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A Certain Generalized Dedekind Sum

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Introduction

Let $[x]$ denote the greatest integer less than or equal to the real number x and $((x))=0$ or $x-[x]-\frac{1}{2}$ according as x is or is not an integer, respectively. L. Carlitz [3] studied some sums of the type $\sum_{k=1}^{p-1}\left[\frac{kq}{p}\right]^n$ and $\sum_{k=1}^{p-1}k^i\left[\frac{kq}{p}\right]^j$, where $(p, q)=1$, and obtained some reciprocity relations. These are related to the classical Dedekind sums which occur in the transformation formula for the logarithm of the Dedekind eta function. Recently, B.C. Berndt and L.A. Goldberg [2] studied the sums of the type $\sum_{k=1}^{p-1}(-1)^{\left[\frac{kq}{p}\right]}$, $\sum_{k=1}^{p-1}(-1)^k\left(\left(\frac{kq}{p}\right)\right)\left(\left(\frac{k}{p}\right)\right)$, etc., which arose in the transformation formula for the logarithm of the theta functions.

On the other hand, in order to decide whether the trigonometric sum $\sum_{k=1}^{p-1}\left|\cot\frac{k\pi}{p}\cdot\cot\frac{kq\pi}{p}\right|$ is a complete invariant for the isometric class of 3-dimensional lens spaces or not, we have to study some properties of the sum $\sum_{k=1}^{q-1}\left[\frac{kp}{q}\right]^2\zeta^{\left[\frac{kp}{q}\right]q}$, where p is a prime number, $1\leqq q<p$ and ζ is a p^{th} root of unity. This sum is the classical Dedekind sum when $\zeta=1$ and this is the reason of the title of this paper.

Throughout this paper, let

$$H_n(p, q)=\sum_{h=1}^{p-1}\left[\frac{hq}{p}\right]^n\zeta^{hq}$$

and

$$K_n(p, q)=\sum_{k=1}^{q-1}\left[\frac{kp}{q}\right]^{n-1}\zeta^{\left[\frac{kp}{q}\right]q},$$

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where p and q are as above and $\zeta=\exp\frac{2\pi\sqrt{-1}}{p}$. We study the exact values of $H_n(p, q)$ and $K_n(p, q)$ in §1. However, we cannot evaluate the values for $n\geqq 2$. Hence, we next study, in §2, the reciprocity relation between them. A result is Theorem 5:

$$p(\zeta^q-1)H_2(p, q)+2q\zeta^q K_2(p, q)=\frac{\zeta^q-1}{\zeta-1}\left(p-2-\frac{2}{\zeta-1}\right)+\frac{2q}{\zeta-1}+(pq-2p+2)q .$$

This relation corresponds to the well known reciprocity formula:

$$6p\sum_{h=1}^{p-1}\left[\frac{hq}{p}\right]^2+6q\sum_{k=1}^{q-1}\left[\frac{kp}{q}\right]^2=(p-1)(2p-1)(q-1)(2q-1) .$$

Although, corresponding to Carlitz [3], (2.2), the reciprocity relation for $n=3$ is expected to exist, we are unable to find it. The only result we have obtained is the reciprocity relation between the real part of $H_3(p, q)$ and the imaginary part of $\zeta^{\frac{q}{2}}K_3(p, q)$:

$$\begin{aligned}&2p(p-1)(\zeta^q-1)\{H_3(p, q)+\overline{H_3(p, q)}\}+6q(q-1)\zeta^{\frac{q}{2}}\{\zeta^{\frac{q}{2}}K_3(p, q)-\overline{\zeta^{\frac{q}{2}}K_3(p, q)}\}\\&\quad=2p(p-1)(q-1)^3(1-\zeta^q)+\frac{6(p-1)q(q-1)(\zeta+\zeta^q)}{\zeta-1}-\frac{12(p-1)(q-1)\zeta(\zeta^q-1)}{(\zeta-1)^2} .\end{aligned}$$

In §3, we show another reciprocity relation which is fundamental in studying the reciprocity formulas and we give some remarks.

