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On a Degenerate Quasilinear Elliptic Equation with Mixed Boundary Conditions

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Introduction

Let Ω be a bounded simply connected domain in $\boldsymbol{R}^n$. The boundary $\partial\Omega$ is assumed to be of class C^1. Let S be a compact C^1 manifold of dimension $n-2$ belonging to $\partial\Omega$. We assume that S divide $\partial\Omega$ into two non-empty relatively open subsets $\partial_1\Omega$ and $\partial_2\Omega$, more precisely,

$$\partial\Omega=\partial_1\Omega\cup\partial_2\Omega\cup S\ , \qquad \partial_1\Omega\cap\partial_2\Omega=\varnothing\ .$$

We assume that the usual function spaces $C^k(\bar{\Omega})$, $C_0^k(\Omega)$, $L^q(\Omega)$, $W^{1,q}(\Omega)$, $W_0^{1,q}(\Omega)$ are known. The norm in $W^{1,q}(\Omega)$ $(L^q(\Omega))$ is written with $\|\ \|_{1,q}$ $(\|\ \|_q)$, respectively. Throughout this paper let $2<p<\infty$, and let all functions be real-valued. We set

$$C_{(0)}^1(\bar{\Omega})=\{u\in C^1(\bar{\Omega});\ u=0 \text{ in a neighborhood of } \overline{\partial_1\Omega}\}\ .$$

The completion of $C_{(0)}^1(\bar{\Omega})$ with respect to the norm $\|\ \|_{1,p}$ is denoted by $V(\Omega)$. The space $V(\Omega)$ is reflexive and separable. The norm in $V(\Omega)$ is denoted by $\|\ \|_V$. Let $V'(\Omega)$ be the dual space of $V(\Omega)$. As is well-known, Poincaré's inequality is valid for all functions in $V(\Omega)$, that is,

$$\|u\|_p\leqq C\|\nabla u\|_p\ , \qquad u\in V(\Omega)\ . \tag{0.1}$$

Hereafter let α be a real number such that

$$\begin{cases}\alpha\geqq 0 & \text{when}\quad p\geqq n\ ,\\ 0\leqq\alpha\leqq\dfrac{n(p-1)}{n-p}-1 & \text{when}\quad p<n\ .\end{cases} \tag{0.2}$$

We denote by (,) the inner product in $L^2(\Omega)$. For $u\in V(\Omega)$ we define

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$A(u)$ by the equality

$$(A(u), v)=(|\nabla u|^{p-2}\nabla u, \nabla v)+(|u|^{\alpha}u, v)\ , \qquad v\in V(\Omega)\ .$$

If $1<q<\infty$, let q^* be the dual number of q, i.e., $q^*=q/(q-1)$. By Sobolev's imbedding theorem and (0.2) the imbedding $V(\Omega)\to L^{(1+\alpha)p^*}(\Omega)$ is continuous. Hence we see that $A(u)\in V'(\Omega)$.

We consider the equation $A(u)=f$ in Ω, that is,

$$(|\nabla u|^{p-2}\nabla u, \nabla v)+(|u|^{\alpha}u, v)=(f, v)\ , \qquad v\in V(\Omega)\ . \tag{0.3}$$

The following inequalities hold: For $\xi, \eta\in \boldsymbol{R}^n$

$$(|\xi|^{p-2}\xi-|\eta|^{p-2}\eta, \xi-\eta)\geqq c_0|\xi-\eta|^p\ , \qquad c_0>0\ , \tag{0.4}$$

and for $a, b\in \boldsymbol{R}^1$

$$(|a|^{\alpha}a-|b|^{\alpha}b, a-b)\geqq 0\ . \tag{0.5}$$

It is known that with the aid of (0.1), (0.4) and (0.5) the following proposition is obtained by the "monotonicity" method (cf., e.g., Chap. 2 in [11]).

PROPOSITION 0.1. *For* $f\in V'(\Omega)$ *there is a unique solution* $u\in V(\Omega)$ *of* (0.3).

For $x\in \boldsymbol{R}^n$ we denote by $\phi(x)$ the distance between x and S. And we denote simply by ∂^k any k-th order derivative with respect to the variables x_j, $1\leqq j\leqq n$. Our aim is to prove the following theorems.

THEOREM 1. *Let f be in $V'(\Omega)$, and let $u\in V(\Omega)$ be a solution of* (0.3). *Then there is a positive number β_0 such that if $\phi^{1-\beta+\beta/p}f\in L^{p^*}(\Omega)$, where $0\leqq\beta\leqq\beta_0$, it holds that $\phi^{-1-\beta/p}u$, $\phi^{-\beta/p}\partial u\in L^p(\Omega)$ and*

$$\|\phi^{-1-\beta/p}u\|_p+\|\phi^{-\beta/p}\partial u\|_p\leqq C(\|\phi^{1-\beta+\beta/p}f\|_{p^*})^{1/(p-1)}\ .$$

Here β_0 and C are independent of f.

REMARK. If $\phi^{1-\beta+\beta/p}f\in L^{p^*}(\Omega)$, $\beta>0$, then $f\in V'(\Omega)$ from Lemma 1.4 in the following section. Hence the assumption of $f\in V'(\Omega)$ is superfluous in Theorem 1.

Let $\theta\cdot\nabla$ be a C^1 vector field on $\bar{\Omega}$ which is tangent to $\partial\Omega$: if $\theta=(\theta_1, \cdots, \theta_n)$, $\theta\cdot\nabla=\sum_{i=1}^n \theta_i\partial_{x_i}$. Let C^{ω} be the class of analytic functions.

THEOREM 2. *Suppose that $\partial\Omega$ and S are both of class C^{ω}. Let f be* *in* $W^{1,p^*}(\Omega)$, *and let* $u \in V(\Omega)$ *be a solution of* (0.3). *Then* $(\theta\cdot\nabla)|\nabla u|^{p/2} \in L^{1+\delta}(\Omega)$ *for some* $\delta>0$ *and*

$$\|(\theta\cdot\nabla)|\nabla u|^{p/2}\|_{1+\delta} \leqq C[(\|f\|_{1,p*})^{p^*}+(\|f\|_{p*})^{(2+\alpha)/(p-1)}]^{1/2},$$

where δ *and* C *are independent of* f.

THEOREM 3. *Under the assumptions in Theorem 2, it holds that* $\phi^{1-\delta}|\nabla u|^{p-1} \in W^{1,p^*}(\Omega)$ *for some* $\delta>0$ *and*

$$\|\phi^{1-\delta}|\nabla u|^{p-1}\|_{1,p*} \leqq C[\|f\|_{1,p*}+(\|f\|_{p*})^{(1+\alpha)/(p-1)}+(\|f\|_{p*})^{(2+\alpha)/p}],$$

where δ *and* C *are independent of* f.

Let $u \in V(\Omega)$ be a solution of (0.3). Then naturally $u=0$ on $\partial_1\Omega$. By Green's formula it follows that $|\nabla u|^{p-2}\sum_j \cos(\nu, x_j)\partial_{x_j}u=0$ on $\partial_2\Omega$ in the weak sense, where ν is the exterior normal of $\partial\Omega$ with respect to Ω. Hence $\sum_j \cos(\nu, x_j)\partial_{x_j}u=0$ on $\partial_2\Omega$ in the same sense, so that u satisfies the mixed boundary conditions of Dirichlet-Neumann type on $\partial\Omega$. Such a mixed boundary condition appears in the book of J. L. Lions [11] (cf. p. 345), where the existence of weak solutions for the non-stationary case was shown. Thus it seems meaningful for us to derive a regularity property up to the boundary for solutions of (0.3). Under mixed boundary conditions, nonlinear equations of another type or more general type were considered by several authors (cf., e.g., [1], [3], [5], [7], [12], [16], [18]). Here we do not state explicitly their results.

In 1968, E. Shamir [17] proved the regularity up to the boundary for solutions of linear elliptic equations, under general mixed boundary conditions. In this connection we note also the work in H. Beirão da Veiga [20]. Recently, results analogous to [17] for linear parabolic equations have been proved by G. M. Lieberman [9]. The method in [17] is to construct a Green's function. Since it is not almost applicable to nonlinear equations, we have to use another method for (0.3). As an application of [17], M. K. V. Murthy and G. Stampacchia [13] proved the regularity for solutions of a single variational inequality with mixed boundary conditions. One of the authors and H. Nagase [4] have considered a system of variational inequalities and they have obtained an analogous result to one of the theorems in [13]. The method in [4] is, in a word, to use a parallel translation with a weight when one constructs a difference of differentiation for weak solutions.

The interior regularity for weak solutions of $A(u)=f$ was discussed by several authors (cf., e.g., [2], [8], [19]) and then the best result has been obtained. Next we consider the equation $A(u)=f$ under the Dirichlet boundary condition throughout $\partial\Omega$. In this case it is enough to replace $V(\Omega)$ with $W_0^{1,p}(\Omega)$ in (0.3). For this M. I. Vishik [21] first obtained a global regularity property by using Galerkin's method. In a series of papers G. N. Jakolev extended and improved the results in [21] (cf., e.g., [6]). His result is as follows: $|\nabla u|^{p-2}\partial u \in W^{1,p^*}(\Omega)$, if $f \in W^{1,p^*}(\Omega)$. In [6] the method of a simple parallel translation was used, which was already applied to linear elliptic equations by L. Nirenberg [14].

The argument in this paper is based on [14], [11, Chap. 2], [6] and particularly [4]. Theorem 2 is proved with the aid of Theorem 1. We obtain Theorem 3 by proving the regularity of weak solutions along the normal direction of $\partial\Omega$.

§ 1. Lemmas.

Throughout this paper, the notation "$\rightharpoonup$" means the weak convergence.

LEMMA 1.1 (Lions [11, p. 12]). *Let $u \in L^q(\Omega)$ $(1<q<\infty)$ and suppose that $\{\|u_j\|_q\}$ is uniformly bounded and $u_j \to u$ pointwise a.e. in Ω. Then $u_j \rightharpoonup u$ in $L^q(\Omega)$.*

It is known that any $u \in W^{1,p}(\Omega)$ has its trace on $\partial\Omega$, so on $\partial_1\Omega$.

LEMMA 1.2. *Let $u \in W^{1,p}(\Omega)$ and $u=0$ on $\partial_1\Omega$. Then $u \in V(\Omega)$.*

PROOF. For $R>0$ we set $\Sigma=\{x \in \boldsymbol{R}^n;\ |x|<R,\ x_n>0\}$. And we define

$$C^1_{(0)}(\bar{\Sigma})=\{u \in C^1(\bar{\Sigma})\ ;\ u=0 \text{ in a neighborhood of } \{|x|=R\} \cup \{x_n=0,\ x_{n-1}\geqq 0\}\}\ . \tag{1.1}$$

By a suitable partition of unity it is enough to prove the following assertion:

Let $u \in W^{1,p}(\Sigma)$. Let $u=0$ on $\{x_n=0,\ x_{n-1}\geqq 0\}$ and near $\{|x|=R\}$. Then there is a sequence $\{u_j\}\subset C^1_{(0)}(\bar{\Sigma})$ such that $u_j \to u$ in $W^{1,p}(\Sigma)$.

Let u satisfy the assumptions in this proposition. We write $x''=(x_1, \cdots, x_{n-2})$. Since $u(x'', x_{n-1}+\varepsilon, x_n) \to u(x)$ in $W^{1,p}(\Sigma)$ as $\varepsilon \to +0$, we may assume that $u=0$ on $\{x_n=0,\ x_{n-1}>-\delta\}$ for some $\delta>0$.

Let us take a function $g(x_{n-1})$ such that $g(x_{n-1})=0$ for $x_{n-1}\leqq -\delta$ and $g(x_{n-1})=-(x_{n-1}+\delta)^2$ for $x_{n-1}>-\delta$. And we set

$$\tilde{\Sigma}=\{|x|<R\} \cap \{x_n>g(x_{n-1})\}$$

(see Figure 1). We extend u throughout $\tilde{\Sigma}$ in such a way that $u=0$ in $\tilde{\Sigma}-\Sigma$. Then $u \in W^{1,p}(\tilde{\Sigma})$ obviously. Further we define an extension of u in $\{|x|<R\}$ in such a way that $u=0$ on $\{|x|=R\}$ and u is in $W^{1,p}(\{|x|<R\})$.

