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Holomorphic Functions on the Complex Sphere

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Introduction.

Let d be a positive integer and $d \geqq 2$ (see Remark in § 3 for $d=0, 1$). Let us denote by $\mathcal{O}(\boldsymbol{C}^{d+1})$ the space of entire functions on $\boldsymbol{C}^{d+1}$. Suppose λ is an arbitrary complex number and $\mathcal{O}_\lambda(\boldsymbol{C}^{d+1})=\{f \in \mathcal{O}(\boldsymbol{C}^{d+1}); \Delta_z f = -\lambda^2 f\}$, where $\Delta_z = \sum_{j=1}^{d+1} (\partial/\partial z_j)^2$.

Let us consider the algebraic variety $M_\rho=\{z \in \boldsymbol{C}^{d+1}; z^2=\rho^2\}$, where $z^2=\sum_{j=1}^{d+1} z_j^2$ and $\rho \in \boldsymbol{C}$. If $\rho=0$, M_0 is a complex cone and has a singularity at $z=0$. On the other hand if $\rho \neq 0$, M_ρ is a complex manifold and holomorphically diffeomorphic to the complex sphere $\widetilde{S}=M_1=\{z \in \boldsymbol{C}^{d+1}; z^2=1\}$ by the transformation $z \rightarrow \rho z$. The space $\mathcal{O}(M_\rho)$ of holomorphic functions on the analytic set M_ρ is equal to $\mathcal{O}(\boldsymbol{C}^{d+1})|_{M_\rho}$ by the Oka-Cartan Theorem B.

Our first main result is as follows:

THEOREM 1. *The restriction mapping $F \rightarrow F|_{M_\rho}$ is a linear topological isomorphism of $\mathcal{O}_\lambda(\boldsymbol{C}^{d+1})$ onto $\mathcal{O}(M_\rho)$ if*

$$(*) \qquad (\lambda\rho/2)^{-n-(d-1)/2} J_{n+(d-1)/2}(\lambda\rho) \neq 0$$

holds for $n=0, 1, 2, \cdots$, where J_ν is the Bessel function of order ν.

If $\rho=0$, the condition $(*)$ holds automatically and Theorem 1 was proved in [7] and [8]. The case $\rho \neq 0$ is proved in this paper (Theorem 2.1). Remark that Theorem 1 holds locally at the origin (see [9] for $\rho=0$ and Corollary 2.4 for $\rho \neq 0$). We may interpret Theorem 1 as saying that the set M_ρ is a uniqueness set for the differential operators $\Delta_z+\lambda^2$. We described in [10] a uniqueness set for more general linear partial differential operators of the second order with constant coefficients.

In [5] the space $\mathrm{Exp}(\widetilde{S})$ was defined to be the restriction to $\widetilde{S}$ of the space $\mathrm{Exp}(\boldsymbol{C}^{d+1})$ of entire functions of exponential type. But this defini-

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tion is unnatural. The space $\mathrm{Exp}(\tilde{S})$ ought to be the space of holomorphic functions defined solely on $\tilde{S}$ and satisfying the exponential type growth condition.

The second main result of this paper is that the above two definitions are equivalent. More precisely we will prove

THEOREM 2. *The restriction mapping* $F \to F|_{\tilde{S}}$ *is a linear topological isomorphism of* $\mathrm{Exp}_\lambda(\boldsymbol{C}^{d+1})$ *onto* $\mathrm{Exp}(\tilde{S})$ *if*

$$(\lambda/2)^{-n-(d-1)/2} J_{n+(d-1)/2}(\lambda) \neq 0$$

holds for $n=0, 1, 2, \cdots$, *where* $\mathrm{Exp}_\lambda(\boldsymbol{C}^{d+1}) = \mathcal{O}_\lambda(\boldsymbol{C}^{d+1}) \cap \mathrm{Exp}(\boldsymbol{C}^{d+1})$ (cf. Theorem 3.1).

In the previous papers [5] and [9] we used the Lie norm to define the exponential type of holomorphic functions. But by means of the dual Lie norm we can obtain more detailed informations on the exponential type as described in Theorem 3.1. Because $\mathrm{Exp}(\tilde{S}: (A: L)) = \mathrm{Exp}(\tilde{S}: (2A: L^*))$ our present results can be stated with the Lie norm in special cases (Corollary 3.5).

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§ 1. Preliminaries.

Let d be a positive integer and $d \geqq 2$. $S = S^d$ denotes the unit sphere in $\boldsymbol{R}^{d+1}$: $S = \{x \in \boldsymbol{R}^{d+1}; \|x\| = 1\}$, where $\|x\|^2 = x_1^2 + x_2^2 + \cdots + x_{d+1}^2$. ds denotes the unique $O(d+1)$ invariant measure on S with $\int_S 1 ds = 1$, where $O(k)$ is the orthogonal group of degree k. $\| \ \|_\infty$ is the sup norm on S. $H_{n,d}$ is the space of spherical harmonics of degree n in $(d+1)$ dimensions. For spherical harmonics, see Müller [6]. For $S_n \in H_{n,d}$, $\tilde{S}_n$ denotes the unique homogeneous harmonic polynomial of degree n on $\boldsymbol{C}^{d+1}$ such that $\tilde{S}_n|_S = S_n$.

