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A Completely Integrable Hamiltonian System of C. Neumann-type on the Complex Projective Space

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Introduction

The C. Neumann problem is a Hamiltonian system which describes the motion of a point on the sphere $S^{n-1}=\{x\in \boldsymbol{R}^n \mid \|x\|=1\}$ under the influence of a quadratic potential $U(x)=(1/2)\sum a_j x_j^2$, $a_1, \cdots, a_n \in \boldsymbol{R}$. It is shown by many authors that the C. Neumann problem is completely integrable (see [5, §1]). In [5], Ratiu showed that the C. Neumann problem is a Hamiltonian system on an adjoint orbit in a semidirect product of Lie algebras, and that its complete integrability follows entirely from Lie algebraic considerations.

In the present note, we define a C. Neumann-type problem on the complex projective space $P^{n-1}=\{[z] \mid z\in \boldsymbol{C}^n, \|z\|=1\}$ (see section 4). It is a Hamiltonian system which describes the motion of a point on P^{n-1} under the influence of a potential $U([z])=(1/2)\sum a_j|z_j|^2$. We then show, following Ratiu [5], that this system is a Hamiltonian system on an adjoint orbit in a semidirect product of Lie algebras (Theorem 3.4 and Proposition 4.1). As a consequence, we can prove that the C. Neumann-type problem on the complex projective space is completely integrable (Theorem 4.3).

§1. Hamiltonian actions and (co-)adjoint orbits.

In this section, we recall a few facts about symplectic geometry (for references, see [1], [2], [3], [4]). Let M be a symplectic manifold with symplectic structure Ω_M. Recall that Ω_M is a non-degenerate closed two-form on M. Each real-valued smooth (i.e., C^∞) function f on M generates a Hamiltonian vector field X_f on M which satisfies $X_f \lrcorner \Omega_M=df$. Let $C^\infty(M)$ be the space of real-valued smooth functions on M. The Poisson

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bracket of $f_1, f_2 \in C^\infty(M)$ is defined by $\{f_1, f_2\}_M = -\Omega_M(X_{f_1}, X_{f_2})$. The Poisson bracket makes $C^\infty(M)$ into a Lie algebra called the Poisson algebra of M. Let $\mathfrak{X}(M)$ denote the Lie algebra consisting of smooth vector fields on M with the usual Lie bracket. Then $f \mapsto X_f$ is a Lie algebra homomorphism of $C^\infty(M)$ into $\mathfrak{X}(M)$.

Let G be a Lie group with Lie algebra $\mathfrak{g}$, and let Φ be a smooth action of G on M. For each $\xi \in \mathfrak{g}$, let X^ξ denote the infinitesimal generator of the action, i.e., X^ξ is a smooth vector field on M defined by

$$(X^\xi f)(m) = \left[\frac{d}{dt} f \circ \Phi(\exp(-t\xi), m)\right]_{t=0}$$

for $f \in C^\infty(M)$, $m \in M$. Recall that $\xi \mapsto X^\xi$ is a Lie algebra homomorphism of $\mathfrak{g}$ into $\mathfrak{X}(M)$. Φ is called a *Hamiltonian action* if

(i) Φ is a symplectic action, i.e., for each $g \in G$, Φ_g leaves Ω_M invariant, and

(ii) there exists a linear map, $\xi \mapsto f^\xi$, of $\mathfrak{g}$ into $C^\infty(M)$ such that $X_{f^\xi} = X^\xi$ and $f^\xi \circ \Phi_{g^{-1}} = f^{\mathrm{Ad}_g(\xi)}$ for $\xi \in \mathfrak{g}$, $g \in G$, where Ad denotes the adjoint action of G.

Here note that $\xi \mapsto f^\xi$ is a Lie algebra homomorphism of $\mathfrak{g}$ into $C^\infty(M)$ (cf. [1, Corollary 4.2.9]). A symplectic manifold with a transitive Hamiltonian action of a Lie group G is called a Hamiltonian G-space. Let $\mathfrak{g}^*$ denote the dual space of $\mathfrak{g}$. The moment map of a Hamiltonian action is a smooth map J of M into $\mathfrak{g}^*$ given by $\langle J(m), \xi\rangle = f^\xi(m)$ for $m \in M$, $\xi \in \mathfrak{g}$, where $\langle\ ,\ \rangle$ denotes the pairing between $\mathfrak{g}^*$ and $\mathfrak{g}$.