§1. Computations.

We can obtain the exact values for the following three cases only.

PROPOSITION 1.

(1) $$H_0(p, q)=-1 ,$$

(2) $$K_1(p, q)=\frac{\zeta^q-\zeta}{(\zeta-1)\zeta^q} ,$$

(3) $$H_1(p, q)=\frac{q}{\zeta^q-1}-\frac{1}{\zeta-1} .$$

PROOF. (1) is clear and to show (2), put

$$\gamma_k=kp-\left[\frac{kp}{q}\right]q .$$

Then $K_1(p, q)=\sum_{k=1}^{q-1}\bar{\zeta}^{\gamma_k}$, where $\bar{\zeta}$ denotes the complex conjugate of ζ. Since the numbers γ_k for $k=1, \cdots, q-1$ are simply the numbers $1, \cdots, q-1$ in some order, we get $K_1(p, q)=\sum_{j=1}^{q-1}\zeta^j=\dfrac{\zeta^q-\zeta}{(\zeta-1)\zeta^q}$.

To show (3), we need the following key lemma. For each integer h $(1\leqq h\leqq p-1)$, put

$$\delta_h=hq-\left[\frac{hq}{p}\right]p .$$

Then we have

$$\left[\frac{(h+1)q}{p}\right]-\left[\frac{hq}{p}\right]=\begin{cases}1 & \text{if } \delta_h+q\geqq p ,\\ 0 & \text{otherwise} .\end{cases}$$

Since $\sum_{h=1}^{p-1}\left\{\left[\frac{(h+1)q}{p}\right]-\left[\frac{hq}{p}\right]\right\}=q$, there exist q such integers that satisfy $\delta_h+q\geqq p$. Put them

$$(1\leqq)\, h_1<h_2<\cdots<h_{q-1}<h_q\, (\leqq p-1) .$$

LEMMA 2. *These integers are represented as follows:*

$$h_k=\left[\frac{kp}{q}\right] \qquad \textit{for} \quad k=1, 2, \cdots, q-1$$

and

$$h_q=p-1 .$$

PROOF. It follows from the inequalities

$$\frac{kp}{q}-1<\left[\frac{kp}{q}\right]<\frac{kp}{q}$$

that

$$k-\frac{q}{p}<\left[\frac{kp}{q}\right]\frac{q}{p}<k$$

and hence we get $\left[\left[\frac{kp}{q}\right]\frac{q}{p}\right]=k-1$ for each $k=1, 2, \cdots, q-1$. Thus

$$\delta_{\left[\frac{kp}{q}\right]}+q=\left[\frac{kp}{q}\right]q-\left[\left[\frac{kp}{q}\right]\frac{q}{p}\right]p+q=kp-\gamma_k-(k-1)p+q\geqq p ,$$

and we conclude that $\left[\frac{kp}{q}\right]=h_k$ $(1\leqq k\leqq q-1)$. The last equation $h_q=p-1$ follows from the fact that $\delta_{p-1}+q=p$ and $h_{q-1}<p-1$.

It follows from this lemma that

$$\sum_{h=1}^{p-1}\left\{\left[\frac{(h+1)q}{p}\right]-\left[\frac{hq}{p}\right]\right\}\zeta^{hq}=\sum_{k=1}^{q}\zeta^{h_k q}=\sum_{k=1}^{q-1}\zeta^{\left[\frac{kp}{q}\right]q}+\zeta^{(p-1)q}.$$

On the other hand, since the left-hand side of this equation is equal to

$$\frac{1}{\zeta^q}\sum_{h=1}^{p-1}\left[\frac{(h+1)q}{p}\right]\zeta^{(h+1)q}-H_1(p, q)$$
$$=\frac{1}{\zeta^q}\left\{\sum_{h=1}^{p-1}\left[\frac{hq}{p}\right]\zeta^{hq}+q\zeta^{pq}-\left[\frac{q}{p}\right]\zeta^q\right\}-H_1(p, q),$$

we get

$$(\zeta^q-1)H_1(p, q)+\zeta^q K_1(p, q)=q-1$$

and hence (3) follows from (2).