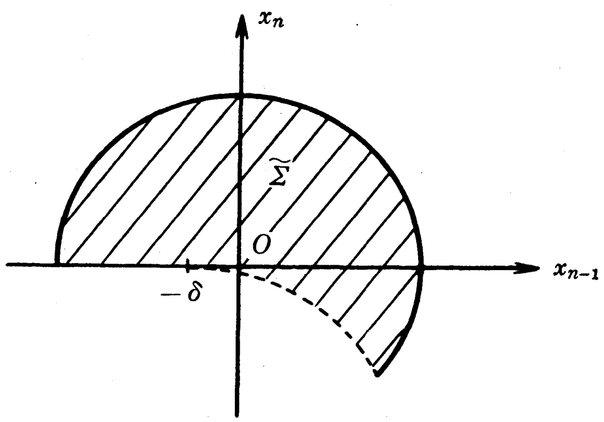

FIGURE 1

Then $u(x'', x_{n-1}, x_n-\varepsilon)\to u(x)$ in $W^{1,p}(\{|x|<R\})$ as $\varepsilon\to+0$. By using the mollifier, we can take a sequence $\{u_j\}\subset C^1_{(0)}(\bar{\Sigma})$ satisfying $u_j\to u$ in $W^{1,p}(\Sigma)$. Thus the above assertion has been proved, so that we have completed the proof of our lemma. Q.E.D.

Let $\boldsymbol{R}^n_+=\{(x_1,\cdots,x_n); x_n>0\}$, and let us define

$$C^1_{(0)}(\overline{\boldsymbol{R}^n_+})=\{u\in C^1(\overline{\boldsymbol{R}^n_+})\ ;\ u=0 \text{ in a neighborhood of } \{x_n=0,\ x_{n-1}\geqq 0\} \text{ and for sufficiently large } |x|\}\ .$$

Hereafter we write $r=|x|$ and $\rho(x)=(x^2_{n-1}+x^2_n)^{1/2}$.

LEMMA 1.3. *Let $t\neq 2$ and $q>1$. Then for any $u\in C^1_{(0)}(\overline{\boldsymbol{R}^n_+})$ it holds that*

$$\int_{x_n\geqq 0}\rho^{-t}|u|^q dx\leqq C\int_{x_n\geqq 0}\rho^{q-t}|\nabla u|^q dx\ ,$$

where C is independent of u.

PROOF. Obviously it is sufficient to prove only the case of $n=2$. Writing $'=d/dr$, we have

$$\int_0^\infty r^{1-t}|u|^q dr=\frac{-1}{2-t}\int_0^\infty (r^{2-t})'|u|^q dr$$
$$=\frac{-q}{2-t}\int_0^\infty r^{2-t}|u|^{q-2}uu'dr\ .$$

Hence

$$\int_{x_2\geqq 0} r^{-t}|u|^q dx\leqq C\int_{x_2\geqq 0} r^{1-t}|u|^{q-1}|\nabla u|dx$$

(by Hölder's inequality)

$$\leqq C\left(\int_{x_2\geqq 0} r^{-t}|u|^q dx\right)^{(q-1)/q}\left(\int_{x_2\geqq 0} r^{q-t}|\nabla u|^q dx\right)^{1/q} .$$

Applying Young's inequality, we complete the proof. Q.E.D.

LEMMA 1.4. *Let $t\neq 2$ and $q>1$. Then for any $u\in C^1_{(0)}(\bar{\Omega})$ it holds that*

$$\int_\Omega \phi^{-t}|u|^q dx \leqq C\left(\int_\Omega \phi^{q-t}|\nabla u|^q dx + \int_\Omega \phi^{q-t}|u|^q dx\right) ,$$

where C is independent of u.

PROOF. Let P_0 be any fixed point in S. There are a neighborhood U of P_0 and a C^1-homeomorphic mapping Φ from U into $(y_1, \cdots, y_n)$-space such that

$$\Phi(P_0)=O , \qquad \Phi(U\cap\Omega)\subset\{y_n>0\} ,$$
$$\Phi(U\cap\partial\Omega)\subset\{y_n=0\} , \quad \Phi(U\cap S)\subset\{y_{n-1}=y_n=0\} ,$$

and

$$c\rho(y)\leqq\phi(x)\leqq c^{-1}\rho(y) , \qquad x\in U ,$$

where c is some positive constant. Let $\eta\in C_0^\infty(U)$. Then from Lemma 1.3

$$\int_{y_n\geqq 0}\rho^{-t}|\eta u|^q dy \leqq C\int_{y_n\geqq 0}\rho^{q-t}|\nabla_y(\eta u)|^q dy .$$

Hence we have

$$\int_\Omega \phi^{-t}|\eta u|^q dx \leqq C\int_\Omega \phi^{q-t}|\nabla_x(\eta u)|^q dx$$
$$\leqq C\left(\int_\Omega \phi^{q-t}|\nabla u|^q dx + \int_\Omega \phi^{q-t}|u|^q dx\right) .$$

By this inequality and a suitable partition of unity Lemma 1.4 is proved. Q.E.D.

§2. Propositions.

For a subdomain D of Ω we denote by $\|\ \|_{1,q,D}$ ($\|\ \|_{q,D}$) the norm in $W^{1,q}(D)$ ($L^q(D)$), respectively. We denote by $L^q_{\mathrm{loc}}(\Omega)$ the space of locally q-integrable functions in Ω. And we define $W^{1,q}_{\mathrm{loc}}(\Omega)=\{u;\, u,\, \partial u\in L^q_{\mathrm{loc}}(\Omega)\}$.

The following proposition is due to M. I. Vishik [21], J. L. Lions [11, p. 112] and G. N. Jakolev [6]. We repeat its proof for the completeness. We follow mainly the method in [6]. However the inequality (2.3) is effective in our proof.

PROPOSITION 2.1. *Let u be the solution in Proposition* 0.1, *and let f be in $W^{1,p^*}_{\mathrm{loc}}(\Omega)$. Then $|\nabla u|^{p-2}\partial u \in W^{1,p^*}_{\mathrm{loc}}(\Omega)$. More precisely, for any subdomain D with $\bar{D}\subset\Omega$ there is a constant C depending on D and f such that*

$$\| \, |\nabla u|^{p-2}\partial u\|_{1,p^*,D}\leqq C \,.$$

PROOF. Let us fix any integer i with $1\leqq i\leqq n$. Let $h>0$ be sufficiently small. For any function v we define

$$v_h(x)=v(x_1, \cdots, x_{i-1}, x_i+h, x_{i+1}, \cdots, x_n) \,,$$
$$(D_h v)(x)=\frac{1}{h}(v_h(x)-v(x)) \,.$$

Let us take a function $\zeta(x)\in C_0^\infty(\Omega)$. If $v\in W^{1,p}(\Omega)$, then $D_h(\zeta^2 v)\in V(\Omega)$ obviously. Hence we have from (0.3)

$$(|\nabla u|^{p-2}\nabla u,\ \nabla D_h(\zeta^2 v))+(|u|^\alpha u,\ D_h(\zeta^2 v))=(f,\ D_h(\zeta^2 v)) \,. \tag{2.1}$$

Evidently

$$(|\nabla u|^{p-2}\nabla u,\ \nabla D_h(\zeta^2 v))=-(D_{-h}(|\nabla u|^{p-2}\nabla u),\ \nabla(\zeta^2 v)) \,,$$
$$(|u|^\alpha u,\ D_h(\zeta^2 v))=-(D_{-h}(|u|^\alpha u),\ \zeta^2 v)$$

and

$$(f,\ D_h(\zeta^2 v))=-(D_{-h}f,\ \zeta^2 v) \,.$$

Hence (2.1) becomes

$$(\zeta^2 D_{-h}(|\nabla u|^{p-2}\nabla u),\ \nabla v)+2(\zeta D_{-h}(|\nabla u|^{p-2}\nabla u),\ v\nabla\zeta)$$
$$+(\zeta^2 D_{-h}(|u|^\alpha u),\ v)=(\zeta^2 D_{-h}f,\ v) \,.$$

Replacing v by $D_{-h}u$, we have

$$(\zeta^2 D_{-h}(|\nabla u|^{p-2}\nabla u),\ D_{-h}\nabla u)+2(\zeta D_{-h}(|\nabla u|^{p-2}\nabla u),\ D_{-h}u\cdot\nabla\zeta)$$
$$+(\zeta^2 D_{-h}(|u|^\alpha u),\ D_{-h}u)=(\zeta^2 D_{-h}f,\ D_{-h}u) \,. \tag{2.2}$$

By P. Lindqvist [10] the following inequality is valid:

(2.3)
$$(|\xi|^{p-2}\xi-|\eta|^{p-2}\eta,\ \xi-\eta) \geqq c_0(|\xi|^{p-2}+|\eta|^{p-2})|\xi-\eta|^2\ , \qquad \xi,\ \eta\in \boldsymbol{R}^n\ ,$$

where c_0 is a positive constant*). Hence we get

(2.4)
$$(\zeta^2 D_{-h}(|\nabla u|^{p-2}\nabla u),\ D_{-h}\nabla u) \geqq c_0\int_\Omega \zeta^2(|\nabla u|^{p-2}+|\nabla u_{-h}|^{p-2})|\nabla D_{-h}u|^2dx\ .$$

If $\gamma\geqq 0$, it holds that for any $\xi,\ \eta\in \boldsymbol{R}^n$

(2.5)
$$|\,|\xi|^\gamma\xi-|\eta|^\gamma\eta|,\ |\,|\xi|^{\gamma+1}-|\eta|^{\gamma+1}|\leqq C(|\xi|^\gamma+|\eta|^\gamma)|\xi-\eta|\ ,$$

which yields

$$|D_{-h}(|\nabla u|^{p-2}\nabla u)|\leqq C(|\nabla u|^{p-2}+|\nabla u_{-h}|^{p-2})|\nabla D_{-h}u|\ .$$

Thus we have

$$|(\zeta D_{-h}(|\nabla u|^{p-2}\nabla u),\ D_{-h}u\cdot\nabla\zeta)|$$
$$\leqq C\int_\Omega |\zeta\nabla\zeta|(|\nabla u|^{p-2}+|\nabla u_{-h}|^{p-2})|\nabla D_{-h}u|\,|D_{-h}u|dx$$

(by Hölder's inequality)

$$\leqq C\Big(\int_\Omega \zeta^2(|\nabla u|^{p-2}+|\nabla u_{-h}|^{p-2})|\nabla D_{-h}u|^2dx\Big)^{1/2}$$
$$\cdot\Big(\int_\Omega |\nabla\zeta|(|\nabla u|^p+|\nabla u_{-h}|^p)dx\Big)^{(p-2)/2p}\Big(\int_\Omega |\nabla\zeta|\,|D_{-h}u|^pdx\Big)^{1/p}$$

(by Young's inequality)

$$\leqq \frac{c_0}{4}\int_\Omega \zeta^2(|\nabla u|^{p-2}+|\nabla u_{-h}|^{p-2})|\nabla D_{-h}u|^2dx$$
$$+C\Big[\int_\Omega |\nabla\zeta|(|\nabla u|^p+|\nabla u_{-h}|^p)dx+\int_\Omega |\nabla\zeta|\,|D_{-h}u|^pdx\Big]\ .$$

Now putting $v=u$ in (0.3), we see that

$$(\|\nabla u\|_p)^p\leqq (f,\ u)\leqq C\|f\|_{V'}\|u\|_V\ ,$$

so that

(2.6)
$$\|\nabla u\|_p\leqq C(\|f\|_{V'})^{1/(p-1)}\ .$$

On the other hand it is clear that

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*) Professor H. Nagase has informed us of this inequality. We are grateful to him for his kindness.

$$\int_\Omega |\nabla\zeta|\,|\nabla u_{-h}|^p dx\,,\ \int_\Omega |\nabla\zeta|\,|D_{-h}u|^p dx \leqq C\int_\Omega |\nabla u|^p dx\,.$$

From the above inequalities we obtain

$$\begin{aligned}&|(\zeta D_{-h}(|\nabla u|^{p-2}\nabla u),\ D_{-h}u\cdot\nabla\zeta)|\\ &\leqq \frac{c_0}{4}\int_\Omega \zeta^2(|\nabla u|^{p-2}+|\nabla u_{-h}|^{p-2})|\nabla D_{-h}u|^2 dx + C(\|f\|_{V'})^{p^*}\,,\end{aligned}$$

where C is a constant depending on ζ and not on h, f. Since $\|\zeta D_{-h}u\|_p \leqq C\|\nabla u\|_p$, it follows that

$$\begin{aligned}&|(\zeta^2 D_{-h}f,\ D_{-h}u)| \leqq C\|\zeta D_{-h}f\|_{p^*}\|\nabla u\|_p\\ &\text{(from (2.6))}\quad \leqq C\|\zeta D_{-h}f\|_{p^*}(\|f\|_{V'})^{1/(p-1)}\,.\end{aligned}$$

And we have from (0.5)

$$(\zeta^2 D_{-h}(|u|^\alpha u),\ D_{-h}u) \geqq 0\,.$$

Combining the above inequalities with (2.2) and (2.4) we conclude that

$$\begin{aligned}&\int_\Omega \zeta^2(|\nabla u|^{p-2}+|\nabla u_{-h}|^{p-2})|\nabla D_{-h}u|^2 dx\\ &\leqq C(\|f\|_{V'})^{1/(p-1)}(\|f\|_{V'}+\|\zeta D_{-h}f\|_{p^*})\,.\end{aligned}\tag{2.7}$$

Here $\|\zeta D_{-h}f\|_{p^*} \leqq C$ from the assumption of $f \in W^{1,p^*}_{\mathrm{loc}}(\Omega)$. On the other hand

$$\begin{aligned}&|D_{-h}(|\nabla u|^{p-2}\nabla u)|^{p^*}\\ &\leqq C(|\nabla u|^{p^*(p-2)}|\nabla D_{-h}u|^{p^*}+|\nabla u_{-h}|^{p^*(p-2)}|\nabla D_{-h}u|^{p^*})\,.\end{aligned}$$

Using Hölder's inequality and noting that $p^*(p-2)=p(2-p^*)$, we have

$$\begin{aligned}&\int_\Omega \zeta^{p^*}|D_{-h}(|\nabla u|^{p-2}\nabla u)|^{p^*}dx\\ &\leqq C\Big[\Big(\int_\Omega |\nabla u|^p dx\Big)^{(2-p^*)/2}\Big(\int_\Omega \zeta^2|\nabla u|^{p-2}|\nabla D_{-h}u|^2 dx\Big)^{p^*/2}\\ &\qquad + \Big(\int_\Omega |\nabla u_{-h}|^p dx\Big)^{(2-p^*)/2}\Big(\int_\Omega \zeta^2|\nabla u_{-h}|^{p-2}|\nabla D_{-h}u|^2 dx\Big)^{p^*/2}\Big]\,.\end{aligned}$$

Combining (2.6), (2.7) with this inequality, we obtain

$$\int_\Omega \zeta^{p^*}|D_{-h}(|\nabla u|^{p-2}\nabla u)|^{p^*}dx \leqq C\,,\tag{2.8}$$

where C depends on f, ζ and not on h.

