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The Universality of the Spaces of Ultradistributions $\mathscr{E}_s(\boldsymbol{T})^\wedge$, $\mathscr{E}_{(s)}(\boldsymbol{T})^\wedge$, $(0<s\leqq\infty)$, $\mathscr{E}_0(\boldsymbol{T})^\wedge$ and $\mathrm{Exp}(\boldsymbol{C}^\times)^\wedge$

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Introduction

L. Waelbroeck [11] proved the universality of the space $\mathscr{E}(V)^\wedge$ of Schwartz-distributions with compact support on a C^∞-manifold V with the δ-function mapping $\delta: V\to\mathscr{E}(V)^\wedge$, i.e., any vector valued C^∞-mapping $f: V\to E$ factors through $\delta: V\to\mathscr{E}(V)^\wedge$ by a uniquely determined linear morphism $f^\sim: \mathscr{E}(V)^\wedge\to E$ as $f=f^\sim\circ\delta$. For the unit circle $\boldsymbol{T}$, we proved in a previous paper [9] that any vector valued C^ω-mapping $f: \boldsymbol{T}\to E$ factors through $\delta: \boldsymbol{T}\to\mathscr{B}(\boldsymbol{T})$ where $\mathscr{B}(\boldsymbol{T})$ is the space of Sato-hyperfunctions on $\boldsymbol{T}$. In the case of Schwartz-distributions, Waelbroeck used a notion of b-spaces, and for Sato-hyperfunctions we used a notion of ib-spaces (intersections of b-spaces).

In this paper we prove the universality for the spaces of ultradistributions of various kinds on the unit circle $\boldsymbol{T}$. We represent these spaces as linear subspaces of $\boldsymbol{C}^{\boldsymbol{Z}}$ (called here sequence spaces) using Fourier coefficients. For a sequence space $E\subset\boldsymbol{C}^{\boldsymbol{Z}}$, Köthe [4] defined the dual (called α-dual by him) by

$$E^\wedge=\{v\in\boldsymbol{C}^{\boldsymbol{Z}}\mid \text{for all } u\in E,\ \textstyle\sum_j |u_j|\,|v_j|<+\infty\}\,.$$

We consider functionals only of *sequential type*, i.e., $v: E\to\boldsymbol{C}$ is represented as $v(u)=\langle u, v\rangle=\sum_j u_jv_j$ with $v\in E^\wedge$. A sequence space E is *perfect* if $E=E^{\wedge\wedge}$. A linear mapping $f: E\to F$ between two sequence spaces is called *sequential* if for every $v\in F^\wedge$, the composed mapping $v\circ f\in E^\wedge$. If E and F are perfect, $f: E\to F$ is sequential if and only if f is represented by an infinite matrix $(f_{ij})\in\boldsymbol{C}^{\boldsymbol{Z}\times\boldsymbol{Z}}$ such that

$$\sum_j |f_{kj}|\,|u_j|<+\infty \qquad\text{and}\qquad \sum_k |v_k|\,|\sum_j f_{kj}u_j|<+\infty$$

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for any $v \in F^{\wedge}$ and $u \in E$.

Thus, we develop the theory without topology replacing completeness by perfectness of the spaces and continuity by sequentialness of the mappings. The universality theorem is presented in an abstract form using two axioms on the space $\mathscr{E}(\boldsymbol{T})$ which is a function space on $\boldsymbol{T}$ but identifiable as a sequence space. We require that each $\varphi \in \mathscr{E}(\boldsymbol{T})$ has absolutely convergent Fourier series, i.e., $\mathscr{E}(\boldsymbol{T}) \subset \mathscr{A}(\boldsymbol{T})$ or in sequential terminology $\mathscr{E}(\boldsymbol{T}) \subset l^1(\boldsymbol{Z})$ (Axiom 2.1). Moreover, we require that $\mathscr{E}(\boldsymbol{T})$ be perfect (Axiom 2.2). The vector valued ultradifferentiable mappings are replaced by $\mathscr{E}(\boldsymbol{T})$-mappings f: $\boldsymbol{T} \to E$, i.e., for any $v \in E^{\wedge}$, the composed mapping $v \circ f \in \mathscr{E}(\boldsymbol{T})$. Then we prove the Abstract Universality Theorem 2.4., i.e., δ: $\boldsymbol{T} \to \mathscr{E}(\boldsymbol{T})$ is a $\mathscr{E}(\boldsymbol{T})$-mapping; for any sequential mapping $f^{\sim}$: $\mathscr{E}(\boldsymbol{T}) \to E$, $f^{\sim} \circ \delta$: $\boldsymbol{T} \to E$ is a $\mathscr{E}(\boldsymbol{T})$-mapping; and any $\mathscr{E}(\boldsymbol{T})$-mapping f: $\boldsymbol{T} \to E$ is of the form $f = f^{\sim} \circ \delta$ with unique $f^{\sim}$.