As we cannot determine the exact values of $\sum_{h=1}^{p-1}\left[\frac{hq}{p}\right]^n$ and $\sum_{k=1}^{q-1}\left[\frac{kp}{q}\right]^n$ for $n\geqq 2$, it must be impossible to evaluate $H_n(p, q)$ and $K_n(p, q)$ for $n\geqq 2$. However, we obtain the following proposition which shows the subtle difference between $\sum_{h=1}^{p-1}\left[\frac{hq}{p}\right]^2$ $\left(\text{or } \sum_{k=1}^{q-1}\left[\frac{kp}{q}\right]^2\right)$ and $H_2(p, q)$ (or $K_2(p, q)$ respectively).

Proposition 3. *The imaginary part of $H_2(p, q)$ and the real part of $\zeta^{\frac{q}{2}}K_2(p, q)$ can be evaluated as follows:*

$$(4)\qquad H_2(p, q)-\overline{H_2(p, q)}=(q-1)\left\{\frac{q(\zeta^q+1)}{\zeta^q-1}-\frac{\zeta+1}{\zeta-1}\right\},$$

$$(5)\qquad \zeta^{\frac{q}{2}}K_2(p, q)+\overline{\zeta^{\frac{q}{2}}K_2(p, q)}=(p-1)\bar{\zeta}^{\frac{q}{2}}\cdot\frac{\zeta^q-\zeta}{\zeta-1}.$$

Proof. Since

$$H_2(p, q)=\sum_{h=1}^{p-1}\left[\frac{(p-h)q}{p}\right]^2\zeta^{(p-h)q}$$
$$=\sum_{h=1}^{p-1}\left\{q-1-\left[\frac{hq}{p}\right]\right\}^2\bar{\zeta}^{hq}$$
$$=(q-1)^2\overline{H_0(p, q)}-2(q-1)\overline{H_1(p, q)}+\overline{H_2(p, q)},$$

we get (4). Similarly,

$$\zeta^{\frac{q}{2}}K_2(p, q)=\sum_{k=1}^{q-1}\left[\frac{(q-k)p}{q}\right]\zeta^{\left[\frac{(q-k)p}{q}\right]q+\frac{q}{2}}$$
$$=(p-1)\overline{\zeta^{\frac{q}{2}}K_1(p, q)}-\overline{\zeta^{\frac{q}{2}}K_2(p, q)},$$

and (5) follows at once.

As for the values of the real part of $H_2(p, q)$ and the imaginary part of $\zeta^{\frac{q}{2}}K_2(p, q)$, we obtain the following relations only.

PROPOSITION 4.

$$2\{H_3(p, q)+\overline{H_3(p, q)}\}=3(q-1)\{H_2(p, q)+\overline{H_2(p, q)}\}+(q-1)^3 , \tag{6}$$

$$\zeta^{\frac{q}{2}}K_3(p, q)-\overline{\zeta^{\frac{q}{2}}K_3(p, q)}=(p-1)\{\zeta^{\frac{q}{2}}K_2(p, q)-\overline{\zeta^{\frac{q}{2}}K_2(p, q)}\} . \tag{7}$$

PROOF. Substituting $p-h$ for h, we get

$$H_3(p, q)=\sum_{h=1}^{p-1}\left\{(q-1)^3-3(q-1)^2\left[\frac{hq}{p}\right]+3(q-1)\left[\frac{hq}{p}\right]^2-\left[\frac{hq}{p}\right]^3\right\}\bar{\zeta}^{hq} ,$$

so that

$$H_3(p, q)+\overline{H_3(p, q)}=(q-1)^3\overline{H_0(p, q)}-3(q-1)^2\overline{H_1(p, q)}+3(q-1)\overline{H_2(p, q)} .$$