Let D be any subdomain with $\bar{D}\subset\Omega$. Then from (2.8)

$$\int_D |D_{-h}(|\nabla u|^{p-2}\nabla u)|^{p^*}dx \leqq C\,, \tag{2.9}$$

where C depends on f, D and not on h. Therefore for any k with $1\leqq k\leqq n$ there are a function $v_k\in L^{p^*}(D)$ and a positive sequence $\{h_\nu\}$ with $h_\nu\to 0$ ($\nu\to\infty$) such that

$$D_{-h_\nu}(|\nabla u|^{p-2}\partial_{x_k}u)\rightharpoonup v_k \qquad \text{in } L^{p^*}(D)\,.$$

For any $\varphi\in C_0^\infty(D)$ we see that

$$\begin{aligned}(D_{-h_\nu}(|\nabla u|^{p-2}\partial_{x_k}u),\varphi)&=-(|\nabla u|^{p-2}\partial_{x_k}u,\, D_{h_\nu}\varphi)\\ &\to -(|\nabla u|^{p-2}\partial_{x_k}u,\, \partial_{x_i}\varphi)\,,\end{aligned}$$

which implies that $v_k=\partial_{x_i}(|\nabla u|^{p-2}\partial_{x_k}u)$. Accordingly

$$\|\partial_{x_i}(|\nabla u|^{p-2}\partial_{x_k}u)\|_{p^*,D}\leqq \varliminf_{\nu\to\infty}\|D_{-h_\nu}(|\nabla u|^{p-2}\partial_{x_k}u)\|_{p^*,D}\,.$$

Combining this inequality with (2.9), we have finished the proof. Q.E.D.

REMARK. If we use another inequality in (2.5), the assertion in Proposition 2.1 holds also for $|\nabla u|^{p-1}$, that is, $|\nabla u|^{p-1}\in W^{1,p^*}_{\mathrm{loc}}(\Omega)$. Hence by Sobolev's imbedding theorem we have

$$|\nabla u|^{p-1}\in L^{np^*/(n-p^*)}_{\mathrm{loc}}(\Omega)\,. \tag{2.10}$$

PROPOSITION 2.2. *Under the assumptions in Proposition* 2.1, $\{|\nabla u|^{p-2}D_h\partial u\}$ *has a convergent subsequence in* $L^{p^*}_{\mathrm{loc}}(\Omega)$.

PROOF. Let $q>1$, and let q be so close to 1 if necessary. Let D be a subdomain with $\bar{D}\subset\Omega$. Then by Hölder's inequality and (2.5),

$$\begin{aligned}&\|D_h(|\nabla u|^{p-1}\partial u)\|_{q,D}\\ &\leqq C((\|\nabla u_h\|_{pq/(2-q),D})^{p/2}+(\|\nabla u\|_{pq/(2-q),D})^{p/2})\|\,|\nabla u|^{(p-2)/2}D_h\nabla u\|_{2,D}\,.\end{aligned}$$

Noting that $n(p-1)p^*/(n-p^*)>p$ and $pq/(2-q)\to p+0$ as $q\to 1+0$, we see that the right-hand side is uniformly bounded by (2.10) and the proof of Proposition 2.1. Hence $|\nabla u|^{p-1}\partial u\in W^{1,q}_{\mathrm{loc}}(\Omega)$.

By the remark previous to this proposition, $\{D_h|\nabla u|^{p-1}\}$ is a convergent sequence in $L^{p^*}_{\mathrm{loc}}(\Omega)$. Using (2.10) and Hölder's inequality, we see that $\{D_h|\nabla u|^{p-1}\cdot\partial u_h\}$ is a convergent sequence in $L^q_{\mathrm{loc}}(\Omega)$. Since

$$|\nabla u|^{p-1}D_h\partial u=D_h(|\nabla u|^{p-1}\partial u)-D_h|\nabla u|^{p-1}\cdot\partial u_h\,,$$

$\{|\nabla u|^{p-1}D_h\partial u\}$ is also convergent in $L^q_{\mathrm{loc}}(\Omega)$. Thus there is a function $v\in L^q_{\mathrm{loc}}(\Omega)$ such that

$$|\nabla u|^{p-1}D_{h_\nu}\partial u \to v \qquad \text{a.e. in } \Omega .$$

From now on we denote by the same $\{h_\nu\}$ any subsequence of positive numbers tending to 0. Setting $\tilde{v}(x)=0$ if $|\nabla u(x)|=0$ and $\tilde{v}(x)=v(x)/|\nabla u(x)|^{p/2}$ if $|\nabla u(x)|\neq 0$, we obtain

$$|\nabla u|^{(p-2)/2}D_{h_\nu}\partial u \to \tilde{v} \qquad \text{a.e. in } \Omega .$$

On the other hand from the proof of Proposition 2.1 there is a function $w \in L^2_{\mathrm{loc}}(\Omega)$ such that

$$|\nabla u|^{(p-2)/2}D_{h_\nu}\partial u \rightharpoonup w \qquad \text{in } L^2_{\mathrm{loc}}(\Omega) .$$

By applying Banach-Sacks theorem to the sequence $\{|\nabla u|^{(p-2)/2}D_{h_\nu}\partial u\}$, we see that $w=\tilde{v}$ in Ω. If we repeat the argument in the book of Lions [11, p. 144], it follows that for any $\varepsilon>0$

$$|\nabla u|^{(p-2)/2}D_{h_\nu}\partial u \to w \qquad \text{in } L^{2-\varepsilon}_{\mathrm{loc}}(\Omega) ,$$

which completes the proof. Q.E.D.

§3. Proof of Theorem 1.

Let $\{\delta_j\}$ be a monotone sequence of positive numbers decreasing to zero, and let us set $\omega_j=\{x; \phi(x)<\delta_j\}$. Then $\{\omega_j\}$ is a sequence of neighborhoods of S tending to S. Setting $\Omega_j=\Omega-\omega_j$, we define

$$C^1_{(0)}(\bar{\Omega}_j)=\{u \in C^1(\bar{\Omega}_j)\, ;\, u=0 \text{ near } \partial\omega_j \cup \overline{\partial_1\Omega}\} .$$

Let $V(\Omega_j)$ be the completion of $C^1_{(0)}(\bar{\Omega}_j)$ with respect to the norm $\|\ \|_{1,p,\Omega_j}$. If we extend u to be zero in ω_j for $u \in V(\Omega_j)$, then $u \in V(\Omega)$ and $u \in V(\Omega_k)$ for $k \geqq j$.

PROOF OF THEOREM 1. First we assume that $f \in W^{1,p^*}_{\mathrm{loc}}(\Omega)$. Replacing Ω with Ω_j, by Proposition 0.1 we can find $u_j \in V(\Omega_j)$ such that $A(u_j)=f$ in Ω_j, that is,

$$(3.1) \qquad (|\nabla u_j|^{p-2}\nabla u_j, \nabla v)+(|u_j|^\alpha u_j, v)=(f, v) , \qquad v \in V(\Omega_j) .$$

Hereafter let β be a positive number which is taken to be close to zero if necessary. Setting $v=\phi^{-\beta}u_j$ in (3.1) particularly, we have

$$(3.2) \qquad (\phi^{-\beta}|\nabla u_j|^{p-2}\nabla u_j, \nabla u_j)+(u_j|\nabla u_j|^{p-2}\nabla\phi^{-\beta}, \nabla u_j) \\ +(|u_j|^\alpha u_j, \phi^{-\beta}u_j)=(f, \phi^{-\beta}u_j) .$$

Since $|\nabla\phi|$ is bounded, it follows that

$$
\begin{aligned}
&|(u_j|\nabla u_j|^{p-2}\nabla\phi^{-\beta},\ \nabla u_j)| \\
&\leqq C\beta\Big(\int_\Omega \phi^{-\beta}|\nabla u_j|^p dx\Big)^{(p-1)/p}\Big(\int_\Omega \phi^{-\beta-p}|u_j|^p dx\Big)^{1/p} \\
&\leqq C\beta\Big(\int_\Omega \phi^{-\beta}|\nabla u_j|^p dx+\int_\Omega \phi^{-\beta-p}|u_j|^p dx\Big) \\
&\text{(by Lemma 1.4)} \\
&\leqq C\beta\Big(\int_\Omega \phi^{-\beta}|\nabla u_j|^p dx+\int_\Omega \phi^{-\beta}|u_j|^p dx\Big)\,.
\end{aligned}
$$

Using Poincaré's inequality and Lemma 1.4 again, we see that

$$|(u_j|\nabla u_j|^{p-2}\nabla\phi^{-\beta},\ \nabla u_j)|\leqq C\beta\int_\Omega \phi^{-\beta}|\nabla u_j|^p dx\,.$$

Combining this with (3.2), we obtain

$$\int_\Omega \phi^{-\beta}|\nabla u_j|^p dx\leqq C|(f,\ \phi^{-\beta}u_j)|\,. \tag{3.3}$$

Clearly

$$|(f,\ \phi^{-\beta}u_j)|\leqq \|\phi^{1-\beta+\beta/p}f\|_{p*}\|\phi^{-1-\beta/p}u_j\|_p\,.$$

From Poincaré's inequality and Lemma 1.4

$$
\begin{aligned}
\|\phi^{-1-\beta/p}u_j\|_p &\leqq C(\|\phi^{-\beta/p}\nabla u_j\|_p+\|\phi^{-\beta/p}u_j\|_p) \\
&\leqq C\|\phi^{-\beta/p}\nabla u_j\|_p\,.
\end{aligned}
$$

These two inequalities and (3.3) yield

$$\|\phi^{-1-\beta/p}u_j\|_p+\|\phi^{-\beta/p}\nabla u_j\|_p\leqq C(\|\phi^{1-\beta+\beta/p}f\|_{p*})^{1/(p-1)}\,. \tag{3.4}$$

Hereafter let us denote by the same notation $\{u_k\}$ any subsequence of $\{u_j\}$. By (3.4) and Sobolev's compact imbedding theorem there is a function $u\in L^p(\Omega)$ such that

$$u_k\to u \qquad \text{in } L^p(\Omega)\,. \tag{3.5}$$

Combining (3.4) with (3.5), we have

$$\phi^{-1-\beta/p}u_k \rightharpoonup \phi^{-1-\beta/p}u \qquad \text{in } L^p(\Omega)$$

and for any i with $1\leqq i\leqq n$

$$\phi^{-\beta/p}\partial_{x_i}u_k \rightharpoonup \phi^{-\beta/p}\partial_{x_i}u \qquad \text{in } L^p(\Omega)\,.$$

From this we see that $u\in V(\Omega)$. Further from (3.4) again we have

(3.6) $$\|\phi^{-1-\beta/p}u\|_p+\|\phi^{-\beta/p}\nabla u\|_p\leqq C(\|\phi^{1-\beta+\beta/p}f\|_{p^*})^{1/(p-1)} .$$