Since the presentation is axiomatic, our ultradistributions start from $\mathscr{E}_{\infty}(\boldsymbol{T})^{\wedge}$ = Schwartz-distributions to all the way up to $\mathscr{E}_0(\boldsymbol{T})^{\wedge}$ = the space of all formal trigonometric series. In between, Sato-hyperfunctions $\mathscr{E}_1(\boldsymbol{T})^{\wedge}$, Morimoto's cohomological hyperfunctions $\mathscr{E}_{(1)}(\boldsymbol{T})^{\wedge}$ and entire functionals $\mathrm{Exp}(\boldsymbol{C}^{\times})^{\wedge}$ are situated as follows: For $0 < s < 1$ and $t > 1$,

$$\mathscr{E}_{(s)}(\boldsymbol{T})^{\wedge} \supset \mathscr{E}_s(\boldsymbol{T})^{\wedge} \supset \mathrm{Exp}(\boldsymbol{C}^{\times})^{\wedge} \supset \mathscr{E}_{(1)}(\boldsymbol{T})^{\wedge} \supset \mathscr{E}_1(\boldsymbol{T})^{\wedge} \supset \mathscr{E}_t(\boldsymbol{T})^{\wedge}$$

where $\mathscr{E}_{(s)}(\boldsymbol{T})^{\wedge}$ is the space of ultradistributions of Gevrey-Beurling type and $\mathscr{E}_s(\boldsymbol{T})^{\wedge}$, $\mathscr{E}_t(\boldsymbol{T})^{\wedge}$ are that of Gevrey-Roumieu type for any $0 < s, t < \infty$.

We would like to thank Prof. M. Morimoto for sending his manuscripts and pointing out some similarities with his own work on $\mathrm{Exp}(\boldsymbol{C}^{\times})$ which is included in § 5. The existence of this paper owes to Prof. H. Komatsu. After his kind advices we succeeded to formulate our abstract universality theorem without appealing any topology which we thought indispensable. Finally, our thanks go to the referee for his kind advices which improved greatly the presentation of this paper.

§ 1. Sequential spaces and sequential linear mappings.

DEFINITION 1.1. A *sequential space* E is a linear subspace of

$$\boldsymbol{C}^{\boldsymbol{Z}} = \{a = (a_j) \mid a_j \in \boldsymbol{C},\ j \in \boldsymbol{Z}\} .$$

The *dual* $E^{\wedge}$ (α-dual in the terminology of G. Köthe) is a sequence space defined by

$$E^{\wedge} = \{a \in \boldsymbol{C}^{\boldsymbol{Z}} \mid \text{for all } u \in E,\ \textstyle\sum_j |u_j|\,|a_j| < +\infty\} .$$

DEFINITION 1.2. A sequence space E is called *perfect* if $E^{\wedge\wedge}=E$.

EXAMPLE 1.3. The space $\boldsymbol{C}^{\boldsymbol{Z}}$ is the largest perfect space for which $(\boldsymbol{C}^{\boldsymbol{Z}})^{\wedge}=\boldsymbol{C}^{(\boldsymbol{Z})}=$ the space of all finite sequences. Since $E\subset F$ implies $E^{\wedge}\supset F^{\wedge}$, $\boldsymbol{C}^{(\boldsymbol{Z})}$ is the smallest perfect space so that any perfect space E contains $\boldsymbol{C}^{(\boldsymbol{Z})}$.

EXAMPLE 1.4. For any space E, its dual $E^{\wedge}$ is perfect. In fact, by definition $E\subset E^{\wedge\wedge}$ so that $E^{\wedge}\supset E^{\wedge\wedge\wedge}$. But since $E^{\wedge}\subset(E^{\wedge})^{\wedge\wedge}$, we have $E^{\wedge}=E^{\wedge\wedge\wedge}$.

DEFINITION 1.5. A sequence space is *normal* if $u=(u_j)\in E$ implies $|u|=(|u_j|)\in E$.

A dual space $E^{\wedge}$ is normal so that any perfect space is normal.

DEFINITION 1.6. A pairing $\langle\ ,\ \rangle_E$ on E is a bilinear function on $E\times E^{\wedge}$ defined by

$$\langle u, v\rangle_E=\sum_j u_j v_j\in \boldsymbol{C}\ ,$$

which converges absolutely by the definition of $E^{\wedge}$.