Since the left-hand side is real, we get

$$H_3(p, q)+\overline{H_3(p, q)}=(q-1)^3H_0(p, q)-3(q-1)^2H_1(p, q)+3(q-1)H_2(p, q) ,$$

and hence

$$2\{H_3(p, q)+\overline{H_3(p, q)}\}=(q-1)^3\{H_0(p, q)+\overline{H_0(p, q)}\}$$
$$-3(q-1)^2\{H_1(p, q)+\overline{H_1(p, q)}\}+3(q-1)\{H_2(p, q)+\overline{H_2(p, q)}\} .$$

Using Proposition 1, (6) follows at once.

We get (7) similarly.

Proposition 3 and Proposition 4 are generalized as follows. Substituting $p-h$ for h, we get

$$H_n(p, q)=\sum_{j=0}^{n}(-1)^j\binom{n}{j}(q-1)^{n-j}\sum_{h=1}^{p-1}\left[\frac{hq}{p}\right]^j\bar{\zeta}^{hq}$$

so that

$$H_n(p, q)-(-1)^n\overline{H_n(p, q)}=\sum_{j=0}^{n-1}(-1)^j\binom{n}{j}(q-1)^{n-j}\overline{H_j(p, q)}$$
$$=-(-1)^n\sum_{j=0}^{n-1}(-1)^j\binom{n}{j}(q-1)^{n-j}H_j(p, q) ,$$

and hence

$$2\{H_n(p, q)-(-1)^n\overline{H_n(p, q)}\}$$
$$=-(-1)^n\sum_{j=0}^{n-1}(-1)^j\binom{n}{j}(q-1)^{n-j}\{H_j(p, q)-(-1)^n\overline{H_j(p, q)}\} . \tag{8}$$

Similarly, we get

$$(9)\quad 2\{\zeta^{\frac{q}{2}}K_n(p, q)+(-1)^n\overline{\zeta^{\frac{q}{2}}K_n(p, q)}\}$$
$$=(-1)^n\sum_{j=1}^{n-1}(-1)^{j-1}\binom{n-1}{j-1}(p-1)^{n-j}\{\zeta^{\frac{q}{2}}K_j(p, q)+(-1)^n\overline{\zeta^{\frac{q}{2}}K_j(p, q)}\}\ .$$

Note that the recursive relations (8) and (9) do not give any available information about the imaginary part of $H_3(p, q)$ and the real part of $\zeta^{\frac{q}{2}}K_3(p, q)$.

§ 2. Reciprocity formulas.

We first study the relation between $H_2(p, q)$ and $K_2(p, q)$. Applying Lemma 2, we get

$$\sum_{h=1}^{p-1}h\left\{\left[\frac{(h+1)q}{p}\right]-\left[\frac{hq}{p}\right]\right\}\zeta^{hq}=\sum_{k=1}^{q}h_k\zeta^{h_kq}$$
$$=\sum_{k=1}^{q-1}\left[\frac{kp}{q}\right]\zeta^{\left[\frac{kp}{q}\right]q}+(p-1)\zeta^{(p-1)q}\ .$$