Let D be any subdomain with $\bar{D}\subset\Omega$. Since we have assumed that $f\in W^{1,p^*}_{\text{loc}}(\Omega)$, we can apply Proposition 2.1. Thus there is a positive integer j_0 $(=j_0(D))$ such that

$$\|\,|\nabla u_j|^{p-2}\nabla u_j\|_{1,p^*,D}\leqq C$$

for $j\geqq j_0$, where C depends on f, D and not on j. Hence we can apply Sobolev's compact imbedding theorem to the sequence $\{|\nabla u_j|^{p-2}\nabla u_j\}$ as follows: for each i, $1\leqq i\leqq n$, there is a function $g_i\in L^{p^*}_{\text{loc}}(\Omega)$ such that for any D with $\bar{D}\subset\Omega$

(3.7) $$\begin{cases}|\nabla u_k|^{p-2}\partial_{x_i}u_k\to g_i & \text{in } L^{p^*}(D)\\ |\nabla u_k|^{p-2}\partial_{x_i}u_k\to g_i & \text{a.e. in } \Omega .\end{cases}$$

This implies that $g_i=(\sum_{l=1}^n h_l^2)^{(p-2)/2}h_i$ for some functions $\{h_i\}$ and

$$\partial_{x_i}u_k\to h_i \qquad \text{a.e. in } \Omega .$$

Clearly $h_i\in L^p(D)$, so that we have by Lemma 1.1

$$\partial_{x_i}u_k\rightharpoonup h_i \qquad \text{in } L^p(D) .$$

From this and (3.5) we see that $h_i=\partial_{x_i}u$. Therefore it follows from (3.7) that

(3.8) $$|\nabla u_k|^{p-2}\partial_{x_i}u_k\to|\nabla u|^{p-2}\partial_{x_i}u \qquad \text{a.e. in } \Omega$$

and

(3.9) $$(|\nabla u_k|^{p-2}\partial_{x_i}u_k,\varphi)\to(|\nabla u|^{p-2}\partial_{x_i}u,\varphi) , \qquad \varphi\in C_0^\infty(\Omega) .$$

Since $\{|\nabla u_j|^{p-2}\partial_{x_i}u_j\}$ are uniformly bounded in $L^{p^*}(\Omega)$, it holds from (3.9) that

$$|\nabla u_k|^{p-2}\partial_{x_i}u_k\rightharpoonup|\nabla u|^{p-2}\partial_{x_i}u \qquad \text{in } L^{p^*}(\Omega) .$$

Hence we obtain

(3.10) $$(|\nabla u_k|^{p-2}\nabla u_k,\nabla v)\to(|\nabla u|^{p-2}\nabla u,\nabla v) , \qquad v\in C^1_{(0)}(\Omega) .$$

The following inequality is valid from Sobolev's imbedding theorem and the assumption (0.2):

$$\|\,|u_j|^\alpha u_j\|_{p^*}\leqq C(\|u_j\|_V)^{1+\alpha} .$$

From this and (3.4) we see that $\{\|\,|u_j|^\alpha u_j\|_{p^*}\}$ are uniformly bounded. More easily, $|u|^\alpha u \in L^{p*}(\Omega)$. Hence by (3.5) and Lemma 1.1 we obtain

$$(|u_k|^\alpha u_k, v) \to (|u|^\alpha u, v) , \qquad v \in C^1_{(0)}(\bar{\Omega}) . \tag{3.11}$$

Let us fix any $v \in C^1_{(0)}(\bar{\Omega})$. Then from (3.1) there is a positive integer j_1 such that for $j \geqq j_1$

$$(|\nabla u_j|^{p-2}\nabla u_j, \nabla v) + (|u_j|^\alpha u_j, v) = (f, v) .$$

Combining (3.10), (3.11) with this equality we conclude that

$$(|\nabla u|^{p-2}\nabla u, \nabla v) + (|u|^\alpha u, v) = (f, v) , \qquad v \in C^1_{(0)}(\bar{\Omega}) ,$$

which holds also for any $v \in V(\Omega)$ as easily seen. That is, u is the solution itself in Theorem 1. In view of (3.6), Theorem 1 is correct under the assumption of $f \in W^{1,p*}_{\mathrm{loc}}(\Omega)$.

Next let us remove the assumption of $f \in W^{1,p*}_{\mathrm{loc}}(\Omega)$. We take a sequence $\{g_j\} \subset C^1_0(\Omega)$ such that $g_j \to \phi^{1-\beta+\beta/p} f$ in $L^{p*}(\Omega)$. If we set $f_j = \phi^{\beta-1-\beta/p} g_j$, then $\phi^{1-\beta+\beta/p} f_j \to \phi^{1-\beta+\beta/p} f$ in $L^{p*}(\Omega)$. It follows by Lemma 1.4 that for $v \in C^1_{(0)}(\bar{\Omega})$

$$\begin{aligned} \|\phi^{\beta-1-\beta/p} v\|_p &\leqq C(\|\phi^{\beta-\beta/p}\nabla v\|_p + \|\phi^{\beta-\beta/p} v\|_p) \\ &\leqq C\|v\|_V , \end{aligned}$$

so that

$$|(f_j - f, v)| \leqq C\|\phi^{1-\beta+\beta/p}(f_j - f)\|_{p*}\|v\|_V .$$

This implies that

$$\|f_j - f\|_{V'} \leqq C\|\phi^{1-\beta+\beta/p}(f_j - f)\|_{p*} .$$

Hence $f_j \to f$ in $V'(\Omega)$. Let $u_j \in V(\Omega)$ be the solution of $A(u_j) = f_j$ in Ω. Then Theorem 1 is valid for each u_j from the first half, so that

$$\|\phi^{-1-\beta/p} u_j\|_p + \|\phi^{-\beta/p}\nabla u_j\|_p \leqq C(\|\phi^{1-\beta+\beta/p} f_j\|_{p*})^{1/(p-1)} . \tag{3.12}$$

It is obvious that for any $v \in V(\Omega)$

$$(|\nabla u_j|^{p-2}\nabla u_j - |\nabla u|^{p-2}\nabla u, \nabla v) + (|u_j|^\alpha u_j - |u|^\alpha u, v) = (f_j - f, v) .$$

Setting $v = u_j - u$ in this equality, we have from (0.4) and (0.5)

$$c_0(\|\nabla(u_j - u)\|_p)^p \leqq \|f_j - f\|_{V'}\|u_j - u\|_V ,$$

so that

$$\|u_j-u\|_V \leqq C(\|f_j-f\|_{V'})^{1/(p-1)} .$$

This yields that $u_j \to u$ in $V(\Omega)$. Therefore it follows from (3.12) that

$$\phi^{-\beta/p}\partial_{x_i}u_k \to \phi^{-\beta/p}\partial_{x_i}u \quad \text{in } L^p(\Omega) , \qquad i=1, \cdots, n .$$

Accordingly,

$$\|\phi^{-\beta/p}\nabla u\|_p \leqq C(\|\phi^{1-\beta+\beta/p}f\|_{p*})^{1/(p-1)} .$$

We have similarly

$$\|\phi^{-1-\beta/p}u\|_p \leqq C(\|\phi^{1-\beta+\beta/p}f\|_{p*})^{1/(p-1)} .$$

Thus we have finished the proof of Theorem 1. Q.E.D.

§ 4. Parallel translations with a weight.

The content of this section is due to [4]. The descriptions of Lemmas 4.2 and 4.3 are slightly different from those in [4].

For some time we consider our lemma in the upper half space $\{x_n \geqq 0\}$. We denote $\{x_n>0\}$ by $\boldsymbol{R}^n_+$. Let γ be any fixed real number with $0<\gamma<1$, which may be chosen so close to 1. Let us define $\tilde{\rho}(x)=(x^2_{n-1}+x^{2\gamma}_n)^{1/2}$. We write often $\tilde{\rho}(x)$ simply with $\tilde{\rho}$. It is easily seen that

$$(4.1) \qquad |\partial_{x_{n-1}}\tilde{\rho}| \leqq C , \qquad |\partial_{x_n}\tilde{\rho}| \leqq C\tilde{\rho}^{1-1/\gamma} .$$

We define the following mapping from $\boldsymbol{R}^n_+$ into itself:

$$\Phi_h : \begin{cases} y_j=x_j & \text{if } j\neq n-1 \\ y_{n-1}=x_{n-1}+h\tilde{\rho} , \end{cases}$$

where h is a sufficiently small positive number. Hereafter we suppose that the y-variable is always connected with the x-variable by the equality $y=\Phi_h(x)$. Thus $\tilde{\rho}(y)=\tilde{\rho}(\Phi_h(x))$. There is a positive constant c such that

$$(4.2) \qquad c\tilde{\rho}(x) \leqq \tilde{\rho}(y) \leqq c^{-1}\tilde{\rho}(x) , \qquad x \in \overline{\boldsymbol{R}^n_+} .$$

In fact the inequality on the right is trivial. That on the left is easily seen from

$$\tilde{\rho}^2 \leqq 2(y^2_{n-1}+y^{2\gamma}_n+h^2\tilde{\rho}^2) .$$

By an easy computation we have

$$\begin{pmatrix} \partial_{x_1}y_1 & \cdots & \partial_{x_1}y_n \\ \cdots & \cdots & \cdots \\ \partial_{x_n}y_1 & \cdots & \partial_{x_n}y_n \end{pmatrix} = \begin{pmatrix} 1 & & & & \\ & \ddots & & 0 & \\ & & 1 & & \\ & 0 & & 1+h\partial_{x_{n-1}}\tilde{\rho} & 0 \\ & & & h\partial_{x_n}\tilde{\rho} & 1 \end{pmatrix},$$

(4.3)

$$\begin{pmatrix} \partial_{y_1}x_1 & \cdots & \partial_{y_1}x_n \\ \cdots & \cdots & \cdots \\ \partial_{y_n}x_1 & \cdots & \partial_{y_n}x_n \end{pmatrix} = \begin{pmatrix} 1 & & & \\ & \ddots & & 0 \\ & & 1 & \\ & 0 & & * \end{pmatrix}$$

where

$$*=\frac{1}{1+h\partial_{x_{n-1}}\tilde{\rho}}\begin{pmatrix} 1 & 0 \\ -h\partial_{x_n}\tilde{\rho} & 1+h\partial_{x_{n-1}}\tilde{\rho} \end{pmatrix}.$$

From (4.1) and (4.3) we get

$$|\partial_{y_{n-1}}\tilde{\rho}| \leqq C\,, \qquad |\partial_{y_n}\tilde{\rho}| \leqq C\tilde{\rho}^{1-1/\tau}\,. \tag{4.4}$$

Let J_h be the Jacobian of Φ_h. We can write $J_h = 1+h\partial_{x_{n-1}}\tilde{\rho}$, so that $c \leqq J_h \leqq c^{-1}$ for some positive constant c. It is easy to see that Φ_h is a one-to-one mapping from $\overline{R^n_+}$ onto itself.

Let us write $x''=(x_1, \cdots, x_{n-2})$ $(y''=(y_1, \cdots, y_{n-2}))$ for $x=(x_1, \cdots, x_n)$ $(y=(y_1, \cdots, y_n))$, respectively. We define

$$(S_h u)(x) = u(x'', x_{n-1}+h\tilde{\rho}, x_n)\,,$$
$$(T_h u)(y) = u(y'', y_{n-1}-h\tilde{\rho}, y_n)\,.$$

Namely, $(S_h u)(x)=u(y)$ and $(T_h u)(y)=u(x)$. Further we define

$$(P_h u)(x) = h^{-1}((S_h u)(x)-u(x))\,,$$
$$(Q_h u)(y) = h^{-1}((T_h u)(y)-u(y))\,.$$

We have always $(Q_h u)(y) = -(P_h u)(x)$. From now on, $(S_h u)(x)$, $(T_h u)(y)$, $(P_h u)(x)$ and $(Q_h u)(y)$ are often written simply by $S_h u$, $T_h u$, $P_h u$ and $Q_h u$, respectively. Clearly

$$P_h(uv) = S_h u\cdot P_h v + vP_h u\,,$$
$$Q_h(uv) = T_h u\cdot Q_h v + vQ_h u\,.$$

If we define

$$F_h(\nabla_x u) = h^{-1}(\nabla_x(S_h u) - S_h\nabla_x u)$$

and

$$G_h(\nabla_y u)=h^{-1}(\nabla_y(T_h u)-T_h\nabla_y u)\ ,$$

then it is easily seen that

$$\begin{cases}|F_h(\nabla_x u)|\leqq C\tilde{\rho}^{1-1/\gamma}|S_h\partial_{x_{n-1}}u| \\ |G_h(\nabla_y u)|\leqq C\tilde{\rho}^{1-1/\gamma}|T_h\partial_{y_{n-1}}u| \end{cases}\tag{4.5}$$

on any bounded subset of $\overline{R^n_+}$, where C is independent of h and u.

For the time being, let us denote by $(\ , \ )$ and $\| \ \|_q$ the inner product in $L^2(R^n_+)$ and the norm in $L^q(R^n_+)$, respectively. Let $(\ , \ )_x$ ($(\ , \ )_y$) be the inner product in $L^2(R^n_+)$ with respect to the x-variable (y-variable), respectively.

