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On the Number of Parameters of Linear Differential Equations with Regular Singularities on a Compact Riemann Surface

Dedicated to Professor Kôtaro Oikawa on his 60th birthday

Michitake KITA

Kanazawa University
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Introduction

Let X be a compact Riemann surface of genus g and let Y be a divisor of X consisting of m distinct points $p_1, \cdots, p_m$ of X. *We suppose that* $m \geqq 1$ *and moreover* $m \geqq 2$ *when* $g=0$. We recall a fundamental fact about linear differential equations with regular singularities; let $\Delta = \{z \in \boldsymbol{C} \mid |z|<1\}$ be a unit disc in $\boldsymbol{C}$ and let

$$\frac{d^n w}{dz^n} + a_1(z)\frac{d^{n-1}w}{dz^{n-1}} + \cdots + a_n(z)w = 0 \tag{1}$$

be a linear differential equation of order n where $a_i(z)$ is holomorphic in $\Delta - \{0\}$. The origin 0 is said to be a *regular singular point* of the equation (1) if the functions $z^i a_i(z)$ $(i=1, 2, \cdots, n)$ are holomorphic at 0. It is well known that this is equivalent to the condition that the equation (1), multiplied by z^n, can be written in the form

$$\left(z\frac{d}{dz}\right)^n w + b_1(z)\left(z\frac{d}{dz}\right)^{n-1} w + \cdots + b_n(z)w = 0 \tag{2}$$

where $b_i(z)$ $(i=1, \cdots, n)$ are holomorphic at 0. Using this fact, we define a linear differential equation on a compact Riemann surface X of order n with regular singularities along Y as follows; let $X = \bigcup_{j=1}^N U_j$ be a sufficiently fine finite open coordinate covering of X such that $p_j \in U_j$ $(j=1, \cdots, m)$ and $z_j(p_j)=0$ for $j=1, \cdots, m$ and z_j is *nowhere zero* in U_j for $j=m+1, \cdots, N$. In each neighbourhood U_j we consider a linear differential equation

$$\left(z_j\frac{d}{dz_j}\right)^n w + b_{j,1}(z_j)\left(z_j\frac{d}{dz_j}\right)^{n-1} w + \cdots + b_{j,n}(z_j)w = 0 \tag{3}$$

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where $b_{j,k}(z_j)$ $(k=1, \cdots, n)$ are holomorphic in U_j. A linear differential equation on X of order n with regular singularities along Y is, by definition, a collection of the equations (3) which are compatible in the sense that any two of them have the same solutions on their common domain of definition. T. Saito [2] decided the number of independent parameters of linear differential equations on X of order n with regular singularities along Y. In this note, we give another proof of the theorem of T. Saito by using the notion of eulerian jet bundles.

§1. Let $z_j(d/dz_j)$ be the *eulerian vector field* on U_j $(j=1, \cdots, N)$. When φ is a holomorphic function in $U_j \cap U_k \neq \emptyset$, by the chain rule of differentiation, we have

$$\begin{pmatrix} \varphi \\ z_j\dfrac{d}{dz_j}\varphi \\ \vdots \\ \left(z_j\dfrac{d}{dz_j}\right)^n \varphi \end{pmatrix} = P_{jk}^{(n)}(z) \begin{pmatrix} \varphi \\ z_k\dfrac{d}{dz_k}\varphi \\ \vdots \\ \left(z_k\dfrac{d}{dz_k}\right)^n \varphi \end{pmatrix}$$

where $P_{jk}^{(n)}(z)$ is the matrix-valued holomorphic functions in $U_j \cap U_k$, of the form

$$P_{jk}^{(n)}(z) = \begin{pmatrix} 1 & & & & 0 \\ 0 & \dfrac{z_j}{z_k}\dfrac{dz_k}{dz_j} & & & \\ 0 & & \left(\dfrac{z_j}{z_k}\dfrac{dz_k}{dz_j}\right)^2 & & \\ \vdots & & & \ddots & \\ 0 & & * & & \left(\dfrac{z_j}{z_k}\dfrac{dz_k}{dz_j}\right)^n \end{pmatrix} \tag{4}$$

which satisfies the cocycle conditions

$$P_{jk}^{(n)}(z) P_{kj}^{(n)}(z) = E_{n+1} \quad \text{on} \quad U_j \cap U_k$$

and

$$P_{ij}^{(n)}(z) P_{jk}^{(n)}(z) = P_{ik}^{(n)}(z) \quad \text{on} \quad U_i \cap U_j \cap U_k .$$