Since the left-hand side is

$$\frac{1-\zeta^q}{\zeta^q}\sum_{h=1}^{p-1}h\left[\frac{hq}{p}\right]\zeta^{hq}+\frac{pq}{\zeta^q}-\frac{\left[\frac{q}{p}\right]\zeta^q}{\zeta^q}-\frac{1}{\zeta^q}\left\{\sum_{h=1}^{p-1}\left[\frac{hq}{p}\right]\zeta^{hq}+q-\left[\frac{q}{p}\right]\zeta^q\right\}$$
$$=\frac{1-\zeta^q}{\zeta^q}\sum_{h=1}^{p-1}h\left[\frac{hq}{p}\right]\zeta^{hq}-\frac{1}{\zeta^q}\sum_{h=1}^{p-1}\left[\frac{hq}{p}\right]\zeta^{hq}+\frac{(p-1)q}{\zeta^q}\ ,$$

we get

$$(1-\zeta^q)\sum_{h=1}^{p-1}h\left[\frac{hq}{p}\right]\zeta^{hq}-H_1(p, q)+(p-1)(q-1)=\zeta^qK_2(p, q)\ .$$

Now, it follows from $\delta_h=hq-\left[\frac{hq}{p}\right]p$ that

$$2pq\sum_{h=1}^{p-1}h\left[\frac{hq}{p}\right]\zeta^{hq}=q^2\sum_{h=1}^{p-1}h^2\zeta^{hq}+p^2\sum_{h=1}^{p-1}\left[\frac{hq}{p}\right]^2\zeta^{hq}-\sum_{h=1}^{p-1}\delta_h^2\zeta^{hq}\ .$$

Since

$$\sum_{h=1}^{p-1}h^2\zeta^{hq}=\frac{1}{1-\zeta^q}\left\{\frac{-2p}{1-\zeta^q}-p^2+2p\right\}$$

and

$$\sum_{h=1}^{p-1} \delta_h^2 \zeta^{hq} = \sum_{h=1}^{p-1} \delta_h^2 \zeta^{s_h} = \frac{1}{1-\zeta}\left\{\frac{-2p}{1-\zeta} - p^2 + 2p\right\} ,$$

we obtain the following reciprocity formula by substituting these values.

THEOREM 5.

$$(10) \qquad p(\zeta^q - 1)H_2(p, q) + 2q\zeta^q K_2(p, q)$$
$$= \frac{\zeta^q - 1}{\zeta - 1}\left(p - 2 - \frac{2}{\zeta - 1}\right) + \frac{2q}{\zeta - 1} + (pq - 2p + 2)q .$$

It seems plausible that $H_3(p, q)$ and $K_3(p, q)$ satisfy a relation similar to (10) (cf. Carlitz [3], (2.2)). However, we have not yet succeeded in getting it. On the other hand, taking the relations (5), (7) and (9) into account, it may be natural to study the relation between the real or imaginary part of $H_n(p, q)$ and $\zeta^{\frac{q}{2}} K_n(p, q)$. As for the cases $n=2$ and $n=3$, we obtain the following

THEOREM 6. *There are reciprocity relations*:

$$(11) \qquad p(\zeta^q - 1)\{H_2(p, q) + \overline{H_2(p, q)}\} + 2q\zeta^{\frac{q}{2}}\{\zeta^{\frac{q}{2}} K_2(p, q) - \overline{\zeta^{\frac{q}{2}} K_2(p, q)}\}$$
$$= p(q-1)^2(1-\zeta^q) + \frac{2q(\zeta + \zeta^q)}{\zeta - 1} - \frac{4\zeta(\zeta^q - 1)}{(\zeta - 1)^2}$$

and

$$(12) \quad 2p(p-1)(\zeta^q - 1)\{H_3(p, q) + \overline{H_3(p, q)}\} + 6q(q-1)\zeta^{\frac{q}{2}}\{\zeta^{\frac{q}{2}} K_3(p, q) - \overline{\zeta^{\frac{q}{2}} K_3(p, q)}\}$$
$$= 2p(p-1)(q-1)^3(1-\zeta^q) + \frac{6(p-1)q(q-1)(\zeta + \zeta^q)}{\zeta - 1} - \frac{12(p-1)(q-1)\zeta(\zeta^q - 1)}{(\zeta - 1)^2} .$$

