture for products of surfaces.

§1. Hodge theory on a product of surfaces.

By an algebraic surface, we mean a smooth projective variety of dimension 2 over the complex numbers. An algebraic surface X has a *Néron-Severi group* (the group of cohomology classes of line bundles) $NS(X)\subset H^2(X, \boldsymbol{Z})$, and a *transcendental lattice* $T(X, \boldsymbol{Z})=H^2(X, \boldsymbol{Z})/NS(X)$. We identify $T(X, \boldsymbol{Z})$ with the orthogonal complement (with respect to the intersection pairing) of $NS(X)/(\text{tors})$ inside $H^2(X, \boldsymbol{Z})/(\text{tors})$; the intersection pairing then restricts to a nondegenerate integral symmetric bilinear form on $T(X, \boldsymbol{Z})$. We let $NS(X, \boldsymbol{Q})$ and $T(X, \boldsymbol{Q})$ denote $NS(X)\otimes\boldsymbol{Q}$ and $T(X, \boldsymbol{Z})\otimes\boldsymbol{Q}$, respectively. The bilinear form on $T(X, \boldsymbol{Z})$ determines a natural isomorphism of rational Hodge structures $\alpha_X\colon T(X, \boldsymbol{Q})\to T(X, \boldsymbol{Q})^*(-2)$, where $V(n)$ denotes the n^{th} Tate twist of the rational Hodge structure V.

For any complex projective variety Z, let $Hdg(Z)$ denote the $\boldsymbol{Q}$-algebra of rational Hodge classes on Z, that is,

$$Hdg(Z)=(\bigoplus H^i(Z, \boldsymbol{Q}))\cap(\bigoplus H^{p,p}(Z)) .$$

$Hdg(Z)$ is a $\boldsymbol{Q}$-subalgebra of the cohomology algebra $\bigoplus H^i(Z, \boldsymbol{Q})$, and the Lefschetz (1, 1) theorem says that $NS(Z, \boldsymbol{Q})=Hdg(Z)\cap H^2(Z, \boldsymbol{Q})$. We let $Alg(Z)$ denote the $\boldsymbol{Q}$-subalgebra of $Hdg(Z)$ generated by classes of algebraic cycles and $Div(Z)$ denote the $\boldsymbol{Q}$-subalgebra of $Alg(Z)$ generated by $NS(Z, \boldsymbol{Q})$ and the fundamental class $\xi_Z\in H^0(Z, \boldsymbol{Q})$. (Elements of $Div(Z)$ are called *$\boldsymbol{Q}$-linear combinations of complete intersections* on Z.)

If $Z=X\times Y$ is a product, the Hodge cycles on the Künneth components may be described by using the natural isomorphism

$$H^{p,p}(X\times Y)\cap(H^i(X, \boldsymbol{Q})\otimes H^j(Y, \boldsymbol{Q}))\simeq\mathrm{Hom}_{\mathrm{Hod}}(H^i(X, \boldsymbol{Q})^*, H^j(Y, \boldsymbol{Q})(p)) ,$$

where $\mathrm{Hom}_{\mathrm{Hod}}$ denotes morphisms of rational Hodge structures. If $Z=X\times Y$ is a product of two algebraic surfaces, we define the *Hodge-Künneth-Transcendence group* of X and Y by

$$\begin{aligned}HKT(X, Y)&=Hdg(X\times Y)\cap(T(X, \boldsymbol{Q})\otimes T(Y, \boldsymbol{Q}))\\&\simeq\mathrm{Hom}_{\mathrm{Hod}}(T(X, \boldsymbol{Q})^*, T(Y, \boldsymbol{Q})(2)) .\end{aligned}$$

(This is a slight modification of a definition of Okamoto [10].) For any algebraic cycle W on $X\times Y$, we let $[W]_{HKT}$ denote the HKT-component of the cohomology class of W, regarded as a morphism of rational Hodge structures $[W]_{HKT}\colon T(X, \boldsymbol{Q})^* \to T(Y, \boldsymbol{Q})(2)$, and let $[W]_{trans}=[W]_{HKT}(-2)\circ \alpha_X\colon T(X, \boldsymbol{Q}) \to T(Y, \boldsymbol{Q})$ be the induced morphism of rational Hodge structures. (We extend this notation by $\boldsymbol{Q}$-linearity to $Alg(X\times Y)$.)