DEFINITION 1.7. $\varphi: E\to \boldsymbol{C}$ is called a *sequential linear functional* if there exists some $a\in E^{\wedge}$ so that $\varphi(u)=\langle u, a\rangle_E$ for all $u\in E$. We abuse the notation $a: E\to \boldsymbol{C}$ for this mapping.

DEFINITION 1.8. A mapping $f: E\to F$ between two sequential spaces is called a *sequential linear mapping* if

i) f is algebraically linear,

ii) for any $v\in F^{\wedge}$, the composed mapping $v\circ f\in E^{\wedge}$.

ADJOINTNESS THEOREM 1.9. *Let E, F be two perfect spaces. Then a linear mapping $f: E\to F$ is sequential if and only if f is represented by an infinite matrix $(f_{kj})\in \boldsymbol{C}^{\boldsymbol{Z}\times\boldsymbol{Z}}$ such that for any $u\in E$, $v\in F^{\wedge}$,*

$$(*)\qquad \sum_j |f_{kj}|\,|u_j|<+\infty\ ,\qquad \sum_k |v_k|\,|\sum_j f_{kj}u_j|<+\infty\ .$$

The adjoint mapping $f^{\wedge}: F^{\wedge}\to E^{\wedge}$ defined by the composite $f^{\wedge}v=v\circ f$ is also sequential and the transposed matrix ${}^t(f_{kj})$ represents $f^{\wedge}$. Mappings f and $f^{\wedge}$ are related by the adjointness relation:

$$\langle fu, v\rangle_F=\langle u, f^{\wedge}v\rangle_E \qquad \text{for any } u\in E,\ v\in F^{\wedge}\ .$$

For the proof of this Adjointness Theorem 1.9, we need the following lemma:

LEMMA. *Let E, F be perfect spaces and $(f_{kj})\in \boldsymbol{C}^{\boldsymbol{Z}\times\boldsymbol{Z}}$ be an infinite* *matrix satisfying* (*), *then for any* $u\in E$ *and* $v\in F^{\wedge}$, *we have*

$$\lim_{n\to\infty}\sum_k |v_k|\cdot|\sum_{|j|\leqq n} f_{kj}u_j|=\sum_k |v_k|\cdot|\sum_j f_{kj}u_j| .$$

PROOF OF LEMMA. For any $u\in E$, let us define its n-cut $u^{(n)}=(u_j^{(n)})$ by

$$u_j^{(n)}=\begin{cases}u_j & (|j|\leqq n)\\ 0 & (|j|>n)\end{cases} \qquad (n=1, 2, \cdots) .$$

Then for any $w\in E^{\wedge}$, we have

$$\langle u-u^{(n)}, w\rangle \to 0 \qquad \text{as } n\to\infty ,$$

because the fact $\sum_k |u_k w_k|<+\infty$ implies that

$$|\langle u-u^{(n)}, w\rangle|\leqq \sum_{|k|>n}|u_k w_k|\to 0 \qquad \text{as } n\to\infty .$$

Now for any $u\in E$ and $v\in F^{\wedge}$, we have

$$\langle fu, v\rangle=\sum_{j,k} f_{kj}u_j v_k=\langle u, v\circ f\rangle$$

and, since E is perfect, i.e.,

$$\sum_k |u_k|\cdot|v\circ f_k|<+\infty ,$$

we can conclude that $v\circ f\in E^{\wedge}$. Moreover, for any $v\in F^{\wedge}$, we have $v\circ f\in E^{\wedge}$, so that

$$\langle fu-fu^{(n)}, v\rangle=\langle u-u^{(n)}, v\circ f\rangle \to 0 \qquad \text{as } n\to\infty .$$

That is, for any $u\in E$ and $v\in F^{\wedge}$,

$$\lim_{n\to\infty}\sum_k |v_k|\cdot\sum_{|j|\leqq n} f_{kj}u_j=\sum_k |v_k|\cdot\sum_j f_{kj}u_j .$$

Using the fact that F is perfect, we can replace $\sum_{|j|\leqq n} f_{kj}u_j$ and $\sum_j f_{kj}u_j$ by their absolute values respectively:

$$\lim_{n\to\infty}\sum_k |v_k|\cdot|\sum_{|j|\leqq n} f_{kj}u_j|=\sum_k |v_k|\cdot|\sum_j f_{kj}u_j| . \qquad \Box$$