Thus the cocycle $\{P_{jk}^{(n)}(z)\}$ determines a holomorphic vector bundle $P_Y^{(n)}$ of rank $n+1$, which is called the *eulerian jet bundle associated to Y*. From the form (4) of transition functions and the choice of the local coordinate z_j $(j=1, \cdots, N)$, it follows that the bundle $P_Y^{(n)}$ contains a subbundle $(K\otimes[Y])^{\otimes n}$ of rank one where K is the canonical bundle of X and $[Y]$ is the line bundle associated to the divisor Y. On U_j, the sheaf $\mathcal{O}(P_Y^{(n)})$ of germs of holomorphic sections of $P_Y^{(n)}$ is identified with $\mathcal{O}_{U_j}^{n+1}$ and we have a homomorphism

$$D^n : \mathcal{O}_{U_j} \longrightarrow \mathcal{O}(P_Y^{(n)})|_{U_j} = \mathcal{O}_{U_j}^{n+1} \tag{5}$$
$$\varphi \longmapsto \begin{pmatrix} \varphi \\ z_j \dfrac{d}{dz_j}\varphi \\ \vdots \\ \left(z_j \dfrac{d}{dz_j}\right)^n \varphi \end{pmatrix} .$$

It follows that these homomorphisms are compatible in the sense that any two of them define the same homomorphism in a common domain of definition and define a sheaf homomorphism

$$D^n : \mathcal{O}_X \longrightarrow \mathcal{O}(P_Y^{(n)}) .$$

By using the eulerian jet bundle associated to Y and the formulation of P. Deligne [1, p. 24], we can formulate the notion of linear differential equations on X of order n with regular singularities along Y as follows:

DEFINITION. A linear differential equation on X of order n with regular singularities along Y is a $\mathcal{O}_X$-homomorphism

$$E : \mathcal{O}(P_Y^{(n)}) \longrightarrow \mathcal{O}((K\otimes[Y])^{\otimes n})$$

such that the restriction of E to the subsheaf $\mathcal{O}((K\otimes[Y])^{\otimes n})$ is the identity: $E|_{((K\otimes[Y])^{\otimes n})} = \text{identity}$. Then a holomorphic function φ near z is a solution of the differential equation E if $E(D^n(\varphi))=0$.

REMARK. Let z be a local coordinate of a small open neighbourhood U. Then we can identify $\mathcal{O}(P_Y^{(n)})|_U$ with $\mathcal{O}_U^{n+1}$ and the homomorphism D^n can be written in the form

$$D^n : \mathcal{O}_U \longrightarrow \mathcal{O}_U^{n+1}$$
$$\varphi \longmapsto {}^t\!\left(\varphi,\ z\frac{d}{dz}\varphi,\ \cdots,\ \left(z\frac{d}{dz}\right)^n \varphi\right) .$$

By the choice of the local coordinate z we can identify the locally free sheaf $\mathcal{O}((K\otimes[Y])^{\otimes n})|_U$ with $\mathcal{O}_U$ and the linear differential equation E can be written in the form

$$E:\mathcal{O}_U^{n+1}\longrightarrow\mathcal{O}_U$$
$$^t(\varphi_n,\cdots,\varphi_0)\longmapsto\sum_{i=0}^{n}b_i(z)\varphi_i(z)$$

where $b_0(z)=1$. Then a solution φ of E in U is a holomorphic function in U which satisfies

$$\left(z\frac{d}{dz}\right)^n\varphi+b_1(z)\left(z\frac{d}{dz}\right)^{n-1}\varphi+\cdots+b_n(z)\varphi=0\ .$$

Thus our definition of linear differential equation E on X of order n with regular singularities along Y is equivalent to the classical one.

