TOKYO J. MATH.
VOL. 10, NO. 1, 1987

A Note on Test Sufficiency in Weakly Dominated Statistical Experiments

Tokitake KUSAMA and Junji FUJII

Waseda University and Osaka City University

Introduction

Let $\mathscr{E}=(X, \underline{A}, P)$ be a statistical experiment or simply an experiment, i.e., X be a set, $\underline{A}$ a σ-field of subsets of X and P a family of probability measures on $\underline{A}$. A set N is called P-null if $p(N)=0$ for all $p\in P$, and written $N=\emptyset\ [P]$. For A and B in $\underline{A}$, we write $A\subset B\ [P]$ if $A-B=\emptyset\ [P]$. A subfield $\underline{B}$ of $\underline{A}$ is called test sufficient if for any $\underline{A}$-measurable test function f, i.e., $0\leqq f\leqq 1$, there exists a $\underline{B}$-measurable test function g such that $\int f\,dp=\int g\,dp$ for all $p\in P$.

An experiment $\mathscr{E}$ is called weakly dominated if there exists a measure λ on $\underline{A}$ such that (a) for each p in P, there exists a density $dp/d\lambda$ and $P\equiv\lambda$, i.e., all the λ-null sets are P-null and vice versa, and (b) for every family $\{A_\gamma;\gamma\in\Gamma\}$ consisting of subsets which are σ-finite with respect to λ, there exists a set U called essential supremum, which satisfies (b-1) $U\in\underline{A}$, (b-2) $A_\gamma\subset U\ [\lambda]$ for all $\gamma\in\Gamma$ and (b-3) if $A\in\underline{A}$ and $A_\gamma\subset A\ [\lambda]$ for all $\gamma\in\Gamma$, then $U\subset A\ [\lambda]$.

An experiment $\mathscr{E}$ is called majorized if for each $p\in P$, there exists a set $S(p)\in\underline{A}$ called an $\mathscr{E}$-support of p, which satisfies
S-1. $p(S(p))=1$, and
S-2. $P\ll p$ on $S(p)$, i.e., if $N\in\underline{A}$, $N\subset S(p)$ and $p(N)=0$, then $N=\emptyset\ [P]$.

A weakly dominated experiment $\mathscr{E}$ is majorized since for each $p\in P$, $\{x\in X;(dp/d\lambda)(x)>0\}$ is an $\mathscr{E}$-support of p.

In a majorized experiment there exists a subclass $\underline{F}$ of $\underline{A}$ called a maximal decomposition, which satisfies
D-1. for each $F\in\underline{F}$, there exists $p\in P$ such that $p(F)>0$ and $F\subset S(p)\ [P]$,
D-2. for any distinct sets F and G in $\underline{F}$, $F\cap G=\emptyset\ [P]$,
D-3. each $p\in P$ is concentrated on a countable number of sets in F and
D-4. if $A\in\underline{A}$ and $A\cap F=\emptyset\ [P]$ for all $F\in\underline{F}$, then $A=\emptyset\ [P]$.

Received April 22, 1986

In a majorized experiment, the smallest pairwise sufficient subfield with supports (PSS) exists (see Ghosh, Morimoto and Yamada [3], Theorem 5), so that it does in a weakly dominated case. In general, sufficiency implies test sufficiency, and test sufficiency implies pairwise sufficiency. In majorized experiments, test sufficiency implies PSS (see Mussmann [4]), and PSS implies pairwise sufficiency. In dominated experiments, all the above notions coincide with each other. In this note, if an experiment is undominated and weakly dominated, we call such an experiment simply weakly dominated. One of the authors and Morimoto show that in weakly dominated experiments, the smallest PSS is not sufficient (see Fujii and Morimoto [2], Theorem 8).

In this note we prove that in weakly dominated experiments, the smallest PSS is not test sufficient. This is an improvement of the above result since sufficiency implies test sufficiency.

§ 1. Theorem.

Let $\mathscr{E}=(X, \underline{A}, P)$ be a weakly dominated experiment, i.e., there exists a measure λ on $\underline{A}$ which satisfies (a), (b) in section 1 and λ is not σ-finite. It is well known that all the maximal decompositions have uncountable cardinalities (see Diepenbrock [1]).

Before proving the theorem, we construct a special maximal decomposition which plays an essential role in the proof.

There exists a well-ordering on P, so that for every non empty subset of P, there exists the least element of it with respect to this well-ordering. Let p_0 be the least element of $P_0=P$, and put $A_0=S(p_0)$. The set $Q_0=\{p \in P; S(p)-A_0 \neq \emptyset\ [P]\}$ is non-empty as $\mathscr{E}$ is undominated, so that there exists the least element q_0 of Q_0, and we put $B_0=S(q_0)-A_0$. Then the class $\{A_0, B_0\}$ satisfies D-1 and D-2 in the definition of a maximal decomposition, that is, $A_0 \neq \emptyset\ [P]$, $B_0 \neq \emptyset\ [P]$, $A_0 \subset S(p_0)\ [P]$, $B_0 \subset S(q_0)\ [P]$ and $A_0 \cap B_0=\emptyset\ [P]$ are satisfied.