PROOF OF ADJOINTNESS THEOREM 1.9. Let $f\colon E\to F$ be sequential, i.e., for any $v\in F^{\wedge}$, the composed mapping $v\circ f\in E^{\wedge}$. Let $e_k=(e_{kj})\in F^{\wedge}$ be the functional defined by

$$e_{kj}=\begin{cases}1 & k=j\\ 0 & k\neq j .\end{cases}$$

Then $e_k \circ f = (f_{kj}) \in E^{\wedge}$, i.e., for any $u \in E$, $\sum_j |f_{kj}| \, |u_j| < +\infty$. For any $u \in E$, $e_k \circ fu$ is the k-th coordinate of $fu \in F$ hence

$$(fu)_k = \sum_j f_{kj} u_j \ ,$$

i.e., f is represented by the matrix $(f_{kj}) \in C^{Z \times Z}$ satisfying (*).

Let (f_{kj}) be an infinite matrix satisfying (*). Taking $u = e_j$ which belongs to E such that $\sum_l f_{kl} e_{jl} = f_{kj}$, we have $\sum_k |v_k| \, |f_{kj}| < +\infty$. Consider, for any finite n, the sum

$$I_n = \sum_{|j| \leq n} |\sum_k v_k f_{kj}| \, |u_j| \ .$$

We can choose $\varepsilon_j \in C$, $|\varepsilon_j| = 1$ such that

$$|\sum_k v_k f_{kj} u_j| = \sum_k v_k f_{kj} \varepsilon_j u_j \ .$$

Since $(\varepsilon_j u_j) \in E$, we have

$$\begin{aligned} I_n &= \sum_{|j| \leq n} \sum_k v_k f_{kj} \varepsilon_j u_j \\ &= |\sum_k v_k \sum_{|j| \leq n} f_{kj} \varepsilon_j u_j| \\ &\leqq \sum_k |v_k| \, |\sum_{|j| \leq n} f_{kj} \varepsilon_j u_j| \ . \end{aligned}$$

Letting $n \to \infty$ we get from Lemma

$$\begin{aligned} \lim_{n \to \infty} I_n &= \sum_j |\sum_k v_k f_{kj}| \, |u_j| \\ &\leqq \lim_{n \to \infty} \sum_k |v_k| \, |\sum_{|j| \leq n} f_{kj} \varepsilon_j u_j| \\ &= \sum_k |v_k| \, |\sum_j f_{kj} \varepsilon_j u_j| < +\infty \ , \end{aligned}$$

i.e., if a matrix (f_{kj}) satisfies (*), it satisfies also

$$(**) \qquad \sum_k |v_k| \, |f_{kj}| < +\infty \ , \qquad \sum_j |\sum_k v_k f_{kj}| \, |u_j| < +\infty$$

for any $u \in E$, $v \in F^{\wedge}$. Now, let $f \colon E \to F$ be defined by $fu = (\sum_j f_{kj} u_j)$ for any $u \in E$, then for any $v \in F^{\wedge}$, the composed mapping $v \circ f \colon E \to C$ is given by

$$v \circ fu = \sum_k v_k (\sum_j f_{kj} u_j) = \sum_j (\sum_k v_k f_{kj}) u_j \ .$$

Since by (**)

$$\sum_j |\sum_k v_k f_{kj}| \, |u_j| < +\infty \ ,$$

$f^\wedge v=(\sum_k v_k f_{kj})\in E^\wedge$, i.e., f is sequential.

Finally, using (*) and (**) we have, for any $u\in E$, $v\in F^\wedge$,

$$\langle fu, v\rangle_F=\sum_k v_k(\sum_j f_{kj}u_j)=\sum_j(\sum_k v_k f_{kj})u_j=\langle u, f^\wedge v\rangle_E$$

as asserted. □

EXAMPLE 1.10. For any two perfect spaces E and F we denote by $L(E, F)$ the space of all sequential linear mappings from E to F. Then it is easy to determine the structure of the following spaces:

$L(\boldsymbol{C}^{(\boldsymbol{Z})}, \boldsymbol{C}^{\boldsymbol{Z}})=\boldsymbol{C}^{\boldsymbol{Z}\times\boldsymbol{Z}}=$ all matrices.

$L(\boldsymbol{C}^{\boldsymbol{Z}}, \boldsymbol{C}^{(\boldsymbol{Z})})=\boldsymbol{C}^{(\boldsymbol{Z}\times\boldsymbol{Z})}=$ all matrices of finite supports.