§2. We show that the eulerian jet bundle $P_Y^{(n)}$ is decomposed into a direct sum of line bundles.

THEOREM 1. *We have*

$$P_Y^{(n)}=1\oplus(K\otimes[Y])\oplus(K\otimes[Y])^{\otimes 2}\oplus\cdots\oplus(K\otimes[Y])^{\otimes n} \tag{6}$$

where 1 *is the trivial line bundle on* X.

PROOF. We shall prove Theorem 1 by the induction on the number n. When $n=1$, by (4) the transition function $P_{jk}^{(1)}(z)$ has the form

$$P_{jk}^{(1)}(z)=\begin{pmatrix}1 & 0\\ 0 & \dfrac{z_j}{z_k}\dfrac{dz_k}{dz_j}\end{pmatrix}\quad\text{in } U_j\cap U_k$$

which shows that $P_Y^{(1)}=1\oplus(K\otimes[Y])$. Supposing that the statement is true for $n-1$, we shall show that it is true for n. Since the line bundle $(K\otimes[Y])^{\otimes n}$ is the subbundle of $P_Y^{(n)}$ and the transition function $P_{jk}^{(n)}(z)$ has the form

$$P_{jk}^{(n)}(z)=\left(\begin{array}{c|c} P_{jk}^{(n-1)} & \begin{matrix}0\\ \vdots\\ 0\end{matrix}\\ \hline *\ \cdots\ * & \left(\dfrac{z_j}{z_k}\dfrac{dz_k}{dz_j}\right)^n\end{array}\right)$$

we have an exact sequence of vector bundles

$$(7)\qquad 0 \longrightarrow (K\otimes[Y])^{\otimes n} \longrightarrow P_Y^{(n)} \longrightarrow P_Y^{(n-1)} \longrightarrow 0\ .$$

Since $\mathcal{O}(P_Y^{(n-1)})$ is locally free, (7) induces an exact $\mathcal{O}_X$-sequence

$$(8)\qquad \begin{aligned} 0 \longrightarrow \mathcal{H}om(P_Y^{(n-1)}, (K\otimes[Y])^{\otimes n}) &\longrightarrow \mathcal{H}om(P_Y^{(n-1)}, P_Y^{(n)}) \\ &\longrightarrow \mathcal{H}om(P_Y^{(n-1)}, P_Y^{(n-1)}) \longrightarrow 0 \end{aligned}$$

where $\mathcal{H}om(V, W)$ is the sheaf of germs of local homomorphisms from the sheaf of germs of holomorphic sections of a holomorphic vector bundle V to that of a holomorphic vector bundle W. Thus we have an exact sequence of cohomology groups

$$(9)\qquad \begin{aligned} H^0(X, \mathcal{H}om(P_Y^{(n-1)}, P_Y^{(n)})) &\xrightarrow{\alpha} H^0(X, \mathcal{H}om(P_Y^{(n-1)}, P_Y^{(n-1)})) \\ &\longrightarrow H^1(X, \mathcal{H}om(P_Y^{(n-1)}, (K\otimes[Y])^{\otimes n}))\ . \end{aligned}$$

We denote by $\mathcal{L}$ the locally free sheaf of germs of holomorphic sections of $K\otimes[Y]$ and by $\mathcal{L}^{\otimes n}$ the tensor product $\mathcal{L}\otimes\cdots\otimes\mathcal{L}$ of $\mathcal{L}$ with itself n times. From the assumption of the induction it follows that

$$(10)\qquad \begin{aligned} \mathcal{H}om(P_Y^{(n-1)}, (K\otimes[Y])^{\otimes n}) &= \mathcal{O}(P_Y^{(n-1)})^*\otimes\mathcal{L}^{\otimes n} \\ &= [\mathcal{O}\oplus\mathcal{L}\oplus\cdots\oplus\mathcal{L}^{\otimes(n-1)}]^*\otimes\mathcal{L}^{\otimes n} \\ &= \mathcal{L}^{\otimes n}\oplus\mathcal{L}^{\otimes(n-1)}\oplus\cdots\oplus\mathcal{L}\ . \end{aligned}$$