Now we shall consider the coadjoint action Ad$'$ of G on $\mathfrak{g}^*$, which is defined by $\langle \mathrm{Ad}'_g(\alpha), \xi\rangle = \langle \alpha, \mathrm{Ad}_{g^{-1}}(\xi)\rangle$ for $g \in G$, $\alpha \in \mathfrak{g}^*$, $\xi \in \mathfrak{g}$. For each $\xi \in \mathfrak{g}$, let $\tilde{\xi}$ denote the infinitesimal generator of the coadjoint action. Let $\mathcal{O}$ be a coadjoint orbit of G. Then $\mathcal{O}$ is a symplectic manifold with the Lie-Kirillov-Kostant-Souriau symplectic structure $\omega_\mathcal{O}$. Recall that $\omega_\mathcal{O}$ is given by $\omega_\mathcal{O}(\tilde{\xi}_\alpha, \tilde{\eta}_\alpha) = -\langle \alpha, [\xi, \eta]\rangle$ for $\alpha \in \mathcal{O}$, $\xi, \eta \in \mathfrak{g}$. The restriction of the coadjoint action to $\mathcal{O}$ is a Hamiltonian action with the Lie algebra homomorphism $\mathfrak{g} \to C^\infty(\mathcal{O})$ given by $\xi \mapsto \langle \cdot, \xi\rangle|_\mathcal{O}$. Thus $\mathcal{O}$ is a Hamiltonian G-space.

THEOREM 1.1 (cf. [4, Theorem 5.4.1]). *Let M be a Hamiltonian G-space with a moment map J. Then the image $J(M)$ of M under J coincides with a coadjoint orbit of G, and $J\colon M \to J(M)$ is a symplectic covering map of Hamiltonian G-spaces.*

If $F \in C^\infty(\mathfrak{g}^*)$, then the Legendre transformation $\mathcal{L}_F\colon \mathfrak{g}^* \to \mathfrak{g}$ associated to F is defined by

$$\langle \beta, \mathscr{L}_F(\alpha)\rangle = \left[\frac{d}{dt} F(\alpha + t\beta)\right]_{t=0}$$

for $\alpha, \beta \in \mathfrak{g}^*$ (cf. [3, §1]). The Poisson bracket $\{\ ,\ \}_{\mathfrak{g}^*}$ is then defined by

$$\{F_1, F_2\}_{\mathfrak{g}^*}(\alpha) = \langle \alpha, [\mathscr{L}_{F_1}(\alpha), \mathscr{L}_{F_2}(\alpha)]\rangle$$

for $F_1, F_2 \in C^\infty(\mathfrak{g}^*)$, $\alpha \in \mathfrak{g}^*$. The Poisson bracket makes $C^\infty(\mathfrak{g}^*)$ into a Lie algebra called the Poisson algebra of $\mathfrak{g}^*$. Recall that the pull-back J^*: $C^\infty(\mathfrak{g}^*) \to C^\infty(M)$ by the moment map J of a Hamiltonian action is a Lie algebra homomorphism. For each $F \in C^\infty(\mathfrak{g}^*)$, let us define $\tilde{F} \in \mathfrak{X}(\mathfrak{g}^*)$ by

$$\tilde{F}_\alpha = \widetilde{\mathscr{L}_F(\alpha)}(\alpha)$$

for $\alpha \in \mathfrak{g}^*$ (cf. [3, §1]). From the definition, it is easy to see that $\tilde{F}|_{\mathscr{O}}$ is tangent to $\mathscr{O}$ for any coadjoint orbit $\mathscr{O}$. Moreover $\tilde{F}|_{\mathscr{O}}$ is the Hamiltonian vector field on $\mathscr{O}$ generated by $F|_{\mathscr{O}}$. It then follows that

$$\{F_1|_{\mathscr{O}}, F_2|_{\mathscr{O}}\}_{\mathscr{O}} = \{F_1, F_2\}_{\mathfrak{g}^*}|_{\mathscr{O}}$$

for $F_1, F_2 \in C^\infty(\mathfrak{g}^*)$.

