TOKYO J. MATH.
VOL. 10, No. 2, 1987

On the Symmetry of a Reflecting Brownian Motion Defined by Skorohod's Equation for a Multi-Dimensional Domain

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Introduction

The existence and uniqueness of solutions to Skorohod's equation for a multi-dimensional domain were discussed by Lions and Sznitman [6] and Saisho [9]. In this paper we prove that a reflecting Brownian motion X obtained by solving Skorohod's equation for a domain D in $\boldsymbol{R}^d$ is symmetric in the sense that $\int fT_t g dx = \int g T_t f dx$ holds for any L^2-functions f and g on $\bar{D}$, where T_t is the semigroup of X. The proof is based on the construction of X by the penalty method, that is, we prove the above result by showing that X can be approximated by symmetric diffusions (with respect to the invariant measures) which are described by stochastic differential equations with smooth drift coefficients of gradient type. The penalty method was used for the study of reflecting diffusions by Lions, Menaldi and Sznitman [5], Menaldi [7] and Menaldi and Robin [8]. Our method is similar to theirs but our approximation result (Theorem 2) is given in a pathwise formulation and improves some results of [6].

§1. Formulation of the problem and the result.

We denote by $B(x, r)$ the open ball in $\boldsymbol{R}^d$ with center x and radius r and write $\langle\cdot, \cdot\rangle$ for the usual inner product in $\boldsymbol{R}^d$. Let D be a domain in $\boldsymbol{R}^d$ and let $x \in \partial D$. Denote by $\mathfrak{N}_{x,r}$ the set of unit vectors $\boldsymbol{n}$ in $\boldsymbol{R}^d$ such that $B(x-r\boldsymbol{n}, r) \cap D = \emptyset$ and by $\mathfrak{N}_x$ the union of $\mathfrak{N}_{x,r}$ as r runs over all positive numbers. An element of $\mathfrak{N}_x$ is called an inward normal vector at x.

Following Lions and Sznitman [6] we introduce conditions for D.

CONDITION (A). There exists a constant $r_0 > 0$ such that $\mathfrak{N}_x = \mathfrak{N}_{x,r_0} \neq \emptyset$ for any $x \in \partial D$.

Received October 27, 1986

CONDITION (B). There exist constants $\delta>0$ and $\beta\in[1,\infty)$ with the following property: for any $x\in\partial D$ there exists a unit vector l_x such that

$$\langle l_x, \boldsymbol{n}\rangle \geqq \frac{1}{\beta} \qquad \text{for any } \boldsymbol{n}\in \bigcup_{y\in B(x,\delta)\cap\partial D} \mathfrak{N}_y .$$

We can prove that the following two conditions for a unit vector $\boldsymbol{n}$ are equivalent (see [6: Remark 1.2]).

$$\boldsymbol{n}\in\mathfrak{N}_{x,r} , \tag{1.1}$$

$$\langle y-x, \boldsymbol{n}\rangle+\frac{1}{2r}|y-x|^2\geqq 0 \qquad \text{for any } y\in\bar{D} . \tag{1.2}$$

Suppose we are given a domain D in $\boldsymbol{R}^d$ satisfying Conditions (A) and (B). It is then easy to see that there exists a unique $\bar{x}\in\bar{D}$ such that $|x-\bar{x}|=\mathrm{dist}(x,\bar{D})$ for any $x\in\boldsymbol{R}^d$ with $\mathrm{dist}(x,\bar{D})<r_0$ and $(\bar{x}-x)/|\bar{x}-x|\in\mathfrak{N}_{\bar{x}}$ if $x\notin\bar{D}$. The notation $\bar{x}$ is used in this sense throughout the paper. For an $\boldsymbol{R}^d$-valued continuous function ϕ with bounded variation we denote by $\|\phi\|_t$ the total variation of ϕ on $[0,t]$, i.e.,

$$\|\phi\|_t=\sup\sum_{k=1}^{n}|\phi(t_k)-\phi(t_{k-1})| ,$$

where the supremum is taken over all partitions $0=t_1<t_2<\cdots<t_n=t$. We also set $\|\phi\|_s^t=\|\phi\|_t-\|\phi\|_s$, $0\leqq s<t$.

Given an $\boldsymbol{R}^d$-valued continuous function w on $[0,\infty)$ with $w(0)\in\bar{D}$, we consider

$$\xi(t)=w(t)+\phi(t) , \qquad t\geqq 0 . \tag{1.3}$$

The problem is to find a pair of $\xi(t)$ and $\phi(t)$ satisfying (1.3) together with the following two conditions:

(1.4) $\xi(t)$ is $\bar{D}$-valued and continuous,

(1.5) $\phi(t)$ is an $\boldsymbol{R}^d$-valued continuous function with bounded variation on each finite interval such that $\phi(0)=0$ and

$$\phi(t)=\int_0^t \boldsymbol{n}(s)d\|\phi\|_s ,$$

$$\|\phi\|_t=\int_0^t 1_{\partial D}(\xi(s))d\|\phi\|_s ,$$

where

$$\boldsymbol{n}(s)\in\mathfrak{N}_{\xi(s)} \qquad \text{if } \xi(s)\in\partial D .$$

When we speak of the equation (1.3) we always consider it under the conditions (1.4) and (1.5). The equation (1.3) is called Skorohod's equation for D. By the results of [3] and [9] there exists a unique solution of (1.3).

We now replace $w(t)$ in (1.3) by a d-dimensional Brownian motion $B(t)$ ($B(0)=0$), i.e., consider

$$X(t)=x+B(t)+\Phi(t) \tag{1.6}$$

where $x\in\bar{D}$ is given. Of course, (1.6) should be solved under additional conditions similar to (1.4) and (1.5). It is easy to see that the solution $X(t)$ of (1.6) is adapted to the filtration generated by the Brownian motion $B(t)$ and that it gives rise to a diffusion process (reflecting Brownian motion) X on $\bar{D}$. Denote by T_t the semigroup of the diffusion X. Then we have the following theorem.