For $z, \zeta \in \boldsymbol{C}^{d+1}$ we put $z \cdot \zeta = \sum_{j=1}^{d+1} z_j \zeta_j$. The Lie norm $L(z)$ on $\boldsymbol{C}^{d+1}$ is defined as follows:

$$L(z) = \{\|z\|^2 + (\|z\|^4 - |z^2|^2)^{1/2}\}^{1/2}, \tag{1.1}$$

where $z \in \boldsymbol{C}^{d+1}$, $\|z\| = (z \cdot \bar{z})^{1/2}$ and $z^2 = z \cdot z$ (see Drużkowski [1]).

We put

$$\tilde{B}(r) = \{z \in \boldsymbol{C}^{d+1}; L(z) < r\} \qquad \text{for} \quad 0 < r \leqq \infty$$

and

$$\tilde{B}[r] = \{z \in \boldsymbol{C}^{d+1}; L(z) \leqq r\} \qquad \text{for} \quad 0 \leqq r < \infty .$$

Let us denote by $\mathcal{O}(\tilde{B}(r))$ the space of holomorphic functions on $\tilde{B}(r)$. $\mathcal{O}(\tilde{B}(\infty))=\mathcal{O}(\boldsymbol{C}^{d+1})$ is the space of entire functions on $\boldsymbol{C}^{d+1}$. We define

$$\mathcal{O}(\tilde{B}[r])=\operatorname*{ind\,lim}_{r'>r}\mathcal{O}(\tilde{B}(r')) .$$

Let N be a norm on $\boldsymbol{C}^{d+1}$. For $A>0$ we put

$$X_{A,N}=\{f\in\mathcal{O}(\boldsymbol{C}^{d+1});\ \sup_{z\in \boldsymbol{C}^{d+1}}|f(z)|\exp(-AN(z))<\infty\}$$

and we define

$$\begin{aligned}
&\operatorname{Exp}(\boldsymbol{C}^{d+1}\colon(A\colon N))=\operatorname*{proj\,lim}_{A'>A}X_{A',N} \qquad \text{for}\quad 0\leqq A<\infty ,\\
&\operatorname{Exp}(\boldsymbol{C}^{d+1}\colon[A\colon N])=\operatorname*{ind\,lim}_{A'<A}X_{A',N} \qquad \text{for}\quad 0<A\leqq\infty .
\end{aligned}$$

$\operatorname{Exp}(\boldsymbol{C}^{d+1})=\operatorname{Exp}(\boldsymbol{C}^{d+1}\colon[\infty\colon N])$ is called the space of entire functions of exponential type.

$\tilde{S}$ is the complex sphere: $\tilde{S}=\{z\in\boldsymbol{C}^{d+1};\ z^2=1\}$. We put for $1<r\leqq\infty$

$$\tilde{S}(r)=\tilde{B}(r)\cap\tilde{S}$$

and for $1\leqq r<\infty$

$$\tilde{S}[r]=\tilde{B}[r]\cap\tilde{S} .$$

Let us denote by $\mathcal{O}(\tilde{S}(r))$ the space of holomorphic functions on $\tilde{S}(r)$ equipped with the topology of uniform convergence on every compact subset of $\tilde{S}(r)$. We put

$$\mathcal{O}(\tilde{S}[r])=\operatorname*{ind\,lim}_{r'>r}\mathcal{O}(\tilde{S}(r')) .$$

For these spaces, see [3] [4] [5].

If f is a function on S, we denote by $S_n(f;\ )$ the n-th spherical harmonic component of f:

$$S_n(f;\alpha)=N(n,d)\int_S f(s)P_{n,d}(\alpha\cdot s)ds \qquad \text{for}\quad \alpha\in S , \tag{1.2}$$

where

$$N(n,d)=\dim H_{n,d}=\frac{(2n+d-1)(n+d-2)!}{n!\,(d-1)!} \tag{1.3}$$

and $P_{n,d}$ is the Legendre polynomial of degree n and of dimension $d+1$. For Legendre polynomials, see for example [6]. We see that $S_n(f;\ )$ belongs to $H_{n,d}$ for $n=0,1,\cdots$.