Let $\kappa: \mathfrak{g} \times \mathfrak{g} \to \boldsymbol{R}$ be a non-degenerate Ad-invariant symmetric bilinear form on $\mathfrak{g}$. Let us now identify $\mathfrak{g}^*$ with $\mathfrak{g}$ by κ. Then the coadjoint action coincides with the adjoint action. The infinitesimal generator of the adjoint action is given by

$$\tilde{\xi}_\alpha = [\alpha, \xi] \in \mathfrak{g} \quad (\approx T_\alpha \mathfrak{g})$$

for $\alpha, \xi \in \mathfrak{g}$. The symplectic structure of an adjoint orbit $\mathscr{O}$ is given by

$$\omega_{\mathscr{O}}(\tilde{\xi}_\alpha, \tilde{\eta}_\alpha) = -\kappa(\alpha, [\xi, \eta])$$

for $\alpha \in \mathscr{O}$, $\xi, \eta \in \mathfrak{g}$. If $F \in C^\infty(\mathfrak{g})$, then $\tilde{F} \in \mathfrak{X}(\mathfrak{g})$ is given by

$$\tilde{F}_\alpha = [\alpha, (\nabla F)_\alpha]$$

for $\alpha \in \mathfrak{g}$, where ∇F denotes the gradient of F with respect to κ. The Poisson bracket of $F_1, F_2 \in C^\infty(\mathfrak{g})$ is given by

$$\{F_1, F_2\}_{\mathfrak{g}}(\alpha) = \kappa(\alpha, [(\nabla F_1)_\alpha, (\nabla F_2)_\alpha])$$

for $\alpha \in \mathfrak{g}$ (cf. [5, p. 322]).

§2. The Ad-semidirect product $U(n)_{\mathrm{Ad}} \times u(n)$.

In this section, according to Ratiu [5, §§2 and 3], we shall prepare a few facts about the Ad-semidirect product $U(n)_{\mathrm{Ad}}\times u(n)$ of $U(n)$ with $u(n)$. $U(n)_{\mathrm{Ad}}\times u(n)$ is a Lie group with underlying manifold $U(n)\times u(n)$ and composition law

$$(g_1, X_1)(g_2, X_2)=(g_1g_2, X_1+\mathrm{Ad}_{g_1}(X_2)) .$$

Its Lie algebra is the ad-semidirect product $u(n)_{\mathrm{ad}}\times u(n)$ of $u(n)$ with $u(n)$. If $(X_1, Y_1), (X_2, Y_2)\in u(n)_{\mathrm{ad}}\times u(n)$, their bracket is given by

$$[(X_1, Y_1), (X_2, Y_2)]=([X_1, X_2], [X_1, Y_2]+[Y_1, X_2]) .$$

The adjoint action of $U(n)_{\mathrm{Ad}}\times u(n)$ on $u(n)_{\mathrm{ad}}\times u(n)$ is given by

$$\mathrm{Ad}_{(g,Z)}(X, Y)=(\mathrm{Ad}_g(X), \mathrm{Ad}_g(Y)+[Z, \mathrm{Ad}_g(X)]) .$$

Let us define $K: u(n)\times u(n)\to \boldsymbol{R}$ by $K(X, Y)=-(1/2)\mathrm{tr}(XY)$. Then K is a non-degenerate Ad-invariant symmetric bilinear form on $u(n)$. The form K_s, called the semidirect product of K with itself and defined by

$$K_s((X_1, Y_1), (X_2, Y_2))=K(X_1, Y_2)+K(Y_1, X_2) ,$$

is a non-degenerate Ad-invariant symmetric bilinear form on $u(n)_{\mathrm{ad}}\times u(n)$. The infinitesimal generator of the adjoint action of $U(n)_{\mathrm{Ad}}\times u(n)$ is given by

$$\widetilde{(X_1, Y_1)}_{(X,Y)}=([X, X_1], [X, Y_1]+[Y, X_1]) .$$

The symplectic structure of an adjoint orbit is given by

$$\omega_{\mathcal{O}}(\widetilde{(X_1, Y_1)}_{(X,Y)}, \widetilde{(X_2, Y_2)}_{(X,Y)})=-K_s((X, Y), ([X_1, X_2], [X_1, Y_2]+[Y_1, X_2]))$$

for $(X, Y)\in\mathcal{O}$, $(X_1, Y_1), (X_2, Y_2)\in u(n)_{\mathrm{ad}}\times u(n)$. Let $F\in C^\infty(u(n)_{\mathrm{ad}}\times u(n))$ and $(X, Y)\in u(n)_{\mathrm{ad}}\times u(n)$. Let $(\nabla_1 F)_{(X,Y)}$ and $(\nabla_2 F)_{(X,Y)}\in u(n)$ denote the gradient with respect to K of the functions $F(\cdot, Y)$ and $F(X, \cdot)$ on $u(n)$ at X and at Y, respectively. Then the gradient of F with respect to K_s is given by