THEOREM 1. *X is symmetric in the sense that*

(i) $\displaystyle\int_{\bar{D}}|T_tf(x)|^2dx\leqq\int_{\bar{D}}|f(x)|^2dx\ , \quad \forall f\in L^2(\bar{D})\ ,$

(ii) $\displaystyle\int_{\bar{D}}f(x)T_tg(x)dx=\int_{\bar{D}}g(x)T_tf(x)dx\ , \quad \forall f,\ g\in L^2(\bar{D})\ ,$

where $L^2(\bar{D})$ is the space of real L^2-functions on $\bar{D}$. In particular, the Lebesgue measure on $\bar{D}$ is an invariant measure for X.

The proof of this theorem is given in § 7.

§ 2. Some lemmas.

Given a domain D satisfying Conditions (A) and (B), we are going to construct a function $U(x)$, $x\in\boldsymbol{R}^d$, with the following properties:

(*i*) $U\in C^1(\boldsymbol{R}^d)\ ,\quad U\geqq 0\ ,$

(*ii*) $U(x)=|x-\bar{x}|^2$ if $\operatorname{dist}(x,\bar{D})\leqq r_0/2$
(r_0 is the constant appearing in Condition (A)),

(*iii*) ∇U is bounded and Lipschitz continuous.

For $\varepsilon>0$ we denote by D_ε the ε neighborhood of D, and for $x\in D_{r_0}$ set

$$u(x)=|x-\bar{x}|\ ,\qquad U_0(x)=|x-\bar{x}|^2\ .$$

LEMMA 2.1. *For $0<\varepsilon<r_0$, there is a constant $\kappa\equiv\kappa(\varepsilon)>0$ such that*

$$|\bar{x}-\bar{y}|\leqq\kappa|x-y|\qquad \textit{if}\quad |x-\bar{x}|,\ |y-\bar{y}|<\varepsilon\ . \tag{2.1}$$

PROOF. If $x=\bar{x}$ and $y=\bar{y}$ the assertion is clear and we may suppose that $x\neq\bar{x}$. Let

$$z=\bar{x}+\frac{x-\bar{x}}{|x-\bar{x}|}r_0$$

and θ be the angle between two vectors $x-z$ and $y-z$, i.e.,

$$\cos\theta=\left\langle\frac{x-z}{|x-z|},\ \frac{y-z}{|y-z|}\right\rangle .$$

First we assume $|x-y|\leqq r_0-\varepsilon$. Then $\theta\leqq\pi/3$. Since Condition (A) implies $\bar{y}\in B(z, r_0)^c\cap B(y, |\bar{x}-y|)$, we easily have

$$|\bar{x}-\bar{y}|\leqq 4r_0\sin(\theta/2) ,$$
$$|x-y|\geqq(r_0-\varepsilon)\sin\theta ,$$

and hence (2.1) holds for x, y with $|x-y|\leqq r_0-\varepsilon$. If $|x-y|>r_0-\varepsilon$, then $|\bar{x}-\bar{y}|\leqq|x-y|+2\varepsilon$ from which (2.1) follows.

LEMMA 2.2. (i) *$u\in C^1(D_{r_0}\setminus\bar{D})$ and*

$$\nabla u(x)=\frac{x-\bar{x}}{|x-\bar{x}|} .$$

(ii) *$U_0\in C^1(D_{r_0})$ and $\nabla U_0(x)=2(x-\bar{x})$.*

PROOF. (ii) follows immediately from (i) so we prove only (i). Let $x\in \boldsymbol{R}^d$ with $0<\mathrm{dist}(x, \bar{D})<r_0$ and set

$$z=x+hv ,$$
$$\theta_1(h)=\frac{|z-\bar{x}|-|x-\bar{x}|}{h} ,$$
$$\theta_2(h)=\frac{|z-\bar{z}|-|x-\bar{z}|}{h} , \qquad 0\neq h\in\boldsymbol{R}^1$$

for $v\in\boldsymbol{R}^d$ with $|v|=1$. Then we can show that

$$\min\{\theta_1(h),\ \theta_2(h)\}\leqq\frac{u(z)-u(x)}{h}\leqq\max\{\theta_1(h),\ \theta_2(h)\} ,$$
$$\lim_{h\to0}\theta_1(h)=\lim_{h\to0}\theta_2(h)=\frac{\langle x-\bar{x},\ v\rangle}{|x-\bar{x}|} ,$$

which proves

$$\nabla u(x)=\frac{x-\bar{x}}{|x-\bar{x}|} .$$

LEMMA 2.3. *There exists a function* $U(x)$, $x \in \boldsymbol{R}^d$, *satisfying the conditions* (*i*), (*ii*) *and* (*iii*).

PROOF. Choose a non-negative C^2-function $\rho(t)$ on $[0, \infty)$ such that

$$\rho(t)=\begin{cases} t & \text{for } \ 0 \leqq t \leqq (r_0/2)^2 \ , \\ \text{suitably defined} & \text{for } \ (r_0/2)^2 < t < (3r_0/4)^2 \ , \\ \text{const.} & \text{for } \ (3r_0/4)^2 \leqq t \ , \end{cases}$$

and then put

$$U(x)=\begin{cases} \rho(U_0(x)) \ , & x \in D_{r_0} \ , \\ \text{const.}, & \text{otherwise,} \end{cases}$$

where const. is the same as one in the definition of $\rho(t)$. Then $U(x)$ satisfies (*i*), (*ii*) and (*iii*).

§ 3. Penalty method.

Given an $\boldsymbol{R}^d$-valued continuous function w on $[0, \infty)$ with $w(0)=0$ and $x_m \in \bar{D}$, $m \geqq 1$, we denote by $\xi_m(t)$ the solution of

$$\xi(t)=x_m+w(t)-\frac{m}{2}\int_0^t \nabla U(\xi(s))ds \ , \tag{3.1}$$

where U is a function of Lemma 2.3, and put

$$\phi_m(t)=-\frac{m}{2}\int_0^t \nabla U(\xi_m(s))ds \ .$$

We also denote by $\xi(t)$ the solution of the following Skorohod's equation for D:

$$\xi(t)=x+w(t)+\phi(t) \ . \tag{3.2}$$

THEOREM 2. *Assume that the domain D satisfies Conditions* (A) *and* (B) *and that $x_m \to x$ as $m \to \infty$. Then ξ_m converges to the solution ξ of* (3.2) *as $m \to \infty$ uniformly on each finite t-interval.*

The following three sections are devoted to the proof of this theorem.