As a consequence we have:

$$\boldsymbol{C}^{(\boldsymbol{Z}\times\boldsymbol{Z})}\subset L(E, F)\subset\boldsymbol{C}^{\boldsymbol{Z}\times\boldsymbol{Z}}$$

for any perfect spaces E and F.

§2. Abstract Universality Theorem.

Let $\mathscr{E}(\boldsymbol{T})$ be a $\boldsymbol{C}$-valued function space on the unit circle $\boldsymbol{T}$. We propose the following axioms for $\mathscr{E}(\boldsymbol{T})$:

AXIOM 2.1. For any $\varphi\in\mathscr{E}(\boldsymbol{T})$, the Fourier coefficients

$$\varphi_j=\frac{1}{2\pi}\int_0^{2\pi}\varphi(t)e^{-ijt}dt$$

are defined and for $\varphi\neq 0$, $0<\sum_j|\varphi_j|<+\infty$, i.e., $\mathscr{E}(\boldsymbol{T})\subset\mathscr{A}(\boldsymbol{T})=$ the space of all functions having absolutely convergent Fourier series. Imbedding $\mathscr{E}(\boldsymbol{T})$ into $\boldsymbol{C}^{\boldsymbol{Z}}$ using Fourier coefficients this means that we require $\mathscr{E}(\boldsymbol{T})\subset l^1(\boldsymbol{Z})$.

AXIOM 2.2. Under the above identification, $\mathscr{E}(\boldsymbol{T})$ is a perfect space.

REMARK. Axioms 2.1 and 2.2 are independent. In fact, $\mathscr{E}(\boldsymbol{T})=L^2(\boldsymbol{T})$ satisfies Axiom 2.2 but not 2.1.

Take any integer $N>0$ and consider

$$\mathscr{E}(\boldsymbol{T})=\{\varphi\in\mathscr{A}(\boldsymbol{T})\mid \text{for any } j\in\boldsymbol{Z}, |j|>N, \varphi_j=0\}.$$

Then $\mathscr{E}(\boldsymbol{T})$ satisfies Axiom 2.1 but not 2.2.

DEFINITION 2.3. A mapping $f\colon\boldsymbol{T}\to E$ from the unit circle $\boldsymbol{T}$ into a sequence space E is a *$\mathscr{E}(\boldsymbol{T})$-mapping* if for any $u\in E^\wedge$, the composed mapping $u\circ f\colon\boldsymbol{T}\to\boldsymbol{C}$ belongs to $\mathscr{E}(\boldsymbol{T})$.

ABSTRACT UNIVERSALITY THEOREM 2.4. *Let $\mathscr{E}(\boldsymbol{T})$ be a function space on $\boldsymbol{T}$ satisfying Axioms 2.1 and 2.2. Then we have:*

(i) *The delta mapping $\delta: \boldsymbol{T} \to \mathscr{E}(\boldsymbol{T})^\wedge$ defined by*

$$\langle \varphi, \delta_t \rangle = \varphi(t) \quad \text{for all} \quad \varphi \in \mathscr{E}(\boldsymbol{T})$$

is a $\mathscr{E}(\boldsymbol{T})$-mapping.

(ii) *If $g^\sim: \mathscr{E}(\boldsymbol{T})^\wedge \to E$ is a sequential linear mapping, then the composed mapping $g^\sim \circ \delta: \boldsymbol{T} \to E$ is a $\mathscr{E}(\boldsymbol{T})$-mapping.*

(iii) *For any $\mathscr{E}(\boldsymbol{T})$-mapping $f: \boldsymbol{T} \to E$, there exists a unique sequential linear mapping $f^\sim: \mathscr{E}(\boldsymbol{T})^\wedge \to E$ so that $f = f^\sim \circ \delta$.*

PROOF. (i) Take $v \in \mathscr{E}(\boldsymbol{T})^{\wedge\wedge} = \mathscr{E}(\boldsymbol{T})$. Then the composed mapping

$$v \circ \delta_t = \langle v, \delta_t \rangle = v(t)$$

is, by the definition of δ-mapping, an element of $\mathscr{E}(\boldsymbol{T})$.

(ii) Let $u \in E^\wedge$. Then by definition, $u \circ g^\sim \in \mathscr{E}(\boldsymbol{T})^{\wedge\wedge} = \mathscr{E}(\boldsymbol{T})$. Hence $u \circ g^\sim \circ \delta_t = \langle u \circ g^\sim, \delta_t \rangle = u \circ g^\sim(t)$ belongs to $\mathscr{E}(\boldsymbol{T})$.