$$(\nabla F)_{(X,Y)}=((\nabla_2 F)_{(X,Y)}, (\nabla_1 F)_{(X,Y)}) .$$

Hence $\tilde{F}\in\mathfrak{X}(u(n)_{\mathrm{ad}}\times u(n))$ is given by

$$\tilde{F}_{(X,Y)}=([X, (\nabla_2 F)_{(X,Y)}], [X, (\nabla_1 F)_{(X,Y)}]+[Y, (\nabla_2 F)_{(X,Y)}]) .$$

Now, let us fix an $A\in u(n)$ and define $E\in C^\infty(u(n)_{\mathrm{ad}}\times u(n))$ by

$$E(X, Y)=\frac{1}{2}K(Y, Y)+K(A, X) .$$

Since $(\nabla_1 E)_{(X,Y)}=A$, $(\nabla_2 E)_{(X,Y)}=Y$, we have

$$\tilde{E}_{(X,Y)}=([X,\ Y],\ [X,\ A])\ .$$

Thus $t\mapsto(X(t),\ Y(t))$ is an integral curve of $\tilde{E}$ if and only if

$$\dot{X}=[X,\ Y]\ ,\qquad \dot{Y}=[X,\ A]\ .$$

These equations are equivalent to the Lax equation:

$$(X+Y\lambda+A\lambda^2)^{\cdot}=[X+Y\lambda+A\lambda^2,\ Y+A\lambda]$$

for any parameter λ (see [5, Lemma 3.1]). Let $\varphi_k(X,\ Y)$ and $\psi_k(X,\ Y)$ be the respective coefficients of λ^{2k-1} and λ^{2k-2} in the expansion of $F_{k,\lambda}(X,\ Y)=(1/(2ki^k))\mathrm{tr}\{(X+Y\lambda+A\lambda^2)^k\}$ for $k=2,\ 3,\ \cdots,\ n$. Then φ_k and ψ_k are real-valued polynomial functions on $u(n)_{\mathrm{ad}}\times u(n)$:

$$\varphi_k(X,\ Y)=\frac{1}{2i^k}\,\mathrm{tr}(A^{k-1}Y)\ ,$$

$$\psi_k(X,\ Y)=\frac{1}{2i^k}\,\mathrm{tr}\Big(A^{k-1}X+\frac{1}{k}\sum A^aYA^bYA^c\Big)\ ,$$

where the sum is taken over all triplets $(a,\ b,\ c)$ of non-negative integers satisfying $a+b+c=k-2$. Note that $\psi_2=E$.

LEMMA 2.1 (cf. [5, Theorem 3.4]). *The functions* $\varphi_2,\ \cdots,\ \varphi_n,\ \psi_2,\ \cdots,\ \psi_n$ *commute with each other in the Poisson bracket* $\{\ ,\ \}_{u(n)_{\mathrm{ad}}\times u(n)}$.

PROOF (cf. [5, § 3]). By Theorem 3.2 in [5], we have

$$\{F_{k,\lambda},\ F_{l,\mu}\}_{u(n)_{\mathrm{ad}}\times u(n)}=0$$

for any parameters λ, μ. It follows that

$$\{\varphi_k,\ F_{l,\mu}\}_{u(n)_{\mathrm{ad}}\times u(n)}=0\ ,\qquad \{\psi_k,\ F_{l,\mu}\}_{u(n)_{\mathrm{ad}}\times u(n)}=0$$

for all μ. Hence $\{\varphi_k,\ \varphi_l\}_{u(n)_{\mathrm{ad}}\times u(n)}=0$, $\{\varphi_k,\ \psi_l\}_{u(n)_{\mathrm{ad}}\times u(n)}=0$ and $\{\psi_k,\ \psi_l\}_{u(n)_{\mathrm{ad}}\times u(n)}=0$.

LEMMA 2.2.