LEMMA (Lieberman [3], Okamoto [10]). *Let X and Y be algebraic surfaces. For each $\alpha \in Hdg(X\times Y)$ there exists some $\beta \in Div(X\times Y)$ such that $\alpha-\beta \in HKT(X, Y)$.*

PROOF. The lemma follows immediately from the following formula:

$$Hdg(X\times Y)\cap(H^i(X, \boldsymbol{Q})\otimes H^j(Y, \boldsymbol{Q})) = \begin{cases}(NS(X, \boldsymbol{Q})\otimes NS(Y, \boldsymbol{Q}))\oplus HKT(X, Y) & \text{if } i=j=2 \\ Div(X\times Y)\cap(H^i(X, \boldsymbol{Q})\otimes H^j(X, \boldsymbol{Q})) & \text{otherwise}\,.\end{cases}$$

This is straightforward to check, except in the cases $|i-j|=2$, $i+j=4$: in those cases, one must use the hard Lefschetz theorem on X or Y. For example, if $i=1$ and $j=3$, let λ be the class of an ample divisor on Y, and let $\xi_X \in H^0(X, \boldsymbol{Q})$ be the fundamental class of X. Then by hard Lefschetz, cup product with $\xi_X\times\lambda$ gives an isomorphism

$$H^1(X, \boldsymbol{Q})\otimes H^1(Y, \boldsymbol{Q}) \xrightarrow{\sim} H^1(X, \boldsymbol{Q})\otimes H^3(Y, \boldsymbol{Q})(1)$$

which preserves the Hodge structure, and hence induces an isomorphism

$$\mathrm{Hom}_{\mathrm{Hod}}(H^1(X, \boldsymbol{Q})^*, H^1(Y, \boldsymbol{Q})(1)) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{Hod}}(H^1(X, \boldsymbol{Q})^*, H^3(Y, \boldsymbol{Q})(2))\,.$$

Thus, every $\alpha \in Hdg(X\times Y)\cap(H^1(X, \boldsymbol{Q})\otimes H^3(Y, \boldsymbol{Q}))$ is of the form $\alpha=(\xi_X\times\lambda)\cup\gamma$ for some $\gamma \in Hdg(X\times Y)\cap(H^1(X, \boldsymbol{Q})\otimes H^1(Y, \boldsymbol{Q}))\subset Div(X\times Y)$; since $\xi_X\times\lambda \in Div(X\times Y)$ as well, we have $\alpha \in Div(X\times Y)$. The other case is similar. Q.E.D.

DEFINITION. Let X and Y be algebraic surfaces. A *cohomological isogeny* between X and Y is an isomorphism of rational Hodge structures $T(X, \boldsymbol{Q}) \to T(Y, \boldsymbol{Q})$. An *isogeny* between X and Y is an irreducible algebraic cycle $W\subset X\times Y$ such that $[W]_{trans}\colon T(X, \boldsymbol{Q}) \to T(Y, \boldsymbol{Q})$ is a cohomological isogeny. We say that an isogeny or cohomological isogeny is *strict* if it maps the intersection form on $T(X, \boldsymbol{Q})$ to the intersection form on $T(Y, \boldsymbol{Q})$, and is *integral* if it is compatible with an isomorphism of integral Hodge structures $T(X, \boldsymbol{Z}) \to T(Y, \boldsymbol{Z})$.

[We wish to warn the reader that the term "isogeny", when applied in particular to $K3$ surfaces, has several conflicting definitions in the literature. Let X and Y be algebraic K3 surfaces. Inose [1] uses the term "isogeny" to mean a rational map of finite degree between X and Y (when the Picard numbers of X and Y are both 20). Shafarevich [12] uses the term "isogeny" to mean an isomorphism $H^2(X, \boldsymbol{Q}) \to H^2(Y, \boldsymbol{Q})$ preserving the Hodge structures and the intersection pairings. Mukai [8] uses the term "isogeny" to mean an algebraic cycle on $X \times Y$ whose cohomology class is a Shafarevich-isogeny. In the terminology of this paper, an Inose-isogeny is an isogeny, a Mukai-isogeny is a $\boldsymbol{Q}$-linear combination of strict isogenies and algebraic cycles W such that $[W]_{trans}=0$, and a Shafarevich-isogeny restricts to a strict cohomological isogeny. (A previous paper of the author [6] used the term "isogeny" to denote a strict cohomological isogeny between an algebraic K3 surface and an abelian surface.)]