(iii) Existence of $f^\sim: \mathscr{E}(\boldsymbol{T})^\wedge \to E$. By hypothesis, $f: \boldsymbol{T} \to E$ is a $\mathscr{E}(\boldsymbol{T})$-mapping so that for any $v \in E^\wedge$, $v \circ f \in \mathscr{E}(\boldsymbol{T})$. Therefore, a sequential linear mapping $E^\wedge \to \mathscr{E}(\boldsymbol{T})$ is defined. Hence by Adjointness Theorem 1.9, there is an adjoint mapping $f^\sim: \mathscr{E}(\boldsymbol{T})^\wedge \to E^{\wedge\wedge} = E$ which is a sequential mapping. By the definition of adjoint mapping $f^\sim$ we have

$$\langle u, v \circ f \rangle_{\mathscr{E}(T)^\wedge} = \langle f^\sim \circ u, v \rangle_E \, .$$

For $u = \delta_t$, this gives

$$\langle \delta_t, v \circ f \rangle_{\mathscr{E}(T)^\wedge} = v \circ f(t) = \langle f(t), v \rangle_E = \langle f^\sim \delta_t, v \rangle_E$$

for any $v \in E^\wedge$. This proves $f = f^\sim \circ \delta$.

Unicity of $f^\sim: \mathscr{E}(\boldsymbol{T})^\wedge \to E$. Suppose $f^\sim \circ \delta = f = 0$. We have to show that $f^\sim = 0$ on $\mathscr{E}(\boldsymbol{T})^\wedge$. Since $f^\sim$ is sequential, by Adjointness Theorem 1.9 there exists $g: E^\wedge \to \mathscr{E}(\boldsymbol{T})$ such that

$$\langle f^\sim u, v \rangle_E = \langle u, gv \rangle_{\mathscr{E}(T)^\wedge} \, .$$

Take $u = \delta_t \in \mathscr{E}(\boldsymbol{T})^\wedge$, then

$$\langle f^\sim \delta_t, v \rangle_E = \langle \delta_t, gv \rangle_{\mathscr{E}(T)^\wedge} = gv(t) = 0$$

for any $v \in E^\wedge$, i.e., $g = 0$ on $E^\wedge$. From $\langle f^\sim u, v \rangle_E = 0$ for any $v \in E^\wedge$, we conclude that $f^\sim u = 0$ for any $u \in \mathscr{E}(\boldsymbol{T})^\wedge$, i.e., $f^\sim = 0$ on $\mathscr{E}(\boldsymbol{T})^\wedge$. □

§ 3. Gevrey classes $\mathscr{E}_s(\boldsymbol{T})$, $\mathscr{E}_{(s)}(\boldsymbol{T})$ for $0 < s \leqq \infty$.

DEFINITION 3.1. Gevrey classes of functions on $\boldsymbol{T} = \boldsymbol{R}/2\pi\boldsymbol{Z}$ are defined as follows: for any $s\in \boldsymbol{R}$, $\varphi\in\mathscr{E}_s(\boldsymbol{T})$ if and only if $\varphi\in C^\infty(\boldsymbol{T})$ and there exist some $A>0$ and $C>0$ so that for any integer $n\geqq 0$

$$\|\varphi^{(n)}\|_\infty=\sup_{0\leqq t\leqq 2\pi}\left|\frac{d^n\varphi(t)}{dt^n}\right|\leqq C\cdot A^n(n!)^s .$$

Similarly, for any $s\in\boldsymbol{R}$, $\varphi\in\mathscr{E}_{(s)}(\boldsymbol{T})$ if and only if $\varphi\in C^\infty(\boldsymbol{T})$ and for any $A>0$ there exists $C_A>0$ so that for any integer $n\geqq 0$

$$\|\varphi^{(n)}\|_\infty=\sup_{0\leqq t\leqq 2\pi}\left|\frac{d^n\varphi(t)}{dt^n}\right|\leqq C_A\cdot A^n(n!)^s .$$

REMARKS. For $s=\infty$, we define

$$\mathscr{E}_\infty(\boldsymbol{T})=\mathscr{E}_{(\infty)}(\boldsymbol{T})=\mathscr{D}(\boldsymbol{T})=C^\infty(\boldsymbol{T})$$

to be the space of infinitely differentiable functions.