§ 4. Convergence of $\mathrm{dist}(\xi_m(t), \bar{D})$.

In this section we prove the following proposition which is used in § 5. We denote by $\Delta_{0,t,h}(w)$ the modulus of uniform continuity of w, that is,

$$\Delta_{0,t,h}(w)=\sup\{|w(t_2)-w(t_1)| : 0\leqq t_1<t_2\leqq t,\ |t_2-t_1|\leqq h\} .$$

PROPOSITION 4.1. *Let T be any positive fixed time. Suppose $x_m\to x\in\bar{D}$ as $m\to\infty$ and ξ_m is the solution of* (3.1). *Then for any $\varepsilon\in(0, r_0)$, there exists a positive integer $M\equiv M(\varepsilon)$ such that $\xi_m(t)\in D_\varepsilon$, $0\leqq t\leqq T$, $\forall m\geqq M$, where r_0 is the constant in Condition* (A).

To prove the proposition, we prepare the following lemma.

LEMMA 4.1. *For any $x\in\partial D_\varepsilon$ with $0<\varepsilon\leqq r_0/2$, consider the equation*

$$\eta(t)=x-\frac{1}{2}\int_0^t\nabla U(\eta(s))ds\ , \qquad t\geqq 0\ . \tag{4.1}$$

Then (4.1) *can be solved uniquely and the solution $\eta(t)$ satisfies*

$$\eta(t)=\bar{x}-(\bar{x}-x)\exp(-t)\ , \tag{4.2}$$

$$|\overline{\eta(t)}-\eta(t)|=\varepsilon\exp(-t)\ . \tag{4.3}$$

PROOF. It is easy to see that (4.2) satisfies the equation (4.1) and the uniqueness of solutions of (4.1) follows from the Lipschitz continuity of ∇U. (4.3) is immediate from the fact $\overline{\eta(t)}=\bar{x}$, $t\geqq 0$ and (4.2). The proof is finished.

Since the equation (3.1) for $0\leqq t\leqq T$ can be written as

$$\begin{aligned}\xi_m(t/m)&=x_m+w(t/m)-\frac{m}{2}\int_0^{t/m}\nabla U(\xi_m(s))ds\\ &=x_m+w(t/m)-\frac{1}{2}\int_0^t\nabla U(\xi_m(s/m))ds\ ,\quad 0\leqq t\leqq mT\ ,\end{aligned}$$

the equation (3.1) for $0\leqq t\leqq T$ is equivalent to the following equation

$$\tilde{\xi}_m(t)=x_m+\tilde{w}(t)-\frac{1}{2}\int_0^t\nabla U(\tilde{\xi}_m(s))ds\ ,\quad 0\leqq t\leqq mT\ , \tag{4.4}$$

where

$$\tilde{\xi}_m(t)=\xi_m(t/m)\ ,\qquad \tilde{w}(t)=w(t/m)\ .$$

LEMMA 4.2. *Suppose that $0<\varepsilon<r_0/2$ and $x_m\in D_{\varepsilon/2}$. Then there exists a positive integer M_1 for which the following holds. For any u in $(0, mT)$ such that $\tilde{\xi}_m(u)\in\partial D_{\varepsilon/2}$, $\{\tilde{\xi}_m(t+u), 0\leqq t\leqq mT-u\}$ hits $\partial D_{\varepsilon/3}$ before it hits ∂D_ε provided that $m\geqq M_1$.*

PROOF. Now we suppose that there exists a time u $(0<u<mT)$ such

that $\tilde{\xi}_m(u) \in \partial D_{\varepsilon/2}$ and consider the following equation:

$$\eta(t) = \tilde{\xi}_m(u) - \frac{1}{2}\int_u^t \nabla U(\eta(s))ds , \qquad t \geqq u . \tag{4.5}$$

By Lemma 4.1 we have

$$\eta(t) = \overline{\tilde{\xi}_m(u)} - \{\overline{\tilde{\xi}_m(u)} - \tilde{\xi}_m(u)\}\exp\{-(t-u)\} ,$$
$$|\overline{\eta(t)} - \eta(t)| = \frac{\varepsilon}{2}\exp\{-(t-u)\}$$

and therefore, if we set

$$u' = \inf\{t > u : \eta(t) \in \partial D_{\varepsilon/4}\} ,$$

we have $u' = u + \log 2$. On the other hand, if $u \leqq t \leqq mT$ we have

$$\tilde{\xi}_m(t) - \eta(t) = \tilde{w}(t) - \tilde{w}(u) - \frac{1}{2}\int_u^t \{\nabla U(\tilde{\xi}_m(s)) - \nabla U(\eta(s))\}ds ,$$

$$|\tilde{\xi}_m(t) - \eta(t)| \leqq |\tilde{w}(t) - \tilde{w}(u)| + L\int_u^t |\tilde{\xi}_m(s) - \eta(s)|ds , \tag{4.6}$$

where L is one half of the Lipschitz constant of ∇U. If we set $M' = [u'/T] + 1$, then $u' \leqq mT$ $(m \geqq M')$ and for $u \leqq t \leqq u'$

$$|\tilde{w}(t) - \tilde{w}(u)| = |w(t/m) - w(u/m)|$$
$$\leqq \Delta_{0,T,1/m}(w) ,$$

which tends to 0 as m goes to ∞ because of the continuity of w. Thus, if we take $m \geqq M'$ so large that

$$\Delta_{0,T,1/m}(w) < \frac{\varepsilon}{12}\exp(-L) ,$$

(4.6) and Gronwall's lemma yield

$$\begin{aligned} |\tilde{\xi}_m(t) - \eta(t)| &< \frac{\varepsilon}{12}\exp(-L)\cdot\exp\{L(t-u)\} \\ &< \frac{\varepsilon}{12} , \qquad u \leqq t \leqq u' . \end{aligned} \tag{4.7}$$

Therefore,

$$\begin{aligned} |\overline{\tilde{\xi}_m(t)} - \tilde{\xi}_m(t)| &\leqq |\overline{\eta(t)} - \eta(t)| + |\eta(t) - \tilde{\xi}_m(t)| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{12} \\ &< \varepsilon , \qquad u \leqq t \leqq u' . \end{aligned} \tag{4.8}$$

On the other hand, we have

$$(4.9)\qquad |\overline{\tilde{\xi}_m(u')}-\tilde{\xi}_m(u')|\leqq|\overline{\eta(u')}-\eta(u')|+|\eta(u')-\tilde{\xi}_m(u')| < \frac{\varepsilon}{4}+\frac{\varepsilon}{12}=\frac{\varepsilon}{3}\,.$$

Thus, $\{\tilde{\xi}_m(t), u\leqq t\leqq mT\}$ hits $\partial D_{\varepsilon/3}$ before it hits ∂D_ε. The proof of the lemma is finished.