Now we construct inductively (p_ξ, P_ξ, A_ξ) and (q_ξ, Q_ξ, B_ξ) for each ordinal ξ. Suppose that (p_η, P_η, A_η) and (q_η, Q_η, B_η) are given for all $\eta<\xi$ and they satisfy D-1 and D-2, that is, for all $\eta<\xi$, $A_\eta \neq \emptyset\ [P]$, $B_\eta \neq \emptyset\ [P]$, $A_\eta \subset S(p_\eta)\ [P]$, $B_\eta \subset S(q_\eta)\ [P]$, and for all $\mu, \gamma \leqq \eta$, $A_\mu \cap B_\gamma=\emptyset\ [P]$, and if $\mu \neq \gamma$, $A_\mu \cap A_\gamma=\emptyset\ [P]$ and $B_\mu \cap B_\gamma=\emptyset\ [P]$.

Here we put $P_\xi=\{p \in P; S(p)-\operatorname{ess\,sup}\{A_\eta \cup B_\eta; \eta<\xi\} \neq \emptyset\ [P]\}$, and define (p_ξ, A_ξ) as follows;

$$p_\xi=\begin{cases}\text{the least element of } P_\xi & \text{if } P_\xi \neq \emptyset\ , \\ p_0 & \text{if } P_\xi=\emptyset\ .\end{cases}$$

$$A_\xi=\begin{cases} S(p_\xi)-\text{ess sup}\{A_\eta\cup B_\eta;\ \eta<\xi\} & \text{if} \quad P_\xi\neq\varnothing\ , \\ \varnothing & \text{if} \quad P_\xi=\varnothing\ . \end{cases}$$

Similarly $Q_\xi=\{p\in P;\ S(p)-(\text{ess sup}\{A_\eta\cup B_\eta;\ \eta<\xi\}\cup A_\xi)\neq\varnothing\ [P]\}$ and (q_ξ, B_ξ) is defined as follows;

$$q_\xi=\begin{cases} \text{the least element of } Q_\xi & \text{if} \quad Q_\xi\neq\varnothing\ , \\ q_0 & \text{if} \quad Q_\xi=\varnothing\ . \end{cases}$$

$$B_\xi=\begin{cases} S(q_\xi)-(\text{ess sup}\{A_\eta\cup B_\eta;\ \eta<\xi\}\cup A_\xi) & \text{if} \quad Q_\xi\neq\varnothing\ , \\ \varnothing & \text{if} \quad Q_\xi=\varnothing\ . \end{cases}$$

Thus for each ordinal ξ, (p_ξ, P_ξ, A_ξ) and (q_ξ, Q_ξ, B_ξ) are defined.

Here we define two ordinals α and β as the least ordinals of $\{\xi;\ P_\xi=\varnothing\}$ and $\{\xi;\ Q_\xi=\varnothing\}$, respectively. Then it easily follows from the assumption that α is an uncountable ordinal and $\alpha=\beta$ or $\alpha=\beta+1$.

It is easily verified that the following class $\underline{F}$ satisfy the conditions D-1 to D-4 in section 1, and hence it is a maximal decomposition.

$$\underline{F}=\begin{cases} \{A_\xi;\ \xi<\alpha\}\cup\{B_\xi;\ \xi<\alpha\} & \text{if} \quad \alpha=\beta\ , \\ \{A_\xi;\ \xi<\alpha\}\cup\{B_\xi;\ \xi<\beta\} & \text{if} \quad \alpha=\beta+1\ . \end{cases}$$

THEOREM. *Let $\mathscr{E}=(X, \underline{A}, P)$ be a weakly dominated experiment. Then the smallest PSS $\underline{D}$ is not test sufficient.*

Proof. First we take the above maximal decomposition $\underline{F}$ and assume that $\alpha=\beta$ without loss of generality.

Now we may reach a contradiction, assuming that $\underline{D}$ is test sufficient. We put $A=\text{ess sup}\{A_\xi;\ \xi<\alpha\}$ and take the indicator function I_A as a test function.

Note that the set A does not belong to $\underline{D}[P]$ since A is an uncountable type set (see Fujii and Morimoto [2], Theorem 5), i.e., neither A nor $X-A$ is σ-finite with respect to λ.

Since $\underline{D}$ is test sufficient, there exists a $\underline{D}$-measurable test function f_A such that

$$(*) \qquad p(A)=\int f_A dp \qquad \text{for all} \quad p \text{ in } P\ .$$

Then f_A has the following property.

$$f_A=1\,[P] \text{ on } A_\xi \quad \text{and} \quad f_A=0\,[P] \text{ on } B_\xi \qquad \text{for all} \quad \xi<\alpha\ .$$

Now we prove this by using the transfinite induction.