If X and Y are algebraic surfaces with geometric genus one, then every morphism of rational Hodge structures $T(X, \boldsymbol{Q}) \to T(Y, \boldsymbol{Q})$ is either the zero map or an isomorphism. Thus, all non-zero elements of $HKT(X, Y)$ become cohomological isogenies after twisting by -2 and composing with α_X. Moreover, by the Lieberman-Okamoto lemma, every algebraic cycle on $X \times Y$ is homologically equivalent (mod torsion) to a $\boldsymbol{Q}$-linear combination of complete intersections and isogenies. The Hodge conjecture for $X \times Y$ is thus equivalent to the statement: "every cohomological isogeny between X and Y is induced by a $\boldsymbol{Q}$-linear combination of isogenies between X and Y."

§2. Associated K3 surfaces.

A *lattice* is a free $\boldsymbol{Z}$-module L of finite rank, together with a non-degenerate integral symmetric bilinear form $b: L \times L \to \boldsymbol{Z}$. A lattice is *even* if $b(v, v) \in 2\boldsymbol{Z}$ for all $v \in L$, and is *unimodular* if the discriminant of b is ± 1. If T and L are lattices, a *primitive embedding of T into L* is an injective $\boldsymbol{Z}$-linear map $\psi: T \to L$ which preserves the bilinear forms such that coker(ψ) is free; if ψ is also surjective, it is called an *isometry between T and L.*

For any algebraic surface X, the intersection pairing b gives $H^2(X, \boldsymbol{Z})/(\text{tors})$ the structure of a lattice, which is unimodular by Poincaré duality. The Wu formula says that $b(v, v) + b(v, c_1(X)) \in 2\boldsymbol{Z}$ for any $v \in H^2(X, \boldsymbol{Z})$. Thus, since $T(X, \boldsymbol{Z}) \subset c_1(X)^{\perp}$, $T(X, \boldsymbol{Z})$ is an even lattice.

If Y is an algebraic K3 surface (that is, a simply-connected complex algebraic surface with $c_1(Y)=0$), then $H^2(Y, \boldsymbol{Z})$ is an even unimodular lattice, which has signature (3, 19) by the Hodge index theorem. This implies (by a theorem of Milnor) that the isometry class of the lattice $H^2(Y, \boldsymbol{Z})$ does not depend on the algebraic K3 surface Y; we fix one lattice Λ in this isometry class and call it the *K3 lattice.*

THEOREM 1. *Let X be an algebraic surface with geometric genus one.*

(i) *There exists an algebraic K3 surface Y and a strict integral cohomological isogeny between X and Y. (We call Y an* associated K3 surface *of X.)*

(ii) *If the minimal model of X is neither a K3 surface nor a logarithmic transform of an elliptic K3 surface, then any two associated K3 surfaces of X are isomorphic.*

For the proof, we need to analyze the bilinear form on $T(X, \boldsymbol{Z})$.

LEMMA 1. *Let X be an algebraic surface whose minimal model is either a K3 surface or a logarithmic transform of an elliptic K3 surface. Then there exists a primitive embedding ϕ: $T(X, \boldsymbol{Z}) \to \Lambda$.*

PROOF: Since $T(X, \boldsymbol{Z})$ is a birational invariant, we may assume without loss of generality that X is minimal. The lemma is trivial in the case that X is a K3 surface, for there exists an isometry σ: $H^2(X, \boldsymbol{Z}) \to \Lambda$, and $\phi = \sigma|_{T(X,\boldsymbol{Z})}$ is a primitive embedding.