(i) $\tilde{\varphi}_k|_{(X,Y)}=([X,\ i(-iA)^{k-1}],\ [Y,\ i(-iA)^{k-1}])\ ,$

(ii) $$\tilde{\psi}_k|_{(X,Y)}=\Big(\Big[X,\ \sum_{j=0}^{k-2}(-iA)^jY(-iA)^{k-j-2}\Big]\ ,$$
$$[X,\ i(-iA)^{k-1}]+\Big[Y,\ \sum_{j=0}^{k-2}(-iA)^jY(-iA)^{k-j-2}\Big]\Big)\ .$$

Proof. Since for any $X', Y' \in u(n)$,

$$K((\nabla_1\varphi_k)_{(X,Y)}, X')=\left[\frac{d}{dt}\varphi_k(X+tX', Y)\right]_{t=0}=0$$

and

$$K((\nabla_2\varphi_k)_{(X,Y)}, Y')=\left[\frac{d}{dt}\varphi_k(X, Y+tY')\right]_{t=0}$$
$$=K(i(-iA)^{k-1}, Y') ,$$

we have $(\nabla_1\varphi_k)_{(X,Y)}=0$ and $(\nabla_2\varphi_k)_{(X,Y)}=i(-iA)^{k-1}$. Hence (i) follows. (ii) is obtained similarly.

§ 3. Hamiltonian actions of $U(n)_{\mathrm{Ad}}\times u(n)$.

Let $\boldsymbol{C}^n=\{z={}^t(z_1, \cdots, z_n)\}$ be the complex n-space with the Hermitian inner product $\langle\ ,\ \rangle$ and the Euclidean inner product $\langle\ ,\ \rangle_R$ given by $\langle z, w\rangle=\sum \bar{z}_j w_j$ and $\langle z, w\rangle_R=\mathrm{Re}\langle z, w\rangle$, respectively. If we put $z^*=(\bar{z}_1, \cdots, \bar{z}_n)$ for $z={}^t(z_1, \cdots, z_n)$, then $\langle z, w\rangle=z^*w=\mathrm{tr}\, wz^*$. The (co-)tangent bundle of the unit sphere $S^{2n-1}=\{z\in \boldsymbol{C}^n\,|\,\langle z, z\rangle=1\}$ in $\boldsymbol{C}^n$ is realized as

$$T^*S^{2n-1}=\{(z, w)\in S^{2n-1}\times \boldsymbol{C}^n\,|\,\langle z, w\rangle_R=0\} .$$

The tangent bundle TT^*S^{2n-1} of T^*S^{2n-1} and the canonical symplectic structure Ω_S on T^*S^{2n-1} are given by

$$TT^*S^{2n-1}=\{(U, V)_{(z,w)}\,|\,(z, w)\in T^*S^{2n-1},\ (U, V)\in \boldsymbol{C}^n\times \boldsymbol{C}^n ,$$
$$\langle z, U\rangle_R=0, \langle z, V\rangle_R+\langle w, U\rangle_R=0\}$$

and

$$\Omega_S((U_1, V_1)_{(z,w)}, (U_2, V_2)_{(z,w)})=\langle U_1, V_2\rangle_R-\langle V_1, U_2\rangle_R ,$$

respectively.

If $f\in C^\infty(T^*S^{2n-1})$, a smooth extension of f onto $\boldsymbol{C}^n\times \boldsymbol{C}^n$ is also denoted by the same letter f. For $(z, w)\in \boldsymbol{C}^n\times \boldsymbol{C}^n$, let us define $f_z=f_z(z, w)$ and $f_w=f_w(z, w)\in \boldsymbol{C}^n$ by

$$(\mathrm{grad}\, f)_{(z,w)}=(f_z(z, w), f_w(z, w)) ,$$

where $\mathrm{grad}\, f$ is the gradient of f with respect to $\langle\ ,\ \rangle_R$.

Lemma 3.1. *Let $f, f_1, f_2\in C^\infty(T^*S^{2n-1})$. Then the Hamiltonian vector field generated by f is given by*

$$\text{(i)}\quad X_f|_{(z,w)}=(f_w-\langle z, f_w\rangle_R z,\ -f_z+\langle z, f_z\rangle_R z-\langle w, f_w\rangle_R z+\langle z, f_w\rangle_R w)_{(z,w)}$$

and the Poisson bracket of f_1 and f_2 is given by

(ii) $$\{f_1, f_2\}_S(z, w) = -\langle f_{1z}, f_{2w}\rangle_R + \langle f_{2z}, f_{1w}\rangle_R + \langle z, f_{1z}\rangle_R\langle z, f_{2w}\rangle_R - \langle z, f_{2z}\rangle_R\langle z, f_{1w}\rangle_R + \langle z, f_{1w}\rangle_R\langle w, f_{2w}\rangle_R - \langle z, f_{2w}\rangle_R\langle w, f_{1w}\rangle_R$$

for $(z, w) \in T^*S^{2n-1}$.