The following facts are easy consequences of the above definitions. For $s<0$, $\mathscr{E}_s(\boldsymbol{T})=\mathscr{E}_{(s)}(\boldsymbol{T})=\boldsymbol{C}$, i.e., the space of all constant functions. For $s=0$, $\mathscr{E}_{(0)}(\boldsymbol{T})=\boldsymbol{C}$ but $\mathscr{E}_0(\boldsymbol{T})$ is the space of all finite linear combinations of e^{ijt} ($i=\sqrt{-1}$ and $j\in\boldsymbol{Z}$). For $s=1$, $\mathscr{E}_1(\boldsymbol{T})=C^\omega(\boldsymbol{T})$ is the space of all analytic functions on $\boldsymbol{T}$.

Following results concerning the Fourier coefficients of Gevrey classes of functions and their duals are proved in our paper [10]:

PROPOSITION 3.2. *Let*

$$\varphi_j=\frac{1}{2\pi}\int_0^{2\pi}\varphi(t)e^{-ijt}dt \qquad (j\in\boldsymbol{Z})$$

and $0<s<\infty$.

(i) $\varphi\in\mathscr{E}_s(\boldsymbol{T})$ *if and only if for some* $B>0$ *and* $K>0$ *we have*

$$|\varphi_j|\leqq K\cdot e^{-B|j|^{1/s}} \qquad (j\in\boldsymbol{Z}) .$$

(ii) $\varphi\in\mathscr{E}_{(s)}(\boldsymbol{T})$ *if and only if for any* $B>0$ *there exists* $K_B>0$ *such that*

$$|\varphi_j|\leqq K_B\cdot e^{-B|j|^{1/s}} \qquad (j\in\boldsymbol{Z}) .$$

We imbed $\mathscr{E}_s(\boldsymbol{T})$ and $\mathscr{E}_{(s)}(\boldsymbol{T})$ into $\boldsymbol{C}^{\boldsymbol{Z}}$ as its subspaces using Fourier coefficients. Then we have

PROPOSITION 3.3. *Suppose* $0<s<\infty$.

(i) $u\in\mathscr{E}_s(\boldsymbol{T})^\wedge$ *if and only if for any* $B>0$ *there exists* $K_B>0$ *such that*

$$|u_j|\leqq K_B\cdot e^{B|j|^{1/s}} \qquad (j\in\boldsymbol{Z}) .$$

(ii) *$u \in \mathscr{E}_{(s)}(\boldsymbol{T})\hat{}$ if and only if for some $B>0$ and $K>0$ we have*

$$|u_j| \leqq K \cdot e^{B|j|^{1/s}} \qquad (j \in \boldsymbol{Z}) .$$

It is known that $C^\infty \subset \mathscr{A}(\boldsymbol{T})$ so that $\mathscr{E}_s(\boldsymbol{T})$ and $\mathscr{E}_{(s)}(\boldsymbol{T})$ satisfy Axiom 2.1. From Propositions 3.2 and 3.3 it follows that $\mathscr{E}_s(\boldsymbol{T})$ and $\mathscr{E}_{(s)}(\boldsymbol{T})$ are perfect, i.e., they satisfy Axiom 2.2. For the case of $s=\infty$, perfectness follows from Köthe [3]:

$\varphi \in \mathscr{D}(\boldsymbol{T})$ if and only if for all $k=0, 1, 2, \cdots$, $\sum_j |j|^k |\varphi_j| < +\infty$,

$u \in \mathscr{D}(\boldsymbol{T})\hat{}$ if and only if for some $k \geqq 0$, $K>0$,

$$|u_j| \leqq K|j|^k \quad (j \in \boldsymbol{Z}) .$$

Hence, we have

THEOREM 3.4. *The space $\mathscr{E}_s(\boldsymbol{T})\hat{}$ with δ: $\boldsymbol{T} \to \mathscr{E}_s(\boldsymbol{T})\hat{}$ is universal for any $\mathscr{E}_s(\boldsymbol{T})$-mapping f: $\boldsymbol{T} \to E$ into any perfect space E. Similar statement holds for $\mathscr{E}_{(s)}(\boldsymbol{T})\hat{}$ for any s: $0<s \leqq \infty$.*

§ 4. Trigonometric polynomials $\mathscr{E}_0(\boldsymbol{T})$.

The space $\mathscr{E}_0(\boldsymbol{T})$ of polynomials in e^{it}, i.e., trigonometric polynomials is identical with the sequence space $\boldsymbol{C}^{(\boldsymbol{Z})}$, hence is perfect so that the space $\mathscr{E}_0(\boldsymbol{T})$ satisfies Axioms 2.1 and 2.2. We have

THEOREM 4.1. *The space $\mathscr{E}_0(\boldsymbol{T})\hat{}$ of all formal trigonometric series with δ: $\boldsymbol{T} \to \mathscr{E}_0(\boldsymbol{T})\hat{}$ given by $\delta_t = (e^{ijt})_{j \in \boldsymbol{Z}}$ is universal for any $\mathscr{E}_0(\boldsymbol{T})$-mapping f: $\boldsymbol{T} \to E$ into any perfect space E.*

In other words, the space $\mathscr{E}_0(\boldsymbol{T})\hat{}$ is universal for any weakly vector valued polynomial mappings.