LEMMA 4.1. *For arbitrary functions u, v it holds that*

$$(u, P_h v)_x=(Q_h u, v)_y+(K_h T_h u, v)_y\ ,$$

where $K_h(y)=-\partial_{x_{n-1}}\tilde{\rho}/(1+h\partial_{x_{n-1}}\tilde{\rho})$.

PROOF. We see that

$$(u, P_h v)_x=h^{-1}\left(\int_{R^n_+}u(x)v(y)dx-\int_{R^n_+}u(x)v(x)dx\right)\ ,$$

$$\int_{R^n_+}u(x)v(x)dx=\int_{R^n_+}u(y)v(y)dy$$

and

$$\int_{R^n_+}u(x)v(y)dx=\int_{R^n_+}T_h u\cdot vJ_h^{-1}dy\ .$$

Obviously $J_h^{-1}=1+hK_h$, so that we obtain the required equality. Q.E.D.

LEMMA 4.2. *Let $1\leqq q<\infty$, and let $u\in W^{1,q}(R^n_+)$. Then it holds that*

$$\|\tilde{\rho}^{-1}P_h u\|_q\leqq C\|\partial_{x_{n-1}}u\|_q$$

and

$$\|\tilde{\rho}^{-1}Q_h u\|_q\leqq C\|\partial_{y_{n-1}}u\|_q\ ,$$

where C is independent of h and u.

PROOF. Without loss of generality we may assume that $u\in C^1(\overline{R^n_+})$ and $u=0$ for large $|x|$. Since

$$(P_h u)(x)=\tilde{\rho}\int_0^1(\partial_{x_{n-1}}u)(x'', x_{n-1}+th\tilde{\rho}, x_n)dt ,$$

it follows that

$$\begin{aligned}(\|\tilde{\rho}^{-1}P_h u\|_q)^q &\leqq \int_0^1\int_\Omega |(\partial_{x_{n-1}}u)(x'', x_{n-1}+th\tilde{\rho}, x_n)|^q dxdt \\ &= \int_0^1\int_\Omega |(\partial_{y_{n-1}}u)(y)|^q J_{th}^{-1} dydt \\ &\leqq C(\|\partial_{x_{n-1}}u\|_q)^q ,\end{aligned}$$

which proves the first inequality. The second inequality is reduced to the first one by using (4.2). Q.E.D.

LEMMA 4.3. *Let* $1\leqq q<\infty$. *Let* $u\in W^{1,q}(R_+^n)$. *Then*

$$\|\tilde{\rho}^{-1}P_h u-\partial_{x_{n-1}}u\|_q\to 0$$

and

$$\|\tilde{\rho}^{-1}Q_h u+\partial_{y_{n-1}}u\|_q\to 0 \qquad \text{as } h\to 0 .$$

PROOF. By Lemma 4.2 it is enough to assume that $u\in C^1(\overline{R_+^n})$ and $u=0$ for large $|x|$. Similarly to the proof of Lemma 4.2 we have

$$\begin{aligned}&(\|\tilde{\rho}^{-1}P_h u-\partial_{x_{n-1}}u\|_q)^q \\ &\leqq \int_0^1\int_\Omega |(\partial_{x_{n-1}}u)(x'', x_{n-1}+th\tilde{\rho}, x_n)-(\partial_{x_{n-1}}u)(x)|^q dxdt .\end{aligned}$$

Clearly the right-hand side tends to zero as $h\to 0$, so that the first inequality has been proved. The second inequality is reduced to the first one, because $\|(\nabla u)(y)-(\nabla u)(x)\|_q\to 0$ as $h\to 0$. Q.E.D.

§ 5. A localization of $\bar{\Omega}$.

Let Ω be the domain in Theorem 2. In this section we prepare a localization of $\partial\Omega$ near S. We denote the origin simply by O and we write $x'=(x_1, \cdots, x_{n-1})$. The origin in the x'-space is denoted also by the same O.

LEMMA 5.1. *Let* P_0 *be any fixed point on* $\partial\Omega$. *Then there is a neighborhood* U *of* P_0 *and a function* $u\in C^\omega(U)$ *such that*

$$u=0 \quad \text{on } \partial\Omega\cap U , \qquad u>0 \quad \text{in } \Omega\cap U$$

and

$$|\nabla u|=1 \quad \text{in } U .$$

PROOF. We may asssume that $P_0=O$. We consider our lemma in a neighborhood of O. By an orthogonal coordinate transformation it is sufficient to assume that Ω and $\partial\Omega$ are expressed by $x_n>\psi(x')$ and $x_n=\psi(x')$, respectively, where $\psi(O)=0$, $\partial_{x_i}\psi(O)=0$ for i, $1\leqq i\leqq n-1$, and $\psi(x')$ is analytic near O.

We take the following coordinate transformation

$$y_i=x_i \quad (i\neq n)\ , \qquad y_n=x_n-\psi(x')\ .$$

Then

$$\begin{aligned}&\partial_{x_i}u=\partial_{y_i}u-\partial_{x_i}\psi\cdot\partial_{y_n}u \qquad (i\neq n)\ ,\\ &\partial_{x_n}u=\partial_{y_n}u\ ,\end{aligned}$$

so that the equation $|\nabla_x u|^2=1$ is equivalent to

$$\Big(1+\sum_{i=1}^{n-1}(\partial_{x_i}\psi)^2\Big)(\partial_{y_n}u)^2+\sum_{i=1}^{n-1}(\partial_{y_i}u)^2-2\Big(\sum_{i=1}^{n-1}\partial_{x_i}\psi\cdot\partial_{y_i}u\Big)\partial_{y_n}u=1\ .$$

If $u>0$ for $y_n>0$, this equation becomes

$$\begin{aligned}\partial_{y_n}u=\Big(1+\sum_{i=1}^{n-1}(\partial_{x_i}\psi)^2\Big)^{-1}\Big[&\sum_{i=1}^{n-1}\partial_{x_i}\psi\cdot\partial_{y_i}u\\ &+\sqrt{\Big(\sum_{i=1}^{n-1}\partial_{x_i}\psi\cdot\partial_{y_i}u\Big)^2+\Big(1+\sum_{i=1}^{n-1}(\partial_{x_i}\psi)^2\Big)\Big(1-\sum_{i=1}^{n-1}(\partial_{y_i}u)^2\Big)}\Big]\ .\end{aligned}\tag{5.1}$$

The initial condition for u is written by

$$u(y',0)=0\ .\tag{5.2}$$

We shall solve the Cauchy problem (5.1) with the initial condition (5.2). We refer to the book of I. G. Petrovskiĭ [15]. Setting $P_i=\partial_{y_i}u$ for $i\neq n$, we write the right-hand side of (5.1) with $F(y',p_1,\cdots,p_{n-1})$. By differentiating the both sides of (5.1), we have

$$\partial_{y_n}^2u=\sum_{i=1}^{n-1}\partial_{p_i}F\cdot\partial_{y_n}\partial_{y_i}u\ .\tag{5.1'}$$

If we set $u_0=u$ and $u_i=\partial_{y_i}u$ for i, $1\leqq i\leqq n$, the Cauchy problems (5.1'), (5.2) become

$$\begin{cases}\partial_{y_n}u_0=u_n\\ \partial_{y_n}u_1=\partial_{y_1}u_n\\ \cdots\cdots\cdots\\ \partial_{y_n}u_{n-1}=\partial_{y_{n-1}}u_n\\ \partial_{y_n}u_n=\sum_{i=1}^{n-1}(\partial_{p_i}F)(y',u_1,\cdots,u_{n-1})\partial_{y_i}u_n\ ,\end{cases}\tag{5.1''}$$

$$(5.2') \qquad \begin{cases} u_0(y', 0)=u_1(y', 0)=\cdots=u_{n-1}(y', 0)=0 \\ u_n(y', 0)=F(y', 0, \cdots, 0) . \end{cases}$$

If (5.1″) and (5.2′) are solvable, we easily see that $u=u_0$ is a solution of (5.1) and (5.2). By Cauchy-Kowalevski's theorem there is a solution u of (5.1″) and (5.2′) such that u is analytic near O (cf., e.g., [15]), so that we complete the proof. Q.E.D.

LEMMA 5.2. *Let U be a neighborhood of O in R^n. Let $(v_1, \cdots, v_n)\in[C^\omega(U)]^n$, and let $(v_1, \cdots, v_n)\neq O$ in U. Then there are another neighborhood U' of O and vector functions $(a_{i1},\cdots, a_{in})\in[C^\omega(U')]^n$, $i=1,\cdots,n-1$, such that $(a_{i1}, \cdots a_{in})\neq O$, $\sum_{k=1}^n a_{ik}v_k=0$ in U' and $\sum_{k=1}^n a_{ik}a_{jk}=0$ in U' if $i\neq j$.*

PROOF. In the following we often omit the phrase "in a neighborhood of O". Let us prove our lemma by induction on n.

Let $n=2$. Since $(v_1, v_2)\neq O$, we may assume that $v_1\neq 0$. It is enough to take $a_2=1$ and $a_1=-a_2v_2/v_1$. Next let us assume that our lemma is correct. Then we shall prove it, when n is replaced with $n+1$.

Without loss of generality we may suppose that $(v_1, \cdots, v_n)\neq O$. From our assumption there are vector functions $(a_{i1}, \cdots, a_{in})\,(\neq O)\in[C^\omega]^n$, $i=1, \cdots, n-1$, which are orthogonal to each other and so to $(v_1, \cdots, v_n)$. Thus n vector functions $(v_1, \cdots, v_n)$ and $(a_{i1}, \cdots, a_{in})$, $i=1, \cdots, n-1$, are linearly independent, which implies that

$$\begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \cdots & \cdots & \cdots \\ a_{n-1,1} & \cdots & a_{n-1,n} \\ v_1 & \cdots & v_n \end{vmatrix} \neq 0 .$$

We define $a_{1,n+1}=\cdots=a_{n-1,n+1}=0$ and $a_{n,n+1}=1$. And we determine $(a_{n1}, \cdots, a_{nn})$ by the system

$$\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \cdots & \cdots & \cdots \\ a_{n-1,1} & \cdots & a_{n-1,n} \\ v_1 & \cdots & v_n \end{pmatrix} \begin{pmatrix} a_{n1} \\ \vdots \\ \vdots \\ a_{nn} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ -v_{n+1} \end{pmatrix} .$$

Then $(a_{n1}, \cdots, a_{n,n+1})$ is orthogonal to $(v_1, \cdots, v_{n+1})$ and to $(a_{i1}, \cdots, a_{i,n+1})$, $i=1, \cdots, n-1$. Naturally $(a_{n1}, \cdots, a_{n,n+1})\in[C^\omega]^n$ and it does not vanish. The lemma is proved. Q.E.D.

LEMMA 5.3. *Let $(v_1, \cdots, v_n)\in[C^\omega(U)]^n$, and let $(v_1, \cdots, v_n)\neq O$ in U,* *where U is a neighborhood of O in $\boldsymbol{R}^n$. Then there are another neighborhood U' of O and a vector function $(u_1, \cdots, u_{n-1}) \in [C^\omega(U')]^{n-1}$ such that $\sum_{k=1}^n \partial_{x_k} u_i \cdot v_k = 0$, $i=1, \cdots, n-1$, and $\nabla u_i \cdot \nabla u_j = \delta_{ij}$ in U'.*

PROOF. If $v_1 = \cdots = v_{n-1} = 0$ particularly, it is enough to take $u_i = x_i$.

We consider the general case. By Lemma 5.2 there are $n-1$ unit vector functions $(a_{i1}, \cdots, a_{in}) \in [C^\omega]^n$ such that they are orthogonal to each other and so to $(v_1, \cdots, v_n)$. If we set $(a_{n1}, \cdots, a_{nn}) = (\sum_{k=1}^n v_k^2)^{-1/2}(v_1, \cdots, v_n)$, then it is in $[C^\omega]^n$ and the matrix $(a_{ij})_{i,j=1}^n$ is orthogonal.