Suppose that X is a logarithmic transform of an elliptic K3 surface. (Our proof in this case is actually an application of Nikulin's technique of discriminant-forms [9], as modified by Looijenga-Wahl [4] in the non-even case, but it is easier to give a direct argument than to set up the machinery.) Let L_1 denote the lattice $H^2(X, \boldsymbol{Z})$/(tors), and let b_1 denote the bilinear form on L_1. Choose $e \in L_1$ such that the $\boldsymbol{Z}$-span of e is a primitive sublattice of L_1 which contains the class of the fiber of the elliptic pencil $|E|$ on X, and let $f \in L_1$ be the class of some fixed multisection of $|E|$ (which exists since X is algebraic). Then there exist integers k, m, and n with $m \neq 0$ such that $c_1(X) \equiv ke$ mod torsion, and the intersection matrix of e and f is

$$\begin{pmatrix} 0 & m \\ m & n \end{pmatrix}.$$

Let M_1 be the $\boldsymbol{Z}$-span of e and f in L_1, and $N = M_1^{\perp}$. Note that $M_1 \subset NS(X)$/(tors) so that $T(X, \boldsymbol{Z}) \subset N$, and this inclusion is primitive.

Define a lattice M_2 of rank 2 by means of a basis x and y such that the bilinear form has matrix

$$\begin{pmatrix} 0 & m \\ m & n-km^2 \end{pmatrix}$$

with respect to this basis, and let b_2 denote the induced bilinear form on $M_2 \oplus N$. Let

$$L_2 = \{(\alpha x + \beta y) + z \in (M_2 \oplus N) \otimes \boldsymbol{Q} \mid \alpha e + \beta f + z \in L_1\} ,$$

and consider a pair of elements $u_i = \alpha_i x + \beta_i y + z_i \in L_2$ for $i=1, 2$ with corresponding elements $v_i = \alpha_i e + \beta_i f + z_i \in L_1$. Then $b_1(v_i, e) = \beta_i m$ is an integer, so that

$$b_2(u_1, u_2) = b_1(v_1, v_2) - km^2 \beta_1 \beta_2$$

is also an integer, and b_2 makes L_2 into a lattice. Moreover,

$$b_2(u_1, u_1) = b_1(v_1, v_1) + b_1(v_1, c_1(X)) - km\beta_1(m\beta_1 + 1) .$$

Now $b_1(v_1, v_1) + b_1(v_1, c_1(X))$ is even by the Wu formula, and $km\beta_1(m\beta_1+1)$ is an even integer since $m\beta_1 \in \boldsymbol{Z}$. Thus, $b_2(u_1, u_1) \in 2\boldsymbol{Z}$ so that L_2 is an even lattice.

Since M_1 and M_2 both have signature (1, 1), the signature of L_2 coincides with that of L_1, which is (3, 19). Moreover, $[L_1 : M_1 \oplus N] = [L_2 : M_2 \oplus N]$ by construction, so that L_2 is unimodular (since L_1 is unimodular). Hence, L_2 is an even unimodular lattice of signature (3, 19), so that there exists an isometry $\sigma : L_2 \to \Lambda$ by Milnor's theorem. Since $T(X, \boldsymbol{Z}) \subset N \subset L_2$, we may restrict σ to $T(X, \boldsymbol{Z})$, which gives a primitive embedding $\sigma|_{T(X,\boldsymbol{Z})}$: $T(X, \boldsymbol{Z}) \to \Lambda$. Q.E.D.

A minimal surface X with geometric genus one satisfies $c_1^2(X) = q(X) = 0$ if and only if X if a K3 surface or a logarithmic transform of an elliptic K3 surface.

LEMMA 2. *Let X be an algebraic surface with geometric genus one, whose minimal model is neither a K3 surface nor a logarithmic transform of an elliptic K3 surface. Then there is a primitive embedding $\phi : T(X, \boldsymbol{Z}) \to \Lambda$. Moreover, if $\phi_1, \phi_2 : T(X, \boldsymbol{Z}) \to \Lambda$ are two primitive embeddings, there is an isometry $\sigma : \Lambda \to \Lambda$ such that $\phi_2 = \sigma \circ \phi_1$.*

PROOF. As in the proof of Lemma 1, we may assume that X is minimal. We first show that there is a unimodular lattice L of rank at most 20, and a primitive embedding $\psi : T(X, \boldsymbol{Z}) \to L$. By Noether's formula, $b_2(X) = 22 - 8q(X) - c_1^2(X)$, so that if $c_1^2(X) \geqq 2$ or $q(X) \geqq 1$, we may take $L = H^2(X, \boldsymbol{Z})$. Our hypothesis on X guarantees that either $c_1^2(X) \geqq 1$ or

$q(X)\geqq 1$. In the remaining case $c_1^2(X)=1$, $q(X)=0$, we take $L=(c_1(X))^{\perp}\subset H^2(X, \boldsymbol{Z})$; this has rank 20, is unimodular since $c_1^2(X)=1$, and contains $T(X, \boldsymbol{Z})$.