Proposition 4.1 can be proved as follows. Define $M\equiv M(\varepsilon)$ by $M=\max(M_1, M_2)$, where M_1 is the constant in Lemma 4.2 and

$$M_2=\max\{m\geqq 1: \mathrm{dist}(x_m, \bar{D})\geqq\varepsilon/2\}$$

and suppose that there exists a time $\tau\in[0, mT]$ such that $\{\tilde{\xi}_m(t), 0\leqq t\leqq mT\}$ hits ∂D_ε at the time τ for some $m>M$. Then there is a time $u<\tau$ such that $\tilde{\xi}_m(u)\in\partial D_{\varepsilon/2}$ and $\tilde{\xi}_m(t)\in D_\varepsilon\backslash\overline{D_{\varepsilon/2}}$, $u<t\leqq\tau$, which contradicts Lemma 4.2.

§ 5. Estimates of $|\phi_m|_s^t$.

The purpose of this section is to prove the following proposition. We use the notation

$$\Delta_{s,t}(w)=\sup\{|w(t_2)-w(t_1)|: s\leqq t_1<t_2\leqq t\}\,.$$

By Proposition 4.1, for $0<\varepsilon<r_0$ there is a positive $M\equiv M(\varepsilon)$ such that

$$\varepsilon_m\equiv\sup_{0\leqq t\leqq T}\mathrm{dist}(\xi_m(t), \bar{D})\leqq\varepsilon\,,\qquad \forall m\geqq M\,.$$

In what follows let $0<\varepsilon<\min(\delta/2, r_0/2)$ and $m\geqq M(\varepsilon)$, where δ is the constant in Condition (B).

PROPOSITION 5.1. *Suppose that D satisfies Conditions* (A) *and* (B). *Then for sufficiently large m we have*

$$(5.1)\qquad |\phi_m|_s^t\leqq K\{\Delta_{s,t}(w)+\varepsilon_m\}\,,\qquad 0\leqq s<t\leqq T\,,$$

where $K>0$ is a constant depending only on the constants r_0, β, δ in Conditions (A) *and* (B), *T and $\{\Delta_{0,T,h}(w): 0<h\leqq T\}$.*

Set

$$T_{m,0}=\inf\{t\geqq 0: \overline{\xi_m(t)}\in\partial D\}\,,$$
$$t_{m,n}=\inf\{t>T_{m,n-1}: |\overline{\xi_m(t)}-\overline{\xi_m(T_{m,n-1})}|\geqq\delta/2\}\,,$$
$$T_{m,n}=\inf\{t\geqq t_{m,n}: \overline{\xi_m(t)}\in\partial D\}\,.$$

To prove the proposition, we prepare three lemmas.

LEMMA 5.1. *If* $T_{m,n-1} \leqq s < t \leqq T_{m,n}$, *we have*

$$|\phi_m|_t^s \leqq \beta\{\Delta_{s,t}(\xi_m) + \Delta_{s,t}(w)\}\ , \tag{5.2}$$

where β *is the constant appearing in Condition* (B).

PROOF. Since $\varepsilon_m < r_0/2$, we have

$$\nabla U(\xi_m(t)) = 2\{\xi_m(t) - \overline{\xi_m(t)}\}\ , \qquad 0 \leqq t \leqq T\ .$$

Thus, if we set

$$l = l_{\overline{\xi_m(T_{m,n-1})}}\ ,$$

$$\boldsymbol{n} = \boldsymbol{n}_{\overline{\xi_m(u)}} = \begin{cases} \{\overline{\xi_m(u)} - \xi_m(u)\}/|\xi_m(u) - \overline{\xi_m(u)}|\ , & \xi_m(u) \notin \bar{D}\ , \\ \mathbf{0}\ , & \text{otherwise}\ , \end{cases}$$

we have $\langle l, \boldsymbol{n}\rangle \geqq 1/\beta$, $u \in [T_{m,n-1}, T_{m,n}]$ provided that $\boldsymbol{n} \neq \mathbf{0}$ and hence

$$\begin{aligned} |\xi_m(t) - \xi_m(s)| &\geqq \langle l, \xi_m(t) - \xi_m(s)\rangle \\ &= \langle l, w(t) - w(s)\rangle + \left\langle l, m\int_s^t |\xi_m(u) - \overline{\xi_m(u)}|\boldsymbol{n} du\right\rangle \\ &\geqq \langle l, w(t) - w(s)\rangle + \beta^{-1} m\int_s^t |\xi_m(u) - \overline{\xi_m(u)}| du \\ &= \langle l, w(t) - w(s)\rangle + \beta^{-1}|\phi_m|_t^s\ . \end{aligned}$$

Thus we obtain (5.2).

LEMMA 5.2. *For any* $\theta > 0$, *we have*

$$\Delta_{s,t}(\xi_m) \leqq \{(1+\theta^{-1})\Delta_{s,t}(w) + \theta|\phi_m|_t^s + 2\varepsilon_m\theta^{-1}\}\exp(\gamma|\phi_m|_t^s)\ , \quad 0 \leqq s < t \leqq T\ , \tag{5.3}$$

where $\gamma = \kappa^2/2r_0$ *and* κ *is the constant in Lemma* 2.1.