As X is compact, we can identify $H^2(X, \boldsymbol{Z})$ with $\boldsymbol{Z}$ naturally and by this identification we consider the Chern class $c(\mathcal{L})$ of $\mathcal{L}$ as a *rational integer*. Let F be a holomorphic line bundle on X. Then it is known that we have $H^1(X, \mathcal{O}(F))=0$ if $c(F\otimes K^*)>0$. As for $\mathcal{L}^{\otimes k}=\mathcal{O}((K\otimes[Y])^{\otimes k})$, we have that

$$c((K\otimes[Y])^{\otimes k}\otimes K^*)=(k-1)(2g-2)+km>0$$

because we suppose that $m\geqq 1$ and moreover $m\geqq 2$ when $g=0$. Hence we have

$$(11)\qquad H^1(X, \mathcal{L}^{\otimes k})=0 \quad \text{for} \quad k=1, \cdots, n\ .$$

Thus, from (10) it follows that

$$H^1(X, \mathcal{H}om(P_Y^{(n-1)}, (K\otimes[Y])^{\otimes n}))=0\ .$$

This means that the homomorphism α in (9) is surjective; hence by the standard argument we see that the exact sequence (9) splits and we have

$$P_Y^{(n)}=P_Y^{(n-1)}\oplus(K\otimes[Y])^{\otimes n}$$

$$=1\oplus(K\otimes[Y])\oplus\cdots\oplus(K\otimes[Y])^{\otimes n}.$$

This completes the induction. Q.E.D.

§3. From the definition of a linear differential equation with regular singularities along Y, it follows the set of all linear differential equations of order n with regular singularities along Y is the affine subspace V of

$$W=H^0(X,\ \mathscr{H}om(P_Y^{(n)},\ (K\otimes[Y])^{\otimes n}))$$

which consists of $E\in W$ such that $E|_{(K\otimes[Y])^{\otimes n}}=\text{identity}$. Since the exact sequence (7) splits, the restriction mapping

$$W=H^0(X,\ \mathscr{H}om(P_Y^{(n)},\ (K\otimes[Y])^{\otimes n}))\longrightarrow H^0(X,\ \mathscr{E}nd((K\otimes[Y])^{\otimes n}))\simeq \boldsymbol{C}$$

is surjective. Thus V is an affine subspace of W of *codimension one*. In a similar way to (10), by using Theorem 1, we have

$$W=\bigoplus_{k=0}^{n}H^0(X,\ \mathscr{L}^{\otimes k}).$$

By the Riemann-Roch theorem we have

$$\dim H^0(X,\ \mathscr{L}^{\otimes k})-\dim H^1(X,\ \mathscr{L}^{\otimes k})=c(\mathscr{L}^{\otimes k})-(g-1).$$

Then, from (11) it follows that for $k\geqq 1$, we have

$$\begin{aligned}\dim H^0(X,\ \mathscr{L}^{\otimes k})&=kc(\mathscr{L})-(g-1)\\&=km+(2k-1)(g-1).\end{aligned}$$

Thus we have

$$\begin{aligned}\dim W&=1+\sum_{k=1}^{n}\dim H^0(X,\ \mathscr{L}^{\otimes k})\\&=1+\frac{n(n+1)}{2}m+n^2(g-1);\end{aligned}$$

hence we have

$$\dim V=\frac{n(n+1)}{2}m+n^2(g-1).$$

Thus we obtain the following

THEOREM 2. *Let X be a compact Riemann surface of genus g and let Y be a divisor of X consisting of m distinct points of X. We suppose that $m\geqq 1$ and moreover $m\geqq 2$ when $g=0$. Then the number of* *independent parameters of linear differential equation on X of order n with regular singularities along Y is equal to*

$$\frac{n(n+1)}{2}m+n^2(g-1) .$$

References

[1] P. DELIGNE, Équations différentielles à points singuliers, Lecture Notes in Math., **163**, Springer-Verlag, 1970.

[2] T. SAITO, A note on the linear differential equation of Fuchsian type with algebraic coefficients, Kodai Math. Sem. Rep., **10** (1958), 58-63.

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