We write $\boldsymbol{e}_i = (0, \cdots, 0, \overset{i}{1}, 0, \cdots, 0)$ and define

$$\boldsymbol{e}_i' = \sum_{j=1}^n a_{ij} \boldsymbol{e}_j , \qquad i=1, \cdots, n .$$

Then they are unit vector functions which are orthogonal to each other, and we have

$$\boldsymbol{e}_i = \sum_{j=1}^n a_{ji} \boldsymbol{e}_j' .$$

Denoting by ∂_i' the differentiation to the direction $\boldsymbol{e}_i'$, we have

$$\begin{aligned}(\partial_i' f)(x) &= \lim_{h \to 0} h^{-1}[f(x+h\boldsymbol{e}_i') - f(x)] \\ &= \sum_j a_{ij}(\partial_{x_j} f)(x) ,\end{aligned}$$

so that

$$\sum_i \partial_i' f \cdot \boldsymbol{e}_i' = \sum_i \partial_{x_i} f \cdot \boldsymbol{e}_i .$$

This implies that $\nabla f \cdot \nabla g$ is invariant by the above coordinate transformation. More precisely, writing $\nabla' f = (\partial_1' f, \cdots, \partial_n' f)$, we have

$$\nabla f \cdot \nabla g = \nabla' f \cdot \nabla' g . \tag{5.3}$$

For any $x \in \boldsymbol{R}^n$ there are functions $\alpha_i(x)$, $i=1, \cdots, n$, such that

$$x = \sum_i x_i \boldsymbol{e}_i = \sum_i \alpha_i(x) \boldsymbol{e}_i' .$$

Clearly $\alpha_i(x) \in C^\omega$. If we set $u_j(x) = \alpha_j(x)$, then

$$u_j(x + h\boldsymbol{e}_i') = \begin{cases} \alpha_i(x) + h & \text{if } j=i \\ \alpha_j(x) & \text{if } j \neq i , \end{cases}$$

so that $\partial_i' u_j = \delta_{ij}$. Therefore, for $1 \leqq i \leqq n-1$,

$$\sum_k \partial_{x_k} u_i \cdot v_k = \sum_j (\sum_k a_{jk} v_k) \partial'_j u_i = \sum_k a_{ik} v_k = 0 .$$

Further from (5.3)

$$\nabla u_i \cdot \nabla u_j = \nabla' u_i \cdot \nabla' u_j = \delta_{ij} .$$

From the above, $(u_1, \cdots, u_{n-1})$ is the required vector function. Q.E.D.

§6. Proof of Theorem 2.

Let us fix any point on S which may be assumed to be the origin. We consider our problem in the neighborhood of O. Thus we often omit the phrase "in a neighborhood of O". By Lemma 5.1 there is a function $\phi \in C^\omega$ with $|\nabla\phi|=1$ such that Ω and $\partial\Omega$ are expressed with $\phi>0$ and $\phi=0$ in a neighborhood of O, respectively. By Lemma 5.3 there is a vector function $(u_1, \cdots, u_{n-1}) \in [C^\omega]^{n-1}$ such that $u_i(O)=0$, $\nabla u_i \cdot \nabla\phi=0$ and $\nabla u_i \cdot \nabla u_j = \delta_{ij}$.

For the above functions $\phi, u_1, \cdots, u_{n-1}$, we consider the following C^ω mapping

$$\Psi_1 : \begin{cases} \xi_1 = u_1 \\ \cdots\cdots \\ \xi_{n-1} = u_{n-1} \\ \xi_n = \phi . \end{cases}$$

It is clear that Ψ_1 is a one-to-one C^ω mapping from a neighborhood of O onto another one. The inverse Ψ_1^{-1} is also of class C^ω and the Jacobian of Ψ_1 is ± 1. Let us write $\xi=(\xi_1, \cdots, \xi_n)$ and $\xi'=(\xi_1, \cdots, \xi_{n-1})$. The image of S lies on $\xi_n=0$, so that S is an $(n-2)$-dimensional C^ω manifold in the ξ'-space. In a neighborhood of O the ξ'-space is divided into two parts and $\partial_1\Omega$ is mapped into either of them.

Let us use again Lemmas 5.1 and 5.3, by replacing n with $n-1$. Then there is a function $\psi(\xi') \in C^\omega$ with $|\nabla_{\xi'}\psi|=1$ such that S and the image $\Psi_1(\partial_1\Omega)$ are expressed with $\psi(\xi')=0$ and $\psi(\xi')>0$, respectively. Further there are functions $v_1(\xi'), \cdots, v_{n-2}(\xi') \in C^\omega$ such that $v_i(O)=0$, $\nabla v_i \cdot \nabla\psi=0$ and $\nabla v_i \cdot \nabla v_j = \delta_{ij}$. For these functions we consider the following C^ω mapping

$$\Psi_2 : \quad \begin{aligned} &\eta_1 = v_1(\xi') \\ &\cdots\cdots \\ &\eta_{n-2} = v_{n-2}(\xi') \\ &\eta_{n-1} = \psi(\xi') \\ &\eta_n = \xi_n . \end{aligned}$$

The composite mapping of Ψ_1 and Ψ_2 is written as follows:

$$\Psi_2 \circ \Psi_1 : \begin{cases} \eta_1 = w_1(x) \\ \cdots\cdots \\ \eta_n = w_n(x) \, . \end{cases}$$

Then $\Psi_2 \circ \Psi_1$ is a C^ω one-to-one mapping from a neighborhood of O onto another one. The inverse $(\Psi_2 \circ \Psi_1)^{-1}$ is also of class C^ω and $w_i(O)=0$, $\nabla w_i \cdot \nabla w_j = \delta_{ij}$. Further Ω, $\partial\Omega$, $\partial_1\Omega$ and S are mapped into $\{\eta_n>0\}$, $\{\eta_n=0\}$, $\{\eta_n=0,\ \eta_{n-1}>0\}$ and $\{\eta_n=\eta_{n-1}=0\}$, respectively. The Jacobian of $\Psi_2 \circ \Psi_1$ is ± 1. For any two functions f and g we have

$$\nabla_\eta f \cdot \nabla_\eta g = \nabla_x f \cdot \nabla_x g \, . \tag{6.1}$$

Let $\rho(x)$ and $\phi(x)$ be the functions in Lemma 1.3 and Theorem 1, respectively. Then it is easily seen that

$$c\rho(\eta) \leqq \phi(x) \leqq c^{-1}\rho(\eta) \, , \qquad c>0 \, ,$$

in a neighborhood of O.

Hereafter let us regard the η-space newly as the x-space. And let us write $(\eta_1, \cdots, \eta_n)$ with $(x_1, \cdots, x_n)$. We define $\Sigma_A = \{x \in \boldsymbol{R}^n;\ |x|<A,\ x_n>0\}$. Let R be any fixed sufficiently small positive number, and let R' be any fixed number with $0<R'<R$. We write $\Sigma=\Sigma_R$ and $\Sigma'=\Sigma_{R'}$.

By (1.1) the function space $C^1_{(0)}(\bar{\Sigma})$ is defined. We denote by $V(\Sigma)$ the completion of $C^1_{(0)}(\bar{\Sigma})$ with the norm in $W^{1,p}(\Sigma)$. Similarly $V(\Sigma')$ is defined. The inner product in $L^2(\Sigma)$ is denoted by $(\ ,\ )_\Sigma$. We remember that the Jacobian of $\Psi_2 \circ \Psi_1$ is ± 1 and (6.1) is valid, so that the solution of (0.3) satisfies

$$(|\nabla u|^{p-2}\nabla u, \nabla v)_\Sigma + (|u|^\alpha u, v)_\Sigma = (f, v)_\Sigma \, , \qquad v \in V(\Sigma') \, , \tag{6.2}$$

where $f \in W^{1,p^*}(\Sigma)$ from the assumption of Theorem 2.

PROOF OF THEOREM 2. Let us take a function $\psi(x) \in C_0^\infty(\{|x|<R\})$ such that $\psi=1$ in $\{|x|<(R+R')/2\}$. In (6.2) we can replace u with ψu, so that we can assume that $u \in V(\Sigma)$ and $u=0$ near $|x|=R$ without loss of generality. Let S_h, T_h, P_h and Q_h be the operators in Section 4. Let γ be a real number with $0<\gamma<1$, which may be chosen so close to 1. Later we see that γ depends on the number β_0 in Theorem 1. Let the y-variable be connected with the x-variable by the mapping Φ_h defined in Section 4.

Hereafter let $w=Q_h u$. Obviously $T_h u=0$, if $y_n=0$ and $y_{n-1}>0$. And $\partial T_h u \in L^p(\Sigma)$ by Theorem 1 and (4.5). Hence we see that $w \in V(\Sigma)$ similarly to the proof of Lemma 1.2. We note that $w=0$ near $|x|=R$. Let R'' be a number with $0<R''<R'$, and let us take a nonnegative function $\zeta\in C_0^\infty(\{|x|<R''\})$. Clearly $S_h w=0$, if $x_n=0$ and $x_{n-1}>0$. From Theorem 1 there is a positive number δ such that $\tilde{\rho}^{-\delta}S_h\partial w\in L^p(\Sigma)$, so that $P_h(\zeta^2 w)\in V(\Sigma')$ by Lemma 1.2 and (4.5).

For brevity we write $(\ ,\ )_\Sigma$ simply with $(\ ,\ )$. We have from (6.2)

$$(6.3)\qquad (|\nabla u|^{p-2}\nabla u,\ \nabla P_h(\zeta^2 w))+(|u|^\alpha u,\ P_h(\zeta^2 w))=(f,\ P_h(\zeta^2 w))\ .$$

First let us estimate the first term on the left-hand side of (6.3). We rewrite

$$(6.4)\qquad \begin{aligned}&(|\nabla u|^{p-2}\nabla u,\ \nabla P_h(\zeta^2 w))\\ &=(|\nabla u|^{p-2}\nabla u,\ P_h\nabla(\zeta^2 w))+(|\nabla u|^{p-2}\nabla u,\ \boldsymbol{F}_h(\nabla(\zeta^2 w)))\ .\end{aligned}$$

By (4.5) we see that

$$(6.5)\qquad \begin{aligned}&|(|\nabla u|^{p-2}\nabla u,\ \boldsymbol{F}_h(\nabla(\zeta^2 w)))|\\ &\leqq C\Big[\int_\Sigma \tilde{\rho}^{1-1/\gamma}|\nabla u|^{p-1}\zeta(y)^2|(\nabla w)(y)|dx\\ &\quad+\int_\Sigma |(\nabla\zeta^2)(y)|\,|w(y)|\tilde{\rho}^{1-1/\gamma}|\nabla u|^{p-1}dx\\ &\equiv I_1+I_2,\quad \text{say}\ .\end{aligned}$$

It follows from (4.2) and (4.5) that

$$\begin{aligned}I_1&\leqq C\int_\Sigma \tilde{\rho}^{1-1/\gamma}\zeta^2|T_h\nabla u|^{p-1}|\nabla w|dy\\ &\leqq C\Big[\int_\Sigma \tilde{\rho}^{1-1/\gamma}\zeta^2|T_h\nabla u|^{p-1}|Q_h\nabla u|dy+\int_\Sigma \tilde{\rho}^{2(1-1/\gamma)}\zeta^2|T_h\nabla u|^p dy\Big]\ .\end{aligned}$$

By Schwarz inequality

$$\begin{aligned}&\int_\Sigma \tilde{\rho}^{1-1/\gamma}\zeta^2|T_h\nabla u|^{p-1}|Q_h\nabla u|dy\\ &\leqq C\Big(\int_\Sigma \zeta^2\tilde{\rho}^{2(1-1/\gamma)}|T_h\nabla u|^p dy\Big)^{1/2}\Big(\int_\Sigma \zeta^2|T_h\nabla u|^{p-2}|Q_h\nabla u|^2 dy\Big)^{1/2}\ .\end{aligned}$$

Using Cauchy's inequality, we have for any $\varepsilon>0$

$$\begin{aligned}&\int_\Sigma \tilde{\rho}^{1-1/\gamma}\zeta^2|T_h\nabla u|^{p-1}|Q_h\nabla u|dy\\ &\leqq \varepsilon\int_\Sigma \zeta^2|T_h\nabla u|^{p-2}|Q_h\nabla u|^2 dy+C\int_\Sigma \tilde{\rho}^{2(1-1/\gamma)}|T_h\nabla u|^p dy\ ,\end{aligned}$$

where the constant C on the right-hand side depends on ε. Since

$\tilde{\rho}^{2(1-1/\gamma)} \leqq C(x_{n-1}^2 + x_n^2)^{1-1/\gamma}$ in Σ, we have the following inequality by returning to the original coordinate system:

$$\int_\Sigma \tilde{\rho}^{2(1-1/\gamma)} |T_h \nabla u|^p dy \leqq C \int_\Omega \phi^{2(1-1/\gamma)} |\nabla u|^p dx$$
$$\text{(by Theorem 1)} \quad \leqq C(\|f\|_{p*})^{p*} .$$

From the above inequalities we obtain

$$I_1 \leqq \varepsilon \int_\Sigma \zeta^2 |T_h \nabla u|^{p-2} |Q_h \nabla u|^2 dy + (\|f\|_{p*})^{p*} . \tag{6.6}$$

On the other hand by Hölder's inequality

$$I_2 \leqq C\Big(\int_\Sigma |Q_h u|^p dy\Big)^{1/p} \Big(\int_\Sigma \tilde{\rho}^{p*(1-1/\gamma)} |\nabla u|^p dx\Big)^{1/p*} .$$