Let α_X: $T(X, \boldsymbol{Z})\to \mathrm{Hom}(T(X, \boldsymbol{Z}), \boldsymbol{Z})$ be the map induced by the bilinear form on $T(X, \boldsymbol{Z})$. The existence of ψ implies that the finite abelian group $\mathrm{coker}(\alpha_X)$ has at most $\mathrm{rank}(\psi(T(X, \boldsymbol{Z}))^{\perp})\leqq 20-\mathrm{rank}(T(X, \boldsymbol{Z}))$ generators; but then since $\mathrm{rank}(\Lambda)=22$, the Nikulin embedding theorem [9] implies that there is a primitive embedding ϕ: $T(X, \boldsymbol{Z})\to \Lambda$, and that any two such differ by an isometry between Λ and Λ. Q.E.D.

PROOF OF THEOREM 1. (i) Let ϕ: $T(X, \boldsymbol{Z})\to\Lambda$ be the primitive embedding constructed in Lemmas 1 and 2. We give Λ a Hodge structure $\Lambda_{\boldsymbol{C}}=\Lambda^{2,0}\oplus\Lambda^{1,1}\oplus\Lambda^{0,2}$ by requiring that ϕ be a morphism of Hodge structures, and $(\mathrm{im}(\phi))^{\perp}\otimes\boldsymbol{C}\subset\Lambda^{1,1}$. By the surjectivity of the period map for algebraic K3 surfaces [2], there is a K3 surface Y and an isomorphism of integral Hodge structures α: $H^2(Y, \boldsymbol{Z})\to\Lambda$ (with respect to our chosen Hodge structure on Λ), which is also an isometry. Then $\alpha(T(Y, \boldsymbol{Z}))=\phi(T(X, \boldsymbol{Z}))$ so that $\alpha^{-1}\circ\phi$: $T(X, \boldsymbol{Z})\to T(Y, \boldsymbol{Z})$ gives the required strict integral cohomological isogeny.

(ii) Suppose that the minimal model of X is neither a K3 surface nor a logarithmic transform of an elliptic K3 surface. Let Y_i be an algebraic K3 surface and β_i: $T(X, \boldsymbol{Z})\to T(Y_i, \boldsymbol{Z})$ be a strict integral cohomological isogeny for $i=1, 2$, and choose an isometry α_i: $H^2(Y_i, \boldsymbol{Z})\to\Lambda$. Then $\alpha_i\circ\beta_i$: $T(X, \boldsymbol{Z})\to\Lambda$ is a primitive embedding for $i=1, 2$; by Lemma 2, there is some isometry σ: $\Lambda\to\Lambda$ such that $\sigma\circ\alpha_1\circ\beta_1=\alpha_2\circ\beta_2$. Let $\gamma=\alpha_2^{-1}\circ\sigma\circ\alpha_1$: $H^2(Y_1, \boldsymbol{Z})\to H^2(Y_2, \boldsymbol{Z})$. Then $\gamma|_{T(Y_1,\boldsymbol{Z})}=\beta_2\circ\beta_1^{-1}$ is an isomorphism of Hodge structures, which implies that γ is itself an isomorphism of Hodge structures. Since γ is also an isometry, by the global Torelli theorem for algebraic K3 surfaces [11], Y_1 is isomorphic to Y_2. Q.E.D.

§ 3. Composition of isogenies.

Let X, Y, and Z be algebraic surfaces, and let $U\subset X\times Y$ and $V\subset Y\times Z$ be isogenies. Define the *composite isogeny* $V\circ U$ to be the image of the fiber product $\{(u, v)\in U\times V\,|\,pr_2(u)=pr_1(v)\}$ under projection to $X\times Z$.