PROOF. As in the proof of Lemma 2.3 (ii) of [9], we easily have

$$\begin{aligned} |\xi_m(t) - \xi_m(s)|^2 &= |w(t) - w(s)|^2 \\ &\quad + 2\int_s^t \langle \xi_m(u) - \xi_m(s), d\phi_m(u)\rangle + 2\int_s^t \langle w(t) - w(u), d\phi_m(u)\rangle\ . \end{aligned}$$

Using the same $\boldsymbol{n}$ as in the proof of Lemma 5.1, we have

$$\int_s^t \langle \xi_m(u) - \xi_m(s), d\phi_m(u)\rangle = \int_s^t \langle \xi_m(u) - \xi_m(s), \boldsymbol{n}\rangle d|\phi_m|_u\ .$$

By (1.2) and Lemma 2.1, we have

$$
\begin{aligned}
&\langle \xi_m(u)-\xi_m(s),\, n\rangle \\
&=\langle \overline{\xi_m(u)}-\overline{\xi_m(s)},\, n\rangle+\langle \xi_m(u)-\overline{\xi_m(u)},\, n\rangle+\langle \overline{\xi_m(s)}-\xi_m(s),\, n\rangle \\
&\leqq \langle \overline{\xi_m(u)}-\overline{\xi_m(s)},\, n\rangle+2\varepsilon_m \\
&\leqq \frac{1}{2r_0}|\overline{\xi_m(u)}-\overline{\xi_m(s)}|^2+2\varepsilon_m \\
&\leqq \frac{\kappa^2}{2r_0}|\xi_m(u)-\xi_m(s)|^2+2\varepsilon_m\,.
\end{aligned}
$$

Thus

$$
\begin{aligned}
&|\xi_m(t)-\xi_m(s)|^2 \\
&\leqq |w(t)-w(s)|^2+\frac{\kappa^2}{r_0}\int_s^t|\xi_m(u)-\xi_m(s)|^2 d\|\phi_m\|_u \\
&\qquad +4\varepsilon_m\|\phi_m\|_t^s+2\int_s^t\langle w(t)-w(u),\, d\phi_m(u)\rangle \\
&\leqq \Delta_{s,t}^2(w)+\{4\varepsilon_m+2\Delta_{s,t}(w)\}\|\phi_m\|_t^s \\
&\qquad +2\gamma\int_s^t|\xi_m(u)-\xi_m(s)|^2 d\|\phi_m\|_u\,.
\end{aligned}
$$

Therefore, by Gronwall's inequality we have

$$
\begin{aligned}
&|\xi_m(t)-\xi_m(s)|^2 \\
&\leqq [\Delta_{s,t}^2(w)+\{4\varepsilon_m+2\Delta_{s,t}(w)\}\|\phi_m\|_t^s]\exp(2\gamma\|\phi_m\|_t^s) \\
&\leqq \left[\Delta_{s,t}^2(w)+\frac{\{2\varepsilon_m+\Delta_{s,t}(w)\}^2}{\theta^2}+\theta^2(\|\phi_m\|_t^s)^2\right]\exp(2\gamma\|\phi_m\|_t^s)\,,
\end{aligned}
$$

from which (5.3) follows.

LEMMA 5.3. *If* $T_{m,n-1}\leqq s<t\leqq T_{m,n}$, *we have*

$$
\Delta_{s,t}(\xi_m)\leqq K_1\{\Delta_{s,t}(w)+\varepsilon_m\}\,, \tag{5.4}
$$

where

$$
\begin{aligned}
&K_1=8\beta\exp\{4\gamma\beta(\|w\|_T+\delta)\}+1\,, \\
&\|w\|_T=\sup\{|w(t)|:0\leqq t\leqq T\}\,.
\end{aligned}
$$

PROOF. By (5.2) and (5.3) we have

$$
\begin{aligned}
&\Delta_{s,t}(\xi_m) \\
&\leqq \{(1+\theta^{-1}+\beta\theta)\Delta_{s,t}(w)+\beta\theta\Delta_{s,t}(\xi_m)+2\varepsilon_m\theta^{-1}\} \\
&\quad\cdot\exp[\gamma\beta\{\Delta_{s,t}(w)+\Delta_{s,t}(\xi_m)\}]\,, \qquad T_{m,n-1}\leqq s<t\leqq T_{m,n}\,.
\end{aligned}
$$

Since

$$\Delta_{s,t}(\xi_m) \leqq 2\left(\varepsilon_m + \frac{\delta}{2}\right) < 2\delta , \quad T_{m,n-1} \leqq s < t \leqq t_{m,n} ,$$

we have

$$\begin{aligned}\eta \leqq \{(1+\theta^{-1}+\beta\theta)\Delta_{s,t}(w) + \beta\theta\eta + 2\varepsilon_m\theta^{-1}\} \\ \cdot \exp\{2\gamma\beta(\|w\|_T + \delta)\} , \quad T_{m,n-1} \leqq s < t \leqq t_{m,n} ,\end{aligned}$$

where $\eta = \Delta_{s,t}(\xi_m)$. Thus, if we set

$$\begin{aligned}\theta &= (2\beta)^{-1} \exp\{-2\gamma\beta(\|w\|_T + \delta)\} , \\ \lambda &= \exp\{2\gamma\beta(\|w\|_T + \delta)\} ,\end{aligned}$$

then

$$\begin{aligned}\eta &\leqq 2\left\{\left(1 + 2\beta\lambda + \frac{1}{2\lambda}\right)\Delta_{s,t}(w) + 4\beta\lambda\varepsilon_m\right\}\lambda \\ &\leqq 8\beta\lambda^2\{\Delta_{s,t}(w) + \varepsilon_m\} ,\end{aligned}$$

that is,

$$\Delta_{s,t}(\xi_m) \leqq K''\{\Delta_{s,t}(w) + \varepsilon_m\} , \quad s, t \in [T_{m,n-1}, t_{m,n}] , \tag{5.5}$$

where $K'' = 8\beta \exp\{4\beta\gamma(\|w\|_T + \delta)\}$. Since $\Delta_{s,t}(\xi_m) = \Delta_{s,t}(w)$ if $s, t \in [t_{m,n}, T_{m,n})$, we have for $T_{m,n-1} \leqq s \leqq t_{m,n} < t < T_{m,n}$

$$\begin{aligned}\Delta_{s,t}(\xi_m) &\leqq \Delta_{s,t_{m,n}}(\xi_m) + \Delta_{t_{m,n},t}(\xi_m) \\ &\leqq K''\{\Delta_{s,t_{m,n}}(w) + \varepsilon_m\} + \Delta_{t_{m,n},t}(w) \\ &\leqq (K''+1)\{\Delta_{s,t}(w) + \varepsilon_m\} ,\end{aligned}$$

which combined with (5.5) proves (5.4).