PROOF. By multiplying ζ^q to the complex conjugate of (10), we get

$$p(1-\zeta^q)\overline{H_2(p, q)} + 2q\overline{K_2(p, q)} = \frac{1-\zeta^q}{\bar{\zeta} - 1}\left(p - 2 - \frac{2}{\bar{\zeta} - 1}\right) + \frac{2q\zeta^q}{\bar{\zeta} - 1} + (pq - 2p + 2)q\zeta^q .$$

Subtracting this from (10), we get

$$p(\zeta^q - 1)\{H_2(p, q) + \overline{H_2(p, q)}\} + 2q\zeta^{\frac{q}{2}}\{\zeta^{\frac{q}{2}} K_2(p, q) - \overline{\zeta^{\frac{q}{2}} K_2(p, q)}\}$$
$$= (\zeta^q - 1)\left\{(p-2)\left(\frac{1}{\zeta - 1} + \frac{1}{\bar{\zeta} - 1}\right) - 2\left\{\frac{1}{(\zeta - 1)^2} + \frac{1}{(\bar{\zeta} - 1)^2}\right\}\right\} + 2q\left(\frac{1}{\zeta - 1} - \frac{\zeta^q}{\bar{\zeta} - 1}\right)$$
$$+ (pq - 2p + 2)q(1 - \zeta^q)$$

so that (11) follows.

Substituting (6) and (7) into (11), we get (12).

Conversely, (10) follows from (11) together with Proposition 3. However, since we have no available information on the imaginary part of $H_3(p, q)$ and the real part of $\zeta^{\frac{q}{2}}K_3(p, q)$ as we noticed at the end of §1, it is impossible to deduce a reciprocity formula between $H_3(p, q)$ and $\zeta^{\frac{q}{2}}K_3(p, q)$ from (12). Also, there would probably be no reciprocity relation for $n \geqq 4$.

§3. Further arguments.

Applying Lemma 2 to

$$\sum_{h=1}^{p-1} h^n\left\{\left[\frac{(h+1)q}{p}\right]-\left[\frac{hq}{p}\right]\right\}\zeta^{hq}$$

and

$$\sum_{h=1}^{p-1}\left\{\left[\frac{(h+1)q}{p}\right]^n-\left[\frac{hq}{p}\right]^n\right\}\zeta^{hq} \qquad (n \geqq 1)$$

respectively, we obtain

$$\zeta^q\left\{\sum_{k=1}^{q-1}\left[\frac{kp}{q}\right]^n\zeta^{\left[\frac{kp}{q}\right]q}+\sum_{h=1}^{p-1}h^n\left[\frac{hq}{p}\right]\zeta^{hq}\right\}-\sum_{h=1}^{p-1}(h-1)^n\left[\frac{hq}{p}\right]\zeta^{hq}=(p-1)^n(q-1) \tag{13}$$

and

$$\zeta^q\sum_{k=1}^{q-1}\{k^n-(k-1)^n\}\zeta^{\left[\frac{kp}{q}\right]q}+(\zeta^q-1)\sum_{h=1}^{p-1}\left[\frac{hq}{p}\right]^n\zeta^{hq}=(q-1)^n \tag{14}$$

respectively. Proposition 1 is a direct consequence of these equations and, as we have already seen, Proposition 3 and Theorem 5 follow easily from these. Instead of proving them, we show the following general equation which reduces to (13) or (14) when we take $i=n$, $j=1$ or $i=0$, $j=n$ respectively.