LEMMA 3. *If $U\subset X\times Y$ and $V\subset Y\times Z$ are isogenies, then $[V\circ U]_{trans}=[V]_{trans}\circ[U]_{trans}$; in particular, $V\circ U$ is an isogeny.*

PROOF. Let $\xi_X\in H^0(X, \boldsymbol{Q})$ and $\xi_Z\in H^0(Z, \boldsymbol{Q})$ be the fundamental classes, and let $[Y]\in H_4(Y, \boldsymbol{Q})$ be the fundamental class in homology. $V\circ U$ may be regarded as the projection to $X\times Z$ of the cycle $(U\times Z)\cap(X\times V)$ on $X\times Y\times Z$. Thus, $[V\circ U]_{HKT}$ is the HKT-part of the projection to $X\times Z$ of the cohomology class $([U]_{HKT}\times\xi_Z)\cup(\xi_X\times[V]_{HKT})$. Now the cup product map

$$(T(X, \boldsymbol{Q})\otimes T(Y, \boldsymbol{Q})\otimes H^0(Z, \boldsymbol{Q})\otimes(H^0(X, \boldsymbol{Q})\otimes T(Y, \boldsymbol{Q})\otimes T(Z, \boldsymbol{Q}))$$
$$\longrightarrow T(X, \boldsymbol{Q})\otimes H^4(Y, \boldsymbol{Q})\otimes T(Z, \boldsymbol{Q})$$

is simply induced by the cup product $T(Y, \boldsymbol{Q})\otimes T(Y, \boldsymbol{Q})\to H^4(Y, \boldsymbol{Q})$, while the projection to $X\times Z$ is induced by evaluating this cup product on the fundamental homology class $[Y]$. This evaluation gives the intersection pairing on $T(Y, \boldsymbol{Q})$, so that when $[V\circ U]_{HKT}$ is regarded as a homomorphism $T(X, \boldsymbol{Q})^*\to T(Y, \boldsymbol{Q})(2)$, we have

$$[V\circ U]_{HKT}=[V]_{HKT}\circ\alpha_Y(2)\circ[U]_{HKT}\ .$$

But then

$$[V\circ U]_{trans}=[V]_{HKT}(-2)\circ\alpha_Y\circ[U]_{HKT}(-2)\circ\alpha_X$$
$$=[V]_{trans}\circ[U]_{trans}\ .$$

Q.E.D.

From this lemma we immediately deduce the following "functoriality property" of our construction.

COROLLARY. *Let X_i be an algebraic surface with geometric genus one, Y_i be an associated K3 surface of X_i, and suppose that there exists a strict integral isogeny between X_i and Y_i for $i=1, 2$. (For example, X_i may be an abelian surface* [13, 5] *or a Todorov surface* [7].) *Then any isogeny between X_1 and X_2 is induced by an isogeny between Y_1 and Y_2.*

As a consequence of this corollary, the problem of constructing isogenies between algebraic surfaces with geometric genus one can be divided into two parts: constructing a strict integral isogeny between a given surface and its associated K3 surface(s), and constructing all isogenies between algebraic K3 surfaces. In the cases in which the associated K3 surface is not unique, to solve the first problem it is in fact sufficient to construct a strict integral isogeny between the given surface and any one of its associated K3 surfaces, as is shown by the following theorem of Mukai.

THEOREM (Mukai [8]). *Let Y_1 and Y_2 be algebraic K3 surfaces, and let α: $T(Y_1, \boldsymbol{Z})\to T(Y_2, \boldsymbol{Z})$ be a strict integral cohomological isogeny. Then there is an algebraic $\boldsymbol{Q}$-cycle $W\in H^4(Y_1\times Y_2, \boldsymbol{Q})$, such that $[W]_{trans}=\alpha$.*

COROLLARY *Let X be an algebraic surface with geometric genus one.*

(i) *There is a unique strict integral isogeny class of associated $K3$ surfaces of X.*

(ii) *If Y_1 and Y_2 are associated $K3$ surfaces of X and there exists a strict integral isogeny between X and Y_1, then there exists a strict integral isogeny between X and Y_2.*

The first part follows from Mukai's theorem combined with Theorem 1, while the second part also requires Lemma 3.

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Present Address:
DEPARTMENT OF MATHEMATICS
DUKE UNIVERSITY
DURHAM, NORTH CAROLINA 27706
U.S.A.