PROOF OF PROPOSITION 5.1. By (5.2) and (5.4) we have

$$\begin{aligned}|\phi_m|_s^t &\leqq \beta\{(K_1+1)\Delta_{s,t}(w) + K_1\varepsilon_m\} \\ &\leqq K'\{\Delta_{s,t}(w) + \varepsilon_m\} , \quad T_{m,n-1} \leqq s < t \leqq T_{m,n} ,\end{aligned} \tag{5.6}$$

where $K' = \beta(K_1 + 1)$. Since

$$\begin{aligned}|\xi_m(t_{m,n}) - \xi_m(T_{m,n-1})| &\geqq |\overline{\xi_m(t_{m,n})} - \overline{\xi_m(T_{m,n-1})}| - 2\varepsilon_m \\ &= \frac{\delta}{2} - 2\varepsilon_m ,\end{aligned}$$

we have from (5.4)

$$\begin{aligned}\frac{\delta}{2} - 2\varepsilon_m &\leqq |\xi_m(t_{m,n}) - \xi_m(T_{m,n-1})| \\ &\leqq K'\{\Delta_{T_{m,n-1},t_{m,n}}(w) + \varepsilon_m\} ,\end{aligned}$$

and hence, if we set $\Delta_m=(\delta-4\varepsilon_m)/(2K')-\varepsilon_m$, then

$$\Delta_m \leqq \Delta_{T_{m,n-1},t_{m,n}}(w) . \tag{5.7}$$

On the other hand it is clear that $\varliminf_{m\to\infty}\Delta_m>0$ and hence there exist $m_0\geqq 1$ and $\Delta>0$ such that $\Delta_m\geqq\Delta$ for $\forall m\geqq m_0$. Thus (5.7) implies

$$\Delta \leqq \Delta_{T_{m,n-1},t_{m,n}}(w) , \qquad \forall m\geqq m_0 . \tag{5.8}$$

Since w is continuous, there exists $h>0$ such that $\Delta_{0,T,h}(w)<\Delta$. Therefore, $T_{m,n}\leqq T$ implies $T_{m,n}-T_{m,n-1}\geqq h$. In fact, if $T_m-T_{m,n-1}<h$, then

$$\begin{aligned}\Delta_{T_{m,n-1},t_{m,n}}(w) &\leqq \Delta_{T_{m,n-1},T_{m,n}}(w)\\ &\leqq \Delta_{0,T,h}(w)<\Delta ,\end{aligned}$$

which contradicts (5.8). Thus if $m\geqq m_0$, $T_{m,n}>T$ for $n>T/h$ and (5.6) implies

$$\begin{aligned}|\phi_m|_s^t &\leqq \left(\frac{T}{h}+1\right)K'\{\Delta_{s,t}(w)+\varepsilon_m\}\\ &=K\{\Delta_{s,t}(w)+\varepsilon_m\}\end{aligned}$$

with $K=(T/h+1)K'$, as was to be proved.

§ 6. Convergence of ξ_m to ξ.

In this section we prove Theorem 2. Before proving the theorem, we prepare two lemmas.

LEMMA 6.1. *For sufficiently large m, n ($m<n$), we have*

$$\begin{aligned}&\langle \xi_m(t)-\xi_n(t),\ d\phi_m(t)-d\phi_n(t)\rangle\\ &\leqq \left\{\frac{\kappa^2}{2r_0}|\xi_m(t)-\xi_n(t)|^2+2\varepsilon_m\right\}(d|\phi_m|_t+d|\phi_n|_t) ,\end{aligned} \tag{6.1}$$

where κ is the constant in Lemma 2.1.

PROOF. It is easy to see that

$$\langle \xi_m(t)-\xi_n(t),\ d\phi_m(t)\rangle=\langle \xi_m(t)-\xi_n(t),\ \boldsymbol{n}\rangle d|\phi_m|_t ,$$

where $\boldsymbol{n}$ is defined as in the proof of Lemma 5.1 with replacement of u by t. Since $\xi_m(t)$, $\xi_n(t)\in D_{\varepsilon_m}$ for large m, n ($m<n$) by Proposition 4.1, using (1.2) and Lemma 2.1 we have

$$\langle \xi_m(t)-\xi_n(t),\ \boldsymbol{n}\rangle \leqq \langle \overline{\xi_m(t)}-\overline{\xi_n(t)},\ \boldsymbol{n}\rangle+2\varepsilon_m$$

$$\leqq \frac{1}{2r_0}|\overline{\xi_m(t)}-\overline{\xi_n(t)}|^2+2\varepsilon_m$$
$$\leqq \frac{\kappa^2}{2r_0}|\xi_m(t)-\xi_n(t)|^2+2\varepsilon_m \ .$$

Therefore,

$$\begin{aligned} (6.2) \qquad & \langle \xi_m(t)-\xi_n(t),\ d\phi_m(t)\rangle \\ & \leqq \left\{\frac{\kappa^2}{2r_0}|\xi_m(t)-\xi_n(t)|^2+2\varepsilon_m\right\} d\|\phi_m\|_t \ . \end{aligned}$$

Similarly we have

$$\begin{aligned} (6.3) \qquad & \langle \xi_n(t)-\xi_m(t),\ d\phi_n(t)\rangle \\ & \leqq \left\{\frac{\kappa^2}{2r_0}|\xi_m(t)-\xi_n(t)|^2+2\varepsilon_m\right\} d\|\phi_n\|_t \ . \end{aligned}$$

Combining (6.2) and (6.3), we have (6.1). The proof is finished.