First we substitute p_0 for p in the above formula $(*)$. Then

$p_0(S(p_0))=\int f_A dp_0$ holds, so $f_A=1\,[P]$ on $S(p_0)=A_0$ since $0\leqq f_A\leqq 1$ and $P\ll p_0$ on $S(p_0)$.

Next we take $p=q_0$ in $(*)$. Left hand side of $(*)=q_0(A)=q_0(A\cap S(q_0))=q_0(A_0\cap S(q_0))$. Right hand side of $(*)=\int f_A dq_0=\int_{S(q_0)\cap A_0} f_A dq_0+\int_{B_0} f_A dq_0=q_0(A_0\cap S(q_0))+\int_{B_0} f_A dq_0$ (because $f_A=1\,[P]$ on A_0). Thus $\int_{B_0} f_A dq_0=0$, so $f_A=0\,[P]$ on B_0 by $B_0\subset S(q_0)\,[P]$.

We fix $\xi<\alpha$. Then the assumptions of induction are $f_A=1\,[P]$ on A_η and $f_A=0\,[P]$ on B_η for all $\eta<\xi$. Again we substitute p_ξ for p in $(*)$, so $p_\xi(A)=\int f_A dp_\xi$. By the property D-3 of $\underline{F}$, there exist a countable number of ordinals $\eta(k)<\xi$ and $\zeta(n)<\xi$ $(k, n=1, 2, \cdots)$ such that $S(p_\xi)-A_\xi=(\cup_k(A_{\eta(k)}\cap(S(p_\xi)-A_\xi)))\cup(\cup_n(B_{\zeta(n)}\cap(S(p_\xi)-A_\xi)))\,[P]$.

$$\begin{aligned}\int f_A dp_\xi&=\int_{S(p_\xi)-A_\xi} f_A dp_\xi+\int_{A_\xi} f_A dp_\xi\\&=\sum_k\int_{A_{\eta(k)}\cap(S(p_\xi)-A_\xi)} f_A dp_\xi+\sum_n\int_{B_{\zeta(n)}\cap(S(p_\xi)-A_\xi)} f_A dp_\xi+\int_{A_\xi} f_A dp_\xi\\&=\sum_k p_\xi(A_{\eta(k)}\cap(S(p_\xi)-A_\xi))+\int_{A_\xi} f_A dp_\xi \quad \text{(by the assumption)}\\&=p_\xi((A\cap S(p_\xi))-A_\xi)+\int_{A_\xi} f_A dp_\xi\,.\end{aligned}$$

On the other hand $p_\xi(A)=p_\xi(A\cap S(p_\xi))=p_\xi((A\cap S(p_\xi))-A_\xi)+p_\xi(A_\xi)$, and hence $p_\xi(A_\xi)=\int_{A_\xi} f_A dp_\xi$. Therefore $f_A=1\,[p_\xi]$ on A_ξ since $0\leqq f_A\leqq 1$, so $f_A=1\,[P]$ on A_ξ holds by $A_\xi\subset S(p_\xi)\,[P]$. Similarly $f_A=0\,[P]$ on B_ξ.

Here we put $N=(f_A\neq I_A)$, and then it follows that $A_\xi\cap N=\emptyset\,[P]$ and $B_\xi\cap N=\emptyset\,[P]$ for all $\xi<\alpha$. This implies $N=\emptyset\,[P]$ since $\underline{F}$ satisfies D-4. Therefore $f_A=I_A\,[P]$, that is, $A\in\underline{D}\,[P]$. This is a contradiction.

REMARK. The proof shows more than the above Theorem. More precisely, for each test sufficient subfield $\underline{B}$, $A=\operatorname{ess\,sup}\{A_\xi;\xi<\alpha\}\in\underline{B}$, and the subfield $\{A\in\underline{A};$ either A or $X-A$ is σ-finite with respect to $\lambda\}$ is PSS but not test sufficient.

ACKNOWLEDGEMENT. We are deeply grateful to H. Morimoto and K. Namba for various useful suggestions and continuous encouragement.

References

[1] F. R. DIEPENBROCK, Characterisierung einer allgemeineren Bedingung als Dominiertheit mit Hilfe von localisierten Massen, Thesis, Univ. of Münster, 1971.

[2] J. FUJII and H. MORIMOTO, Sufficiency and pairwise sufficiency in majorized experiments, to appear in Sankhyā Ser. A, **48** (1986).
[3] J.K. GHOSH, H. MORIMOTO and S. YAMADA, Neyman factorization and minimality of pairwise sufficient subfields, Ann. Statist., **9** (1981), 514–530.
[4] D. MUSSMANN, Schwache dominierte Familien von Massen, Thesis, Univ. of München, 1971.

Present Address:
DEPARTMENT OF MATHEMATICS
WASEDA UNIVERSITY
OKUBO, SHINJUKU-KU, TOKYO 160
AND
DEPARTMENT OF MATHEMATICS
OSAKA CITY UNIVERSITY
SUGIMOTO-CHO, SUMIYOSHI-KU, OSAKA 558