§ 5. Entire functions of exponential type Exp($\boldsymbol{C}^\times$).

DEFINITION 5.1. A function φ on $\boldsymbol{C}^\times = \boldsymbol{C} \setminus \{0\}$ is entire of exponential type if there are $a \geqq 0$, $b \geqq 0$ such that

$$\|\varphi\|_{a,b} = \sup_{z \neq 0} (|\varphi(z)| e^{-a|z| - b|z|^{-1}}) < +\infty .$$

Exp($\boldsymbol{C}^\times$) is the space of such functions.

DEFINITION 5.2 Fourier coefficients of $\varphi \in \mathrm{Exp}(\boldsymbol{C}^\times)$ are defined by

$$\varphi_j = \frac{1}{2\pi} \int_0^{2\pi} \varphi(e^{it}) e^{-ijt} dt ,$$

φ being identified with its restriction on $\boldsymbol{T}$.

The following proposition holds (see Helgason [1], Morimoto [5]):

PROPOSITION 5.3. (i) *$\varphi \in \mathrm{Exp}(\boldsymbol{C}^\times)$ if and only if for some $K>0$ and $B>0$,*

$$|\varphi_j| \leqq K \frac{B^{|j|}}{|j|!} \qquad (j \in \boldsymbol{Z}) ,$$

(ii) *$u \in \mathrm{Exp}(\boldsymbol{C}^\times)^\wedge$ if and only if for all $B>0$,*

$$\sum_j |u_j| \frac{B^{|j|}}{|j|!} < +\infty .$$

From this it follows that the space $\mathrm{Exp}(\boldsymbol{C}^\times)$ considered as a sequential space is perfect hence satisfies Axioms 2.1 and 2.2. Hence we have

THEOREM 5.4. *The space of entire functionals $\mathrm{Exp}(\boldsymbol{C}^\times)^\wedge$ with δ: $\boldsymbol{T} \to \mathrm{Exp}(\boldsymbol{C}^\times)^\wedge$ is universal for any $\mathrm{Exp}(\boldsymbol{C}^\times)$-mapping f: $\boldsymbol{T} \to E$ to any perfect space E, i.e., $\mathrm{Exp}(\boldsymbol{C}^\times)^\wedge$ is universal for any vector valued weakly entire functions of exponential type.*

Here, we show connections with ultradistributions of Gevrey classes:

THEOREM 5.5. *For any s $(0<s<1)$, we have*

$$\mathscr{E}_s(\boldsymbol{T}) \subset \mathrm{Exp}(\boldsymbol{C}^\times) \subset \mathscr{E}_{(1)}(\boldsymbol{T}) .$$

PROOF. The first inclusion. From Propositions 3.2 (i) and 5.3 (i), it is sufficient to show that for $B>0$, $0<s<1$,

$$e^{-B|j|^{1/s}} = o\left(\frac{1}{|j|!}\right) \qquad \text{for } |j| \to \infty .$$

Let us write

$$C_j = |j|! \cdot e^{-B|j|^{1/s}} .$$

Using the Stirling formula:

$$|j|! \sim |j|^{|j|} e^{-|j|} \sqrt{2\pi |j|} = e^{|j| \cdot \log |j| - |j| + (1/2) \cdot \log 2\pi |j|} ,$$

we have

$$C_j \sim e^{|j| \cdot \log |j| - |j| + (1/2) \cdot \log 2\pi |j| - B|j|^{1/s}} \longrightarrow 0$$

for $|j| \to \infty$ as required.

The second inclusion. From Propositions 5.3 (i) and 3.2 (ii) it is sufficient to show that for any $B>0$,

$$\frac{B^{|j|}}{|j|!}=o(e^{-|j|}) \qquad \text{for } |j|\to\infty .$$

Let us write this time,

$$C_j=\frac{B^{|j|}}{|j|!}\cdot e^{|j|} .$$

Then,

$$C_j \sim e^{|j|\cdot B-|j|\cdot\log|j|+|j|-(1/2)\cdot\log 2\pi|j|+|j|} \longrightarrow 0$$

for $|j|\to\infty$ as required. □

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