PROPOSITION 7. *For each nonnegative integer i and positive integer j, we have a sort of reciprocity formula:*

$$\zeta^q\left\{\sum_{k=1}^{q-1}\{k^j-(k-1)^j\}\left[\frac{kp}{q}\right]^i\zeta^{\left[\frac{kp}{q}\right]q}+\sum_{h=1}^{p-1}h^i\left[\frac{hq}{p}\right]^j\zeta^{hq}\right\}-\sum_{h=1}^{p-1}(h-1)^i\left[\frac{hq}{p}\right]^j\zeta^{hq}=(p-1)^i(q-1)^j\,. \tag{15}$$

PROOF. Applying Lemma 2, we get

$$\sum_{h=1}^{p-1} h^i\left\{\left[\frac{(h+1)q}{p}\right]^j-\left[\frac{hq}{p}\right]^j\right\}\zeta^{hq}=\sum_{k=1}^{q} h_k^i\left\{\left(\left[\frac{h_kq}{p}\right]+1\right)^j-\left[\frac{h_kq}{p}\right]^j\right\}\zeta^{h_kq}$$
$$=\sum_{k=1}^{q-1}\left[\frac{kp}{q}\right]^i\{k^j-(k-1)^j\}\zeta^{\left[\frac{kp}{q}\right]q}+(p-1)^i\left\{\left(\left[\frac{(p-1)q}{p}\right]+1\right)^j-\left[\frac{(p-1)q}{p}\right]^j\right\}\zeta^{(p-1)q}$$
$$=\sum_{k=1}^{q-1}\{k^j-(k-1)^j\}\left[\frac{kp}{q}\right]^i\zeta^{\left[\frac{kp}{q}\right]q}+\frac{(p-1)^i\{q^j-(q-1)^j\}}{\zeta^q}\ .$$

Since the left-hand side reduces to

$$\sum_{h=1}^{p-1}(h+1)^i\left[\frac{(h+1)q}{p}\right]^j\zeta^{(h+1)q-q}-\sum_{h=1}^{p-1}\{(h+1)^i-h^i\}\left[\frac{(h+1)q}{p}\right]^j\zeta^{hq}-\sum_{h=1}^{p-1}h^i\left[\frac{hq}{p}\right]^j\zeta^{hq}$$
$$=\frac{1-\zeta^q}{\zeta^q}\sum_{h=1}^{p-1}h^i\left[\frac{hq}{p}\right]^j\zeta^{hq}+\frac{p^iq^j}{\zeta^q}-\left[\frac{q}{p}\right]^j$$
$$-\frac{1}{\zeta^q}\left\{\sum_{h=1}^{p-1}\{h^i-(h-1)^i\}\left[\frac{hq}{p}\right]^j\zeta^{hq}+\{p^i-(p-1)^i\}q^j\zeta^{pq}-\left[\frac{q}{p}\right]^j\zeta^q\right\}$$
$$=-\sum_{h=1}^{p-1}h^i\left[\frac{hq}{p}\right]^j\zeta^{hq}+\frac{1}{\zeta^q}\sum_{h=1}^{p-1}(h-1)^i\left[\frac{hq}{p}\right]^j\zeta^{hq}+\frac{(p-1)^iq^j}{\zeta^q}\ ,$$

(15) follows immediately.

Note that (15) is valid when $\zeta=1$ and hence we obtain the following reciprocity formula.

PROPOSITION 8. *For each nonnegative integer i and positive integer j, there exists a reciprocity formula*:

$$\sum_{k=1}^{q-1}\{k^j-(k-1)^j\}\left[\frac{kp}{q}\right]^i+\sum_{h=1}^{p-1}\{h^i-(h-1)^i\}\left[\frac{hq}{p}\right]^j=(p-1)^i(q-1)^j\ . \tag{16}$$

This relation plays an important role in studying the reciprocity formulas for the classical Dedekind sums. In fact, from (16) together with the fact that

$$\sum_{k=1}^{q-1}\left(kp-\left[\frac{kp}{q}\right]q\right)^s=\sum_{k=1}^{q-1}k^s\ ,$$

we can deduce all the known reciprocity formulas such as in Carlitz [3], § 2 and § 3. Consequently, (15) should be available to find the reciprocity formula for $H_3(p, q)$ and $K_3(p, q)$ which the author is eager to procure.