PROOF. It is easy to verify that the right-hand side of (i) is tangent to T^*S^{2n-1}. Let $(U, V)_{(z,w)} \in TT^*S^{2n-1}$. Then $\langle z, U\rangle_R = 0$ and $\langle z, V\rangle_R + \langle w, U\rangle_R = 0$. By the direct calculation, we then have

$$\begin{aligned}&\Omega_S((f_w - \langle z, f_w\rangle_R z - f_z + \langle z, f_z\rangle_R z - \langle w, f_w\rangle_R z + \langle z, f_w\rangle_R w)_{(z,w)}, (U, V)_{(z,w)}) \\ &= \langle f_z, U\rangle_R + \langle f_w, V\rangle_R = [(d/dt)f(z + tU, w + tV)]_{t=0} = df((U, V)_{(z,w)}),\end{aligned}$$

from which (i) follows.

Now, let us define an action Φ of $U(n)_{\mathrm{Ad}} \times u(n)$ on T^*S^{2n-1} by

$$\Phi_{(g,X)}(z, w) = (gz, gw - iXgz + \langle gz, iXgz\rangle gz).$$

Φ is well-defined and real-analytic. From the definition, the geometric meaning of the action is easily observed. The infinitesimal generator is given by

$$X^{(X,Y)}|_{(z,w)} = (-Xz, -Xw + iYz - \langle z, iYz\rangle z)_{(z,w)}$$

for $(X, Y) \in u(n)_{\mathrm{ad}} \times u(n)$, $(z, w) \in T^*S^{2n-1}$. Let us define $f^{(X,Y)} \in C^\infty(T^*S^{2n-1})$ by

$$f^{(X,Y)}(z, w) = -\langle Xz, w\rangle_R - \frac{1}{2}\langle z, iYz\rangle.$$

LEMMA 3.2. *Φ: $(U(n)_{\mathrm{Ad}} \times u(n)) \times T^*S^{2n-1} \to T^*S^{2n-1}$ is a Hamiltonian action with Lie algebra homomorphism: $(X, Y) \mapsto f^{(X,Y)}$. The moment map J: $T^*S^{2n-1} \to u(n)_{\mathrm{ad}} \times u(n)$ $(\approx (u(n)_{\mathrm{ad}} \times u(n))^*)$ is given by*

$$J(z, w) = (izz^*, zw^* - wz^*).$$

PROOF. Since $f_z^{(X,Y)} = Xw - iYz$ and $f_w^{(X,Y)} = -Xz$, we have by Lemma 3.1 that the Hamiltonian vector field generated by $f^{(X,Y)}$ coincides with $X^{(X,Y)}$. It then follows that Φ is symplectic, since $U(n)_{\mathrm{Ad}} \times u(n)$ is connected. By the direct calculation we have

$$f^{(X,Y)} \circ \Phi_{(g,Z)^{-1}} = f^{\mathrm{Ad}(g,Z)(X,Y)}$$

for $(X, Y) \in u(n)_{\mathrm{ad}} \times u(n)$, $(g, Z) \in U(n)_{\mathrm{Ad}} \times u(n)$. As to the moment map, it follows from the definition that

$$\begin{aligned}K_s(J(z, w), (X, Y)) &= f^{(X,Y)}(z, w) = -\langle Xz, w\rangle_R - \frac{1}{2}\langle z, iYz\rangle \\ &= K(izz^*, Y) + K(zw^* - wz^*, X) \\ &= K_s((izz^*, zw^* - wz^*), (X, Y)).\end{aligned}$$