LEMMA 6.2. *ξ_m, ϕ_m converge uniformly in $t\in[0, T]$ as $m\to\infty$ for any T $(0<T<\infty)$.*

PROOF. By Lemma 6.1 we have for sufficiently large m, n $(m<n)$,

$$\begin{aligned} & \langle \xi_m(t)-\xi_n(t),\ d\phi_m(t)-d\phi_n(t)\rangle \\ & \leqq \left\{\frac{\kappa^2}{2r_0}|\xi_m(t)-\xi_n(t)|^2+2\varepsilon_m\right\}(d\|\phi_m\|_t+d\|\phi_n\|_t) \ . \end{aligned}$$

On the other hand, we have

$$\begin{aligned} (6.4) \qquad |\xi_m(t)-\xi_n(t)|^2 &= |x_m-x_n|^2+|\phi_m(t)-\phi_n(t)|^2 \\ & \quad +2\langle x_m-x_n,\ \phi_m(t)-\phi_n(t)\rangle \\ &= |x_m-x_n|^2+2\int_0^t \langle \phi_m(s)-\phi_n(s),\ d\phi_m(s)-d\phi_n(s)\rangle \\ & \quad +2\int_0^t \langle x_m-x_n,\ d\phi_m(s)-d\phi_n(s)\rangle \\ &= |x_m-x_n|^2+2\int_0^t \langle \xi_m(s)-\xi_n(s),\ d\phi_m(s)-d\phi_n(s)\rangle \\ &\leqq |x_m-x_n|^2+2\int_0^t \frac{\kappa^2}{2r_0}|\xi_m(s)-\xi_n(s)|^2(d\|\phi_m\|_s+d\|\phi_n\|_s) \\ & \quad +4\varepsilon_m(\|\phi_m\|_t+\|\phi_n\|_t) \ . \end{aligned}$$

By Proposition 5.1 and Gronwall's lemma, (6.4) implies that $\xi_m(t)$ converges uniformly in $t\in[0, T]$ as $m\to\infty$. Therefore, if we denote the limit by ξ and setting

$$\phi(t)=\xi(t)-x-w(t) ,$$

we have that $\phi_m(t)$ converges to $\phi(t)$ uniformly in $t \in [0, T]$ as m tends to ∞. The proof is finished.

PROOF OF THEOREM 2. By Lemma 6.2, (ξ_m, ϕ_m) converges to (ξ, ϕ) as $m \to \infty$ and hence it is enough to show that (ξ, ϕ) satisfies the Skorohod equation (1.3). But this can be done, by a method similar to that of the proof of Theorem 4.1 of [9], by checking

$$|\phi|_t=\int_0^t 1(\xi(s) \in \partial D)d|\phi|_s , \tag{6.5}$$

$$\phi(t)=\int_0^t \boldsymbol{n}(s)d|\phi|_s , \qquad \boldsymbol{n}(s) \in \mathfrak{N}_{\xi(s)}, \ \xi(s) \in \partial D . \tag{6.6}$$

§ 7. Proof of Theorem 1.

Let D satisfy Conditions (A) and (B) as before and let $(\Omega, \mathscr{F}, P)$ be a probability space, $X(0) \in \bar{D}$ be an $\mathscr{F}_0$-measurable random variable and $B(t)$ be a d-dimensional $\mathscr{F}_t$-Brownian motion with $B(0)=0$, where $\{\mathscr{F}_t\}$ is a filtration such that $\mathscr{F}_0$ contains all P-negligible sets and $\mathscr{F}_t = \bigcap_{\varepsilon>0} \mathscr{F}_{t+\varepsilon}$. We adopt the following notation:

$C_0(\boldsymbol{R}^d)$ (resp. $C_0(\bar{D})$)
= the space of continuous functions in $\boldsymbol{R}^d$ (resp. on $\bar{D}$) with compact supports,
$\hat{C}(\boldsymbol{R}^d)$ = the space of continuous functions in $\boldsymbol{R}^d$ vanishing at infinity.

LEMMA 7.1. *Let $V \in C^1(\boldsymbol{R}^d)$ and suppose that ∇V is bounded and Lipschitz continuous. Denote by T_t^V the semigroup of the diffusion process in $\boldsymbol{R}^d$ obtained by solving the stochastic differential equation*

$$dX(t)=dB(t)-\frac{1}{2}\nabla V(X(t))dt . \tag{7.1}$$

Then

(i) $\|T_t^V f\|_\nu \leqq \|f\|_\nu , \quad \forall f \in L^2(\nu) ,$

(ii) $\langle\!\langle T_t^V f, g\rangle\!\rangle_\nu = \langle\!\langle f, T_t^V g\rangle\!\rangle_\nu , \quad \forall f, g \in L^2(\nu) ,$

where $d\nu=\exp\{-V\}dx$ and $\|\cdot\|_\nu$, $\langle\!\langle\cdot, \cdot\rangle\!\rangle_\nu$ are the L^2-norm and L^2-inner product, respectively, in the space $L^2(\nu)$.

PROOF. A more general result containing this lemma is found in Fukushima and Stroock [4], but we give a proof here because it is particularly simple in the present special case. We first notice that T_t^V is a strongly continuous semigroup on $\hat{C}(\boldsymbol{R}^d)$ ([1: Theorem 5.11]). Denote by $A \equiv A_V$ the generator of the strongly continuous semigroup and by $L \equiv L_V$ the differential operator $(1/2)\Delta-(1/2)\nabla V\cdot\nabla$. Then Theorem 1.6 of [2: Chap. 8] (see also its proof) $(L, C_0^2(\boldsymbol{R}^d))$ is a core for A. Also it is easy to check $\langle\langle Lf, g\rangle\rangle_\nu=\langle\langle f, Lg\rangle\rangle_\nu$ for any $f, g \in C_0^2(\boldsymbol{R}^d)$. Therefore

$$\langle\langle Af, g\rangle\rangle_\nu=\langle\langle f, Ag\rangle\rangle_\nu\ , \qquad \forall f, g \in \mathscr{D}(A) \tag{7.2}$$

provided that $\nu(\boldsymbol{R}^d)<\infty$. Now the proof of the lemma is divided into two cases.