Applying Lemma 4.2 and Theorem 1 to the right-hand side we have

$$I_2 \leqq C(\|f\|_{p*})^{p*} .$$

Combining (6.5), (6.6) with this inequality, we conclude that

$$\begin{aligned} &|(|\nabla u|^{p-2} \nabla u,\ F_h(\nabla(\zeta^2 w)))| \\ &\leqq \varepsilon \int_\Sigma \zeta^2 |T_h \nabla u|^{p-2} |Q_h \nabla u|^2 dy + C(\|f\|_{p*})^{p*} . \end{aligned} \tag{6.7}$$

Next we estimate the first term on the right-hand side of (6.4). By Lemma 4.1

$$\begin{aligned} &(|\nabla u|^{p-2} \nabla u,\ P_h \nabla(\zeta^2 w)) \\ &= (Q_h(|\nabla u|^{p-2} \nabla u),\ \nabla(\zeta^2 w)) + (K_h T_h(|\nabla u|^{p-2} \nabla u),\ \nabla(\zeta^2 w)) \\ &= (\zeta^2 Q_h(|\nabla u|^{p-2} \nabla u),\ Q_h \nabla u) + (\zeta^2 Q_h(|\nabla u|^{p-2} \nabla u),\ \nabla Q_h u - Q_h \nabla u) \\ &\quad + (Q_h u \cdot \nabla \zeta^2,\ Q_h(|\nabla u|^{p-2} \nabla u)) + (\zeta^2 K_h T_h(|\nabla u|^{p-2} \nabla u),\ \nabla Q_h u) \\ &\quad + (K_h Q_h u \cdot \nabla \zeta^2,\ T_h(|\nabla u|^{p-2} \nabla u)) \\ &\equiv \sum_{i=1}^{5} J_i, \quad \text{say} . \end{aligned} \tag{6.8}$$

We have from (2.3)

$$J_1 \geqq c_0 \int_\Sigma \zeta^2 (|T_h \nabla u|^{p-2} + |\nabla u|^{p-2}) |Q_h \nabla u|^2 dy .$$

By (2.5) and (4.5)

$$J_2 \leqq C \int_\Sigma \zeta^2 \tilde{\rho}^{1-1/\gamma} |T_h \nabla u| (|T_h \nabla u|^{p-2} + |\nabla u|^{p-2}) |Q_h \nabla u| dy .$$

Repeating the procedure in the proof of (6.6), we easily see that

$$J_2 \leqq \varepsilon \int_\Sigma \zeta^2(|T_h\nabla u|^{p-2}+|\nabla u|^{p-2})|Q_h\nabla u|^2 dy + C(\|f\|_{p*})^{p*} .$$

Similarly

$$J_3,\ J_4 \leqq \varepsilon \int_\Sigma \zeta^2(|T_h\nabla u|^{p-2}+|\nabla u|^{p-2})|Q_h\nabla u|^2 dy + C(\|f\|_{p*})^{p*} .$$

More easily we have

$$J_5 \leqq C(\|f\|_{p*})^{p*} .$$

From the above inequalities and (6.8) it follows that

$$(6.9) \qquad (c_0-3\varepsilon)\int_\Sigma \zeta^2(|T_h\nabla u|^{p-2}+|\nabla u|^{p-2})|Q_h\nabla u|^2 dy \leqq (|\nabla u|^{p-2}\nabla u,\ P_h\nabla(\zeta^2 w)) + C(\|f\|_{p*})^{p*} .$$

Now let us estimate the second term on the left-hand side of (6.3). From (0.5) and Lemma 4.1

$$\begin{aligned} &(|u|^\alpha u,\ P_h(\zeta^2 w)) \\ &= (Q_h(|u|^\alpha u),\ \zeta^2 Q_h u) + (K_h T_h(|u|^\alpha u),\ \zeta^2 Q_h u) \\ &\geqq -C\int_\Sigma \zeta^2 T_h(|u|^{1+\alpha})|Q_h u| dy . \end{aligned}$$

By Hölder's inequality

$$\int_\Sigma T_h(|u|^{1+\alpha})|Q_h u| dy \leqq C\Big(\int_\Sigma |u|^{(1+\alpha)p*} dx\Big)^{1/p*}\Big(\int |Q_h u|^p dy\Big)^{1/p} .$$

And by Sobolev's imbedding theorem

$$\Big(\int_\Sigma |u|^{(1+\alpha)p*} dx\Big)^{1/(1+\alpha)p*} \leqq C\Big(\int_\Sigma |\nabla u|^p dx\Big)^{1/p} ,$$

so that we have from (2.6) and Lemma 4.2

$$\int_\Sigma T_h(|u|^{1+\alpha})|Q_h u| dy \leqq C(\|f\|_{p*})^{(2+\alpha)/(p-1)} .$$

Therefore it holds that

$$(6.10) \qquad (|u|^\alpha u,\ P_h(\zeta^2 w)) \geqq -C(\|f\|_{p*})^{(2+\alpha)/(p-1)} .$$

Lastly we estimate the right-hand side of (6.3). Repeating the above procedure, we see that

$$(f, P_h(\zeta^2 w)) = (Q_h f, \zeta^2 Q_h u) + (K_h T_h f, \zeta^2 Q_h u)$$
$$\leqq C\|f\|_{1,p^*}\left(\int_\Sigma |\nabla u|^p dx\right)^{1/p}$$
$$\leqq C(\|f\|_{1,p^*})^{p^*} .$$

Let us combine (6.3), (6.4), (6.7), (6.9), (6.10) with this inequality. Further let us put $\varepsilon = c_0/5$. Then it follows that

$$\int_\Sigma \zeta^2(|T_h \nabla u|^{p-2} + |\nabla u|^{p-2})|Q_h \nabla u|^2 dy \leqq C[(\|f\|_{1,p^*})^{p^*} + (\|f\|_{p^*})^{(2+\alpha)/(p-1)}] . \tag{6.11}$$

Clearly $\zeta(x)^2 \leqq C(\zeta(y)^2 + h^2)$ and $h|P_h \nabla u| \leqq |S_h \nabla u| + |\nabla u|$. In the same manner as the above we have thus

$$h^2 \int_\Sigma (|S_h \nabla u|^{p-2} + |\nabla u|^{p-2})|P_h \nabla u|^2 dx \leqq C(\|f\|_{p^*})^{p^*} .$$

From this inequality and (6.11) we finally conclude that

$$\int_\Sigma \zeta^2(|S_h \nabla u|^{p-2} + |\nabla u|^{p-2})|P_h \nabla u|^2 dx \leqq CA , \tag{6.12}$$

where CA is the right-hand side of (6.11).

By (2.5) we have

$$|P_h|\nabla u|^{p/2}| \leqq C(|S_h \nabla u|^{(p-2)/2} + |\nabla u|^{(p-2)/2})|P_h \nabla u| ,$$

so that it follows from (6.12) that

$$\int_\Sigma \zeta^2 |P_h|\nabla u|^{p/2}|^2 dx \leqq CA . \tag{6.13}$$

From this there are a function $v \in L^2(\Sigma)$ and a sequence $\{h_\nu\}$ with $h_\nu \to 0$ $(\nu \to \infty)$ such that

$$\zeta P_{h_\nu} |\nabla u|^{p/2} \to v \quad \text{in } L^2(\Sigma) . \tag{6.14}$$

By Lemma 4.1

$$(\zeta P_{h_\nu} |\nabla u|^{p/2}, \varphi) = (|\nabla u|^{p/2}, Q_h(\zeta\varphi)) + (K_{h_\nu} T_{h_\nu}(\zeta\varphi), |\nabla u|^{p/2}) ,$$
$$\varphi \in C_0^\infty(\Sigma) .$$

Since $K_h \to -\partial_{x_{n-1}} \tilde{\rho}$ as $h \to 0$, we have from Lemma 4.3

$$
\begin{aligned}
&(\zeta P_{h_\nu}|\nabla u|^{p/2}, \varphi) \\
&\quad \to -(|\nabla u|^{p/2}, \tilde{\rho}\partial_{x_{n-1}}(\zeta\varphi)) - (\partial_{x_{n-1}}\tilde{\rho}\cdot\zeta\varphi, |\nabla u|^{p/2}) \\
&\quad = -(|\nabla u|^{p/2}, \partial_{x_{n-1}}(\tilde{\rho}\zeta\varphi)) .
\end{aligned}
$$

Combining this with (6.14), we have $v=\zeta\tilde{\rho}\partial_{x_{n-1}}|\nabla u|^{p/2}$. From (6.13) and (6.14) it holds that

$$
\int_\Sigma \zeta^2\tilde{\rho}^2(\partial_{x_{n-1}}|\nabla u|^{p/2})^2 dx \leqq CA . \tag{6.15}
$$

Now we define for any function v

$$
\begin{cases} v_h^{(i)}(x)=v(x_1, \cdots, x_{i-1}, x_i+h, x_{i+1}, \cdots, x_n) \\ (D_h^{(i)}v)(x)=h^{-1}(v_h^{(i)}(x)-v(x)) . \end{cases} \tag{6.16}
$$

By replacing P_h with $D_h^{(i)}$ for $1\leqq i\leqq n-2$, we repeat the above procedure. Then we can obtain

$$
\int_\Sigma \zeta^2(\partial_{x_i}|\nabla u|^{p/2})^2 dx \leqq CA . \tag{6.17}
$$

Further we define for $\varepsilon>0$

$$
\kappa_\varepsilon(t)=\begin{cases} 0 & \text{if } t<\varepsilon \\ t-\varepsilon & \text{if } t\geqq\varepsilon . \end{cases}
$$

In the above arguments we replace P_h and ζ^2 by $D_h^{(n)}$ and $\kappa_\varepsilon(x_n)^2\zeta(x)^2$, respectively. Then we can obtain the following inequality in place of (6.15):

$$
\int_\Sigma \kappa_\varepsilon(x_n)^2\zeta^2(\partial_{x_n}|\nabla u|^{p/2})^2 dx \leqq CA .
$$

If we take $\varepsilon\to 0$, this inequality becomes

$$
\int_\Sigma x_n^2\zeta^2(\partial_{x_n}|\nabla u|^{p/2})^2 dx \leqq CA . \tag{6.18}
$$

Let $\theta\cdot\nabla$ be a C^1 vector field in $\bar{\Sigma}$, which is tangent to $\{x_n=0\}$. Writing $\theta=(\theta_1, \cdots, \theta_n)$, we have $|\theta_n(x)|\leqq Cx_n$. From (6.15), (6.17) and (6.18) we obtain

$$
\int_\Sigma \zeta^2\tilde{\rho}^2((\theta\cdot\nabla)|\nabla u|^{p/2})^2 dx \leqq CA . \tag{6.19}
$$

Let $1<s<2$. By Hölder's inequality

(6.20)
$$\int_\Sigma \zeta^s|(\theta\cdot\nabla)|\nabla u|^{p/2}|^s dx \leqq \left(\int_\Sigma \tilde{\rho}^{-2s/(2-s)}dx\right)^{(2-s)/2}\left(\int_\Sigma \zeta^2\tilde{\rho}^2((\theta\cdot\nabla)|\nabla u|^{p/2})^2 dx\right)^{s/2}.$$

If s is close to 1, it holds that

(6.21)
$$\int_\Sigma \tilde{\rho}^{-2s/(2-s)}dx<\infty .$$

Indeed setting $t=s/(2-s)$ and $z=x_{n-1}/x_n^\gamma$, we have

$$\int_0^1 \frac{dx_{n-1}}{(x_{n-1}^2+x_n^{2\gamma})^t}\leqq x_n^{\gamma(1-2t)}\int_0^\infty \frac{dz}{(z^2+1)^t} ,$$

where we note that $\gamma<1$. Thus if t is close to 1, namely, s is so, we get

$$\int_0^1\int_0^1 \frac{dx_{n-1}dx_n}{(x_{n-1}^2+x_n^{2\gamma})^t}<\infty ,$$

which yields (6.21). Combining (6.19), (6.20) with (6.21), we finally conclude that

(6.22)
$$\int_\Sigma \zeta^s|(\theta\cdot\nabla)|\nabla u|^{p/2}|^s dx\leqq CA^{s/2} .$$

More easily we can prove (6.22) in a neighborhood of each point of $\bar{\Omega}-S$. Let us put $\delta=s-1$. Returning to the original coordinate system and using (6.22), we complete the proof of Theorem 2. Q.E.D.

§7. Proof of Theorem 3.