Hence we have

$$J(z, w)=(izz^*, zw^*-wz^*) .$$

The complex projective space $P^{n-1}=P^{n-1}(\boldsymbol{C})$ is a quotient space of S^{2n-1} by the S^1-action $z\mapsto e^{it}z$. Thus $P^{n-1}=\{[z]\mid z\in S^{2n-1}\}$, where $[z]$ denotes the equivalence class. Its cotangent bundle T^*P^{n-1} with the canonical symplectic structure Ω_P is identified with a reduced symplectic manifold as follows: Let $\phi\in C^\infty(T^*S^{2n-1})$ be defined by $\phi(z, w)=\langle iz, w\rangle_{\boldsymbol{R}}$. The Hamiltonian vector field X_ϕ induces an S^1-action $(z, w)\mapsto(e^{it}z, e^{it}w)$ on T^*S^{2n-1}. By the reduction procedure, we then obtain a symplectic manifold

$$\begin{aligned}\phi^{-1}(\{0\})/\sim &=\{[z, w]\mid \phi(z, w)=0\}\\ &=\{[z, w]\mid \langle z, z\rangle=1,\ \langle z, w\rangle=0\}\\ &=T^*P^{n-1}\end{aligned}$$

with the reduced symplectic structure equal to Ω_P, where $[z, w]$ denotes the equivalence class $\{(e^{it}z, e^{it}w)\mid t\in\boldsymbol{R}\}$ (cf. [2, p. 377, Example 3]). The canonical projection $T^*P^{n-1}\to P^{n-1}$ is given by $[z, w]\mapsto[z]$. Since the action Φ leaves the function ϕ invariant, it is reduced to give an action of $U(n)_{\mathrm{Ad}}\times u(n)$ on T^*P^{n-1}, which we also denote by Φ. Thus

$$\Phi_{(g,X)}([z, w])=[gz, gw-iXgz+\langle gz, iXgz\rangle gz]$$

for $(g, X)\in U(n)_{\mathrm{Ad}}\times u(n)$, $[z, w]\in T^*P^{n-1}$. Since $f^{(X,Y)}$ is invariant under the S^1-action $(z, w)\mapsto(e^{it}z, e^{it}w)$, it induces a function on T^*P^{n-1}, which we also denote by $f^{(X,Y)}$;

$$f^{(X,Y)}([z, w])=-\langle Xz, w\rangle_{\boldsymbol{R}}-\frac{1}{2}\langle z, iYz\rangle .$$

Then as a corollary of Lemma 3.2, we have

PROPOSITION 3.3. *Φ: $(U(n)_{\mathrm{Ad}}\times u(n))\times T^*P^{n-1}\to T^*P^{n-1}$ is a Hamiltonian action with Lie algebra homomorphism: $(X, Y)\mapsto f^{(X,Y)}$. The moment map J: $T^*P^{n-1}\to u(n)_{\mathrm{ad}}\times u(n)$ $(\approx(u(n)_{\mathrm{ad}}\times u(n))^*)$ is given by*

$$J([z, w])=(izz^*, zw^*-wz^*) .$$

If $z=(i/\sqrt{n})^t(1, 1, \cdots, 1)$, then $[z, 0]\in T^*P^{n-1}$ and $J([z, 0])=((1/n)^t(1, 1, \cdots 1)(1, 1, \cdots, 1), 0)$.

THEOREM 3.4. *$[z, w]\mapsto J([z, w])$ is a symplectic diffeomorphism of T^*P^{n-1} onto the adjoint orbit of $U(n)_{\mathrm{Ad}}\times u(n)$ through $((i/n)^t(1, 1, \cdots, 1)\times(1, 1, \cdots, 1), 0)$ with the Lie-Kirillov-Kostant-Souriau symplectic structure.*

Moreover we have

$$J \circ \Phi_{(g,X)} = \mathrm{Ad}_{(g,X)} \circ J$$

for $(g, X) \in U(n)_{\mathrm{Ad}} \times u(n)$.

PROOF. By Theorem 1.1, it suffices to show that Φ is transitive and that J is injective. Since the latter is easily shown, we have only to show the former. If $[z, w], [z', w'] \in T^*P^{n-1}$, then $\langle z, z\rangle = \langle z', z'\rangle = 1$, $\langle z, w\rangle = \langle z', w'\rangle = 0$. Choose $g \in U(n)$ such that $z = gz'$. If we put $X = i\{(w - gw')z^* + z(w - gw')^*\}$, then $X \in u(n)$ and $\Phi_{(g,X)}([z', w']) = [z, w]$. Hence Φ is transitive.

§4. Proof of the complete integrability.