In the next place, corresponding to Lemma 2, we consider the difference

$$\left[\frac{(k+1)p}{q}\right]-\left[\frac{kp}{q}\right]=\begin{cases}\left[\frac{p}{q}\right]+1 & \text{if } \gamma_k+\gamma_1\geqq q ,\\ \left[\frac{p}{q}\right] & \text{otherwise} ,\end{cases}$$

for each integer k $(1\leqq k\leqq q-1)$.

Since

$$\sum_{k=1}^{q-1}\left\{\left[\frac{(k+1)p}{q}\right]-\left[\frac{kp}{q}\right]\right\}=(q-1)\left[\frac{p}{q}\right]+\#\{k;\ \gamma_k+\gamma_1\geqq q\}$$

and the left-hand side is equal to $p-\left[\frac{p}{q}\right]$, there exist $p-\left[\frac{p}{q}\right]q=\gamma_1$ such integers that satisfy $\gamma_k+\gamma_1\geqq q$. We put these integers as

$$k_1<k_2<\cdots<k_{\gamma_1-1}<k_{\gamma_1} .$$

Then we obtain the following lemma of which proof is similar to that of Lemma 2.

LEMMA 9. *These integers are represented as follows:*

$$k_i=\left[\frac{iq}{\gamma_1}\right] \qquad \textit{for} \quad i=1,\ \cdots,\ \gamma_1-1$$

and

$$k_{\gamma_1}=q-1 .$$

Our proof of reciprocity formulas using Lemma 2 is essentially equivalent to the well known method to prove the reciprocity formulas for the classical Dedekind sums (see, for example, Rademacher and Whiteman [5], (3.5)). However, Lemma 9 is not efficient for studying the reciprocity formulas. In fact, applying Lemma 9 to $\sum_{k=1}^{q-1}\left\{\left[\frac{(k+1)p}{q}\right]-\left[\frac{kp}{q}\right]\right\}\zeta^{\left[\frac{kp}{q}\right]_q}$, for example, we get

$$(\zeta^{-\left[\frac{p}{q}\right]_q}-1)\sum_{k=1}^{q-1}\left[\frac{kp}{q}\right]\zeta^{\left[\frac{kp}{q}\right]_q}+(1-\zeta^q)\sum_{i=1}^{\gamma_1}\left[\frac{k_ip}{q}\right]\zeta^{\left[\frac{k_ip}{q}\right]_q}+\left\{\left[\frac{p}{q}\right](1-\zeta^q)-\zeta^q\right\}\sum_{i=1}^{\gamma_1}\zeta^{\left[\frac{k_ip}{q}\right]_q}$$
$$=\left[\frac{p}{q}\right]\sum_{i=1}^{q-1}\zeta^{\left[\frac{kp}{q}\right]_q}-p\zeta^{-\left[\frac{p}{q}\right]_q}+\left[\frac{p}{q}\right] .$$

As this example shows, Lemma 9 reduces $K_n(p, q)$ to the sums $K_m(p, q)$ $(m=0, 1, \cdots, n-1)$ and the sums of the type

$$\sum_{i=1}^{\gamma_1-1}\left[\left[\frac{iq}{\gamma_1}\right]\frac{p}{q}\right]^j \zeta^{[[\frac{iq}{\gamma_1}]\frac{p}{q}]_q}$$
$$=\sum_{i=1}^{\gamma_1-1}\left\{i-1+\left[\frac{p}{q}\right]\left[\frac{iq}{\gamma_1}\right]\right\}^j \zeta^{iq+[\frac{iq}{\gamma_1}][\frac{p}{q}]_q-q} \quad (j=0,1,\cdots,n-1)$$

which have γ_1-1 terms definitely fewer than $q-1$. Therefore, although $K_n(p, q)$ may be evaluated in some cases such as $\gamma_1=1$ or 2 or so, these sums seem to be more complicated than $K_n(p, q)$ itself.

The application of the results of this paper or the application of Lemma 9 will be given in the forthcoming papers.

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