Case 1. $\nu(\boldsymbol{R}^d)<\infty$. In this case (7.2) implies

$$\langle\langle u, (\lambda-A)v\rangle\rangle_\nu=\langle\langle(\lambda-A)u, v\rangle\rangle_\nu \tag{7.3}$$

for any $u=(\lambda-A)^{-1}f$ and $v=(\lambda-A)^{-1}g$ with $f, g \in \hat{C}(\boldsymbol{R}^d)$ and $\lambda>0$. But (7.3) is nothing but $\langle\langle(\lambda-A)^{-1}f, g\rangle\rangle_\nu=\langle\langle f, (\lambda-A)^{-1}g\rangle\rangle_\nu$, i.e.,

$$\left\langle\left\langle\int_0^\infty e^{-\lambda t}T_t^V f dt,\ g\right\rangle\right\rangle_\nu=\left\langle\left\langle f,\ \int_0^\infty e^{-\lambda t}T_t^V g dt\right\rangle\right\rangle_\nu$$

from which (ii) follows for $f, g \in \hat{C}(\boldsymbol{R}^d)$ and then for $f, g \in L^2(\nu)$. Setting $g=1$ in (ii) we see that ν is invariant under T_t^V from which (i) follows after an application of the Schwarz inequality.

Case 2. $\nu(\boldsymbol{R}^d)=\infty$. In this case we approximate V by V_n for which the result of Case 1 can be applied. Choose a function $\chi_n \in C^2([0, \infty))$ such that

$$\chi_n(t)=\begin{cases}1 & \text{for } 0\leqq x\leqq n\ ,\\ \text{between 0 and 1} & \text{for } n<x<n+1\ ,\\ 0 & \text{for } n+1\leqq x\ ,\end{cases}$$

and set

$$V_n(x)= V(x)\chi_n(|x|)+\{1-\chi_n(|x|)\}|x|^{1/2}\ ,$$
$$\nu_n(dx)=\exp\{-V_n\}dx\ .$$

Then $V_n \in C^1(\boldsymbol{R}^d)$, ∇V_n is bounded and Lipschitz continuous and $\nu_n(\boldsymbol{R}^d)<\infty$. Therefore the result of Case 1 implies

$$\|T_t^{V_n}f\|_{\nu_n}\leqq\|f\|_{\nu_n}\ , \qquad \forall f \in L^2(\nu_n)\ , \tag{7.4}$$

$$\langle\langle T_t^{V_n}f, g\rangle\rangle_{\nu_n}=\langle\langle f, T_t^{V_n}g\rangle\rangle_{\nu_n}\ , \qquad f, g \in L^2(\nu_n)\ . \tag{7.5}$$

On the other hand we easily see that $T_t^{V_n} f \to T_t^V f$ (bounded convergence) as $n \to \infty$ for any $f \in \hat{C}(\boldsymbol{R}^d)$. Therefore, letting n tend to ∞ in (7.4) with $f \in C_0(\boldsymbol{R}^d)$ and using Fatou's inequality, we obtain (i) first for $f \in C_0(\boldsymbol{R}^d)$ and then for $f \in L^2(\nu)$. (ii) can also be derived from (7.5) by the same approximation method. The proof is finished.

PROOF OF THEOREM 1. Let X_m be the diffusion process determined by

$$dX(t) = dB(t) - \frac{m}{2} \nabla U(X(t))dt \tag{7.6}$$

and denote by $T_t^{(m)}$ the corresponding semigroup. Let $\mu_m(x) = \exp\{-mU(x)\}$ and $\mu_m(dx) = \mu_m(x)dx$. Then by Lemma 7.1, we have

$$\|T_t^{(m)} f\|_{\mu_m} \leqq \|f\|_{\mu_m}, \qquad \forall f \in L^2(\mu_m). \tag{7.7}$$

$$\langle\!\langle T_t^{(m)} f, g \rangle\!\rangle_{\mu_m} = \langle\!\langle f, T_t^{(m)} g \rangle\!\rangle_{\mu_m}, \qquad f, g \in L^2(\mu_m). \tag{7.8}$$

If we write the solution of (1.6) (resp. (7.6) with $X(0)=x$) by $X(t, x)$ (resp. $X_m(t, x)$), then Theorem 2 implies that $X_m(t, x) \to X(t, x)$ as $m \to \infty$ ($x \in \bar{D}$) and (7.8) yields

$$\begin{aligned} &\int_{\boldsymbol{R}^d} E[f(X_m(t, x))] g(x) \mu_m(x) dx \\ &= \int_{\boldsymbol{R}^d} f(x) E[g(X_m(t, x))] \mu_m(x) dx, \qquad f, g \in C_0(\boldsymbol{R}^d). \end{aligned} \tag{7.9}$$

Therefore noting that $\mu_m(x) = 1$ on $\bar{D}$ and $\mu_m(x) \to 0$ in $\boldsymbol{R}^d \setminus \bar{D}$, and using Lebesgue's dominated convergence theorem, we have for $f, g \in C_0(\boldsymbol{R}^d)$

$$\int_{\bar{D}} E[f(X(t, x))] g(x) dx = \int_{\bar{D}} f(x) E[g(X(t, x))] dx.$$

Since any function in $C_0(\bar{D})$ can be extended to a function in $C_0(\boldsymbol{R}^d)$ by Tietze's theorem, we have

$$\langle\!\langle T_t f, g \rangle\!\rangle = \langle\!\langle f, T_t g \rangle\!\rangle, \qquad f, g \in C_0(\bar{D}). \tag{7.10}$$

Choosing a sequence $\{g_n\}$ in $C_0(\bar{D})$ such that $0 \leqq g_n \uparrow 1$ on $\bar{D}$ as $n \uparrow \infty$, we see that $\langle\!\langle T_t f, 1 \rangle\!\rangle = \langle\!\langle f, 1 \rangle\!\rangle$, $f \in C_0(\bar{D})$, and this means that the Lebesgue measure on $\bar{D}$ is invariant for the diffusion X. (i) follows from this fact. Since $C_0(\bar{D})$ is dense in $L^2(\bar{D})$, (ii) follows from (7.10). The proof is finished.

REMARK. 7.1. Under Condition (A) the Lebesgue measure of ∂D is zero, because .

$$\overline{\lim_{\varepsilon \downarrow 0}} \frac{\text{the Lebesgue measure of } B(x, \varepsilon) \cap \partial D}{\text{the Lebesgue measure of } B(x, \varepsilon)} < 1$$

for each $x \in \partial D$.

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