Let $(\ |\ )$ be the canonical Riemannian metric on P^{n-1}. Then $([z, w] | [z, w']) = \langle w, w'\rangle_R$ for $[z, w], [z, w'] \in T^*P^{n-1}$. Let us fix $a_1, a_2, \cdots, a_n \in \boldsymbol{R}$, and define a Hamiltonian function $H \in C^\infty(T^*P^{n-1})$ by

$$H([z, w]) = \frac{1}{2}([z, w] | [z, w]) + \frac{1}{2}\sum_{j=1}^{n} a_j |z_j|^2 .$$

The Hamiltonian system $(T^*P^{n-1}, \Omega_P, H)$ describes the motion of a point on P^{n-1} under the influence of the potential $U([z]) = (1/2)\sum a_j|z_j|^2$. We shall call this system the C. Neumann-type problem on the complex projective space. Let $\varphi_k, \psi_k \in C^\infty(u(n)_{\mathrm{ad}} \times u(n))$ be the functions defined in Section 2 with $A = \mathrm{diag}(ia_1, \cdots, ia_n)$.

PROPOSITION 4.1 (cf. [5, Theorem 3.4]). *The functions* $\varphi_2 \circ J, \cdots, \varphi_n \circ J$, $\psi_2 \circ J (= H), \cdots, \psi_n \circ J$ *are constants of the motion in involution for the C. Neumann-type problem* $(T^*P^{n-1}, \Omega_P, H)$ *on the complex projective space.*

PROOF. Since the pull-back $J^*: C^\infty(u(n)_{\mathrm{ad}} \times u(n)) \to C^\infty(T^*P^{n-1})$ is a Lie algebra homomorphism, it follows from Lemma 2.1 that the functions $\varphi_2 \circ J, \cdots, \varphi_n \circ J, \psi_2 \circ J, \cdots, \psi_n \circ J$ are all in involution.

LEMMA 4.2. *Let* $A = \mathrm{diag}(ia_1, \cdots, ia_n)$ *with* $a_1, \cdots, a_n$ *all distinct. Then* $d(\varphi_2 \circ J), \cdots, d(\varphi_n \circ J), d(\psi_2 \circ J), \cdots, d(\psi_n \circ J)$ *are linearly independent everywhere on an open dense set in* T^*P^{n-1}.

PROOF. Put $z = (1/\sqrt{n})^t(1, 1, \cdots, 1)$ and $X = izz^*$. Then $[z, 0] \in T^*P^{n-1}$ and $(X, 0) = J([z, 0])$. By Lemma 2.2, $\tilde{\varphi}_k|_{(X,0)} = ([X, i(-iA)^{k-1}], 0)$ and $\tilde{\psi}_k|_{(X,0)} = (0, [X, i(-iA)^{k-1}])$. Since

$$[X, i(-iA)^{k-1}] = \frac{1}{n}\begin{pmatrix} 0 & a_1^{k-1}-a_2^{k-1} & a_1^{k-1}-a_3^{k-1} & \cdots & a_1^{k-1}-a_n^{k-1} \\ a_2^{k-1}-a_1^{k-1} & 0 & a_2^{k-1}-a_3^{k-1} & \cdots & a_2^{k-1}-a_n^{k-1} \\ & & \cdots\cdots\cdots\cdots & & \\ a_n^{k-1}-a_1^{k-1} & a_n^{k-1}-a_2^{k-1} & a_n^{k-1}-a_3^{k-1} & \cdots & 0 \end{pmatrix},$$

we have that $[X, i(-iA)^{k-1}]$, $k=2, 3, \cdots, n$, are linearly independent if and only if $a_1, a_2, \cdots, a_n$ are all distinct. Hence $\tilde{\varphi}_2, \cdots, \tilde{\varphi}_n, \tilde{\psi}_2, \cdots, \tilde{\psi}_n$ are linearly independent at $(X, 0)$. It then follows that $d(\varphi_2 \circ J), \cdots, d(\varphi_n \circ J)$, $d(\psi_2 \circ J), \cdots, d(\psi_n \circ J)$ are linearly independent at $[z, 0]$. Since $\varphi_k \circ J$, $\psi_k \circ J$ are real-analytic, this completes the proof.

From Proposition 4.1 and Lemma 4.2, we have our main result (a "complex projective space-version" of [5, Theorem 4.3]):

THEOREM 4.3. *Let $a_1, \cdots, a_n$ be all distinct. Then the motion of a point on the complex projective space P^{n-1} under the influence of the potential $U([z]) = (1/2) \sum a_j |z_j|^2$ is completely integrable. Its $2(n-1)$ generically independent integrals in involution are given by $\varphi_2 \circ J, \cdots, \varphi_n \circ J$, $\psi_2 \circ J, \cdots, \psi_n \circ J$.*

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