Since $|\partial\phi^{1-\delta}|\leqq C\phi^{-\delta}$, by Theorem 1 $\partial\phi^{1-\delta}|\nabla u|^{p-1}\in L^{p^*}(\Omega)$ for sufficiently small δ. Thus it is enough to prove the inequality

(7.1)
$$\|\phi^{1-\delta}\partial|\nabla u|^{p-1}\|_{p^*}\leqq C[\|f\|_{1,p^*}+(\|f\|_{p^*})^{(1+\alpha)/(p-1)}+(\|f\|_{p^*})^{(2+\alpha)/p}] .$$

PROOF OF (7.1). Let ζ and Σ be the same ones as in the proof of Theorem 2. We return to the procedure in the proof of Theorem 2. Let $\rho=\rho(x)=(x_{n-1}^2+x_n^2)^{1/2}$. Since

$$|P_h|\nabla u|^{p-1}|\leqq C(S_h|\nabla u|^{p-2}+|\nabla u|^{p-2})|P_h\nabla u|$$

and $p^*/2+(p-2)/(2(p-1))=1$, we have by Hölder's inequality

$$\int_\Sigma(\zeta\rho^{-\delta}|P_h|\nabla u|^{p-1}|)^{p^*}dx\leqq\left(\int_\Sigma \rho^{-2p\delta/(p-2)}|S_h\nabla u|^p dx + \int_\Sigma \rho^{-2p\delta/(p-2)}|\nabla u|^p dx\right)^{(p-2)/(2(p-1))}\left(\int_\Sigma \zeta^2(|S_h\nabla u|^{p-2}+|\nabla u|^{p-2})|P_h\nabla u|^2 dx\right)^{p^*/2} .$$

Hence it follows from Theorem 1 and (6.12) that

$$\int_{\Sigma}(\zeta\rho^{-\delta}|P_k|\nabla u|^{p-1}|)^{p^*}dx \leq C[(\|f\|_{1,p^*})^{p^*}+(\|f\|_{p^*})^{(2+\alpha)/(p-1)}]\,, \tag{7.2}$$

where we have used the inequality $\rho(y)\leq C\rho(x)^r$. We write again with CA the right-hand side of (7.2). Repeating the same argument as in the previous section and noting that $\rho\leq C\tilde{\rho}$, we obtain from (7.2)

$$\int_{\Sigma}(\zeta\rho^{1-\delta}|\partial_{x_{n-1}}|\nabla u|^{p-1}|)^{p^*}dx\leq CA\,. \tag{7.3}$$

Next let $D_k^{(i)}$ be the operator in (6.16). Since from (2.5)

$$|D_k^{(i)}|\nabla u|^{p-1}|\leq C(|\nabla u|^{p-2}+|\nabla u_k^{(i)}|^{p-2})|\nabla D_k^{(i)}u|\,, \tag{7.4}$$

we get the following inequality more easily than (7.2):

$$\int_{\Sigma}(\zeta|D_k^{(i)}|\nabla u|^{p-1}|)^{p^*}dx\leq CA\,, \qquad 1\leq i\leq n-2\,,$$

so that

$$\int_{\Sigma}(\zeta|\partial_{x_i}|\nabla u|^{p-1}|)^{p^*}dx\leq CA\,, \qquad i=1,\cdots,n-2\,. \tag{7.5}$$

We estimate finally the integral

$$\int_{\Sigma}(\zeta\rho^{1-\delta}|\partial_{x_n}|\nabla u|^{p-1}|)^{p^*}dx\,.$$

Let D be a subdomain of Σ with $\bar{D}\subset\Sigma$. First let us prove that

$$\int_{D}(|\,|\nabla u_k^{(n)}|^{p-2}-|\nabla u|^{p-2}|\,|D_k^{(n)}\partial_{x_i}u|)^{p^*}dx\to 0\,, \quad \text{as } h\to 0\,, \qquad 1\leq i\leq n\,. \tag{7.6}$$

Let κ be a number such that $0<\kappa\leq\min(p-2,1)$. We take κ so that it is close to 0, if necessary. We have from (2.5)

$$|\,|\nabla u_k^{(n)}|^{p-2}-|\nabla u|^{p-2}|\leq C(|\nabla u_k^{(n)}|^{p-2-\kappa}+|\nabla u|^{p-2-\kappa})|\nabla(u_k^{(n)}-u)|^{\kappa}\,.$$

Let q be a number such that $p^*/2+\kappa p^*/p+1/q=1$. By Hölder's inequality

$$\begin{aligned}&\int_{D}[(|\nabla u_k^{(n)}|^{p-2-\kappa}+|\nabla u|^{p-2-\kappa})|\nabla(u_k^{(n)}-u)|^{\kappa}|D_k^{(n)}\partial_{x_i}u|]^{p^*}dx\\ &\leq C\Big(\int_{D}|\nabla u_k^{(n)}|^{(p-2-2\kappa)p^*q/2}dx+\int_{D}|\nabla u|^{(p-2-2\kappa)p^*q/2}dx\Big)^{1/q}\\ &\quad\cdot\Big(\int_{D}|\nabla(u_k^{(n)}-u)|^{p}dx\Big)^{\kappa p^*/p}\Big(\int_{D}(|\nabla u_k^{(n)}|^{p-2}+|\nabla u|^{p-2})|D_k^{(n)}\partial_{x_i}u|^{2}dx\Big)^{p^*/2}\,.\end{aligned}$$

Since $(p-2-2\kappa)p^*q=2p$, we have the following inequality by using (2.6) and (2.7):

$$\int_D (||\nabla u_h^{(n)}|^{p-2}-|\nabla u|^{p-2}|\,|D_h^{(n)}\partial_{x_i}u|)^{p^*}dx$$
$$\leqq C(f, D)\Big(\int_D |\nabla(u_h^{(n)}-u)|^p dx\Big)^{\kappa p^*/p},$$

where the constant $C(f, D)$ depends on both f and D. Therefore (7.6) has been proved.

Now we easily see that

$$D_h^{(n)}(|\nabla u|^{p-2}\partial_{x_n}u)=|\nabla u_h^{(n)}|^{p-2}D_h^{(n)}\partial_{x_n}u$$
$$+(p-2)\partial_{x_n}u\cdot\int_0^1 |t\nabla u_h^{(n)}+(1-t)\nabla u|^{p-4}(t\nabla u_h^{(n)}+(1-t)\nabla u)\cdot\nabla D_h^{(n)}u dt\,.$$

We define F_h as follows:

$$D_h^{(n)}(|\nabla u|^{p-2}\partial_{x_n}u)=|\nabla u|^{p-2}D_h^{(n)}\partial_{x_n}u+(p-2)\partial_{x_n}u\cdot|\nabla u|^{p-4}\nabla u\cdot\nabla D_h^{(n)}u+F_h\,.$$

If we repeat the arguments in the proof of Proposition 2.2, similarly to (7.6) we see that $F_h\to 0$ in $L^{p^*}(D)$ as $h\to 0$. From the above equality,

$$|\nabla u|^{p-2}|D_h^{(n)}\partial_{x_n}u|\leqq C(|D_h^{(n)}(|\nabla u|^{p-2}\partial_{x_n}u)|+|\nabla u|^{p-2}\sum_{i=1}^{n-1}|D_h^{(n)}\partial_{x_i}u|+|F_h|)\,.$$

From this and (7.4) we have

$$(7.7)\qquad \int_D (\zeta\rho^{1-\delta}|D_h^{(n)}|\nabla u|^{p-1}|)^{p^*}dx\leqq C\Big[\int_D (\zeta\rho^{1-\delta}|D_h^{(n)}(|\nabla u|^{p-2}\partial_{x_n}u)|)^{p^*}dx$$
$$+\sum_{i=1}^{n-1}\int_D (\zeta\rho^{1-\delta}|\nabla u|^{p-2}|D_h^{(n)}\partial_{x_i}u|)^{p^*}dx+\int_D (\zeta\rho^{1-\delta}|F_h|)^{p^*}dx$$
$$+\int_D (\zeta\rho^{1-\delta}||\nabla u_h^{(n)}|^{p-2}-|\nabla u|^{p-2}|\,|D_h^{(n)}\nabla u|)^{p^*}dx\Big]\,.$$

Now from Proposition 2.2 there are two functions $v_1, v_2\in L_{loc}^{p^*}(D)$ such that

$$|\nabla u|^{p-2}D_{h_\nu}^{(n-1)}\partial_{x_n}u\to v_1\,,$$
$$|\nabla u|^{p-2}D_{h_\nu}^{(n)}\partial_{x_{n-1}}u\to v_2 \qquad \text{in } L^{p^*}(D)\,.$$

And from (6.12) there is a function $w\in L^{p^*}(D)$ such that

$$|\nabla u|^{p-2}P_{h_\nu}\partial_{x_n}u\rightharpoonup w \qquad \text{in } L^{p^*}(D)\,.$$

We prove that $\tilde\rho v_1=\tilde\rho v_2=w$ in D. Since $|\nabla u|\varphi\in L^p(D)$ for any $\varphi\in C_0^\infty(D)$, it holds that

$$(P_{h_\nu}\partial_{x_n}u,\ |\nabla u|^{p-1}\varphi)\to(w,\ |\nabla u|\varphi)\ .$$

On the other hand

$$\begin{aligned}&(P_h\partial_{x_n}u,\ |\nabla u|^{p-1}\varphi)\\&=(\partial_{x_n}P_hu,\ |\nabla u|^{p-1}\varphi)-(\partial_{x_n}\tilde\rho\cdot S_h\partial_{x_{n-1}}u,\ |\nabla u|^{p-1}\varphi)\ .\end{aligned}$$

Hence it follows from Lemma 4.3 and the remark continuing from Proposition 2.1 that

$$\begin{aligned}&(P_h\partial_{x_n}u,\ |\nabla u|^{p-1}\varphi)\\&\quad\to-(\tilde\rho\partial_{x_{n-1}}u,\ \partial_{x_n}(|\nabla u|^{p-1}\varphi))-(\partial_{x_n}\tilde\rho\cdot\partial_{x_{n-1}}u,\ |\nabla u|^{p-1}\varphi)\\&\quad=-(\partial_{x_{n-1}}u,\ \partial_{x_n}(\tilde\rho|\nabla u|^{p-1}\varphi))\ .\end{aligned}$$

Obviously

$$\begin{aligned}&(\tilde\rho D_h^{(n)}\partial_{x_{n-1}}u,\ |\nabla u|^{p-1}\varphi)\\&\quad\to-(\partial_{x_{n-1}}u,\ \partial_{x_n}(\tilde\rho|\nabla u|^{p-1}\varphi))\ .\end{aligned}$$

Therefore $|\nabla u|\tilde\rho v_2=|\nabla u|w$ in D, which implies that $\tilde\rho(x)v_2(x)=w(x)$ if $|\nabla u(x)|\neq0$. Naturally $v_2(x)=w(x)=0$ if $|\nabla u(x)|=0$. Accordingly $\tilde\rho v_2=w$ in D. More easily we can prove that $v_1=v_2$ in D.

Let us denote v_1 and v_2 by $|\nabla u|^{p-2}\partial_{x_{n-1}}\partial_{x_n}u$. Then from the argument in the beginning of this section we have

$$\int_D(\zeta\rho^{1-\delta}|\nabla u|^{p-2}|\partial_{x_{n-1}}\partial_{x_n}u|)^{p^*}dx\leqq CA\ .$$

More easily we see that

$$\int_D(\zeta\rho^{1-\delta}|\nabla u|^{p-2}|\partial_{x_i}\partial_{x_n}u|)^{p^*}dx\leqq CA\ ,$$

where $i\leqq n-2$. Hence it follows from (7.6) and (7.7) that

$$\begin{aligned}&\int_D(\zeta\rho^{1-\delta}|\partial_{x_n}|\nabla u|^{p-1}|)^{p^*}dx\\&\leqq C\Big[A+\int_D(\zeta\rho^{1-\delta}|\partial_{x_n}(|\nabla u|^{p-2}\partial_{x_n}u)|^{p^*}dx\Big]\ .\end{aligned}$$

On the other hand from (6.2)

$$\begin{aligned}&|\partial_{x_n}(|\nabla u|^{p-2}\partial_{x_n}u)|\\&\leqq C(|f|+|u|^{1+\alpha}+\sum_{i=1}^{n-1}|\partial_{x_i}(|\nabla u|^{p-2}\partial_{x_i}u)|)\ .\end{aligned}$$

Therefore

$$\int_D (\zeta\rho^{1-\delta}|\partial_{x_n}|\nabla u|^{p-1}|)^{p^*}dx \leqq C[A+(\|u\|_{(1+\alpha)p^*})^{(1+\alpha)p^*}] .$$

By Sobolev's imbedding theorem and (2.6), $(\|u\|_{(1+\alpha)p^*})^{(1+\alpha)p^*} \leqq C(\|f\|_{p^*})^{(1+\alpha)p^*/(p-1)}$. Accordingly we finally obtain

$$\int_D (\zeta\rho^{1-\delta}|\partial_{x_n}|\nabla u|^{p-1}|)^{p^*}dx \leqq C[A+(\|f\|_{p^*})^{(1+\alpha)p^*/(p-1)}] .$$

By taking $D\rightarrow\Sigma$ and taking a partition of unity on $\bar{\Omega}$, we complete the proof of (7.1). Thus the proof of Theorem 3 is finished. Q.E.D.

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