$\mathcal{O}(\tilde{S}(r))$ and $\mathcal{O}(\tilde{S}[r])$ can be characterized by the behavior of the spherical harmonic development as follows:

Lemma 1.1 (Morimoto [5] Theorems 5.1 and 5.2). *If S_n is the n-th spherical harmonic component of f, then*

$$f \in \mathcal{O}(\tilde{S}(r)) \Leftrightarrow \limsup_{n\to\infty} \|S_n\|_\infty^{1/n} \leqq 1/r \qquad (1 < r \leqq \infty), \tag{1.4}$$

$$f \in \mathcal{O}(\tilde{S}[r]) \Leftrightarrow \limsup_{n\to\infty} \|S_n\|_\infty^{1/n} < 1/r \qquad (1 \leqq r < \infty). \tag{1.5}$$

Put $\Lambda_+ = \{(n, k) \in \boldsymbol{Z}_+^2;\ n \equiv k \pmod 2 \text{ and } n \geqq k\}$, where $\boldsymbol{Z}_+ = \{0, 1, 2, \cdots\}$. For any $F \in \mathcal{O}(\tilde{B}(r))$ we can determine uniquely $S_{n,k}(F;\ ) \in H_{k,d}$ for every $(n, k) \in \Lambda_+$ in such a way that

$$F(z) = \sum_{(n,k)\in\Lambda_+} (\sqrt{z^2})^{n-k} \tilde{S}_{n,k}(F; z). \tag{1.6}$$

The right hand side of (1.6) converges uniformly on every compact subset of $\tilde{B}(r)$. The $S_{n,k}(F;\ )$ is called the (n, k)-component of F (see [3]).

For $\lambda \in \boldsymbol{C}$ we put $\mathcal{O}_\lambda(\tilde{B}(r)) = \{f \in \mathcal{O}(\tilde{B}(r));\ \Delta_z f(z) = -\lambda^2 f(z)\}$ and $\mathcal{O}_\lambda(\tilde{B}[r]) = \text{ind}\lim_{r'>r} \mathcal{O}_\lambda(\tilde{B}(r'))$, where $\Delta_z = (\partial/\partial z_1)^2 + (\partial/\partial z_2)^2 + \cdots + (\partial/\partial z_{d+1})^2$.

§ 2. Holomorphic functions on $\tilde{S}$.

We recall the definition of the Bessel function of order ν ($\nu \neq -1, -2, \cdots$):

$$J_\nu(t) = (t/2)^\nu \sum_{k=0}^\infty \frac{(-1)^k (t/2)^{2k}}{k!\,\Gamma(\nu+k+1)}. \tag{2.1}$$

Our first main theorem in this paper is the following:

Theorem 2.1. *Let $\lambda \in \boldsymbol{C}$ and*

$$(\lambda/2)^{-n-(d-1)/2} J_{n+(d-1)/2}(\lambda) \neq 0$$

for any $n \in \boldsymbol{Z}_+$. Then the restriction mapping $F \to F|_{\tilde{S}}$ defines the following linear topological isomorphism:

$$\alpha_\lambda\colon \mathcal{O}_\lambda(\boldsymbol{C}^{d+1}) \longrightarrow \mathcal{O}(\tilde{S}). \tag{2.2}$$

In order to prove the theorem we need the following

Lemma 2.2 (cf. [3] [4] [9]). *Let $F \in \mathcal{O}_\lambda(\tilde{B}(r))$ (resp. $F \in \mathcal{O}_\lambda(\tilde{B}[r])$) and $S_{n,k}$ be the (n, k)-component of F. Then we have*

$$S_{n,k} = \frac{(i\lambda/2)^{n-k}\Gamma(k+(d+1)/2)}{\Gamma((n-k)/2+1)\Gamma((n+k+d+1)/2)} S_{k,k} \tag{2.3}$$

for $(n, k) \in \Lambda_+$ and

(2.4) $$\limsup_{n\to\infty}\|S_{n,n}\|_\infty^{1/n}\leqq 1/r$$

(*resp.*

(2.4′) $$\limsup_{n\to\infty}\|S_{n,n}\|_\infty^{1/n}<1/r)\ .$$

Conversely if we are given a sequence of spherical harmonics $\{S_{n,k}\}_{(n,k)\in\Lambda_+}$ *satisfying* (2.3) *and* (2.4) (*resp.* (2.3) *and* (2.4′)) *and if we put for* $z\in\tilde{B}(r)$ (*resp.* $z\in\tilde{B}[r]$)

(2.5) $$F(z)=\sum_{(n,k)\in\Lambda_+}(\sqrt{z^2})^{n-k}\tilde{S}_{n,k}(z)\ ,$$

then the right hand side of (2.5) *converges uniformly and absolutely on every compact subset of* $\tilde{B}(r)$ (*resp. the right hand side of* (2.5) *converges in the topology of* $\mathcal{O}_\lambda(\tilde{B}[r])$) *and* F *belongs to* $\mathcal{O}_\lambda(\tilde{B}(r))$ (*resp.* F *belongs to* $\mathcal{O}_\lambda(\tilde{B}[r])$). *Furthermore we have*

$$\tilde{S}_{n,k}(z)=\tilde{S}_{n,k}(F;z)\qquad for\quad (n,k)\in\Lambda_+\ .$$

PROOF OF THEOREM 2.1. It is clear that $\mathcal{O}_\lambda(C^{d+1})|_{\tilde{S}}\subset\mathcal{O}(C^{d+1})|_{\tilde{S}}=\mathcal{O}(\tilde{S})$. Let $F\in\mathcal{O}_\lambda(C^{d+1})$. From (1.6) and (2.3) we have for $z\in C^{d+1}$

(2.6) $$F(z)=\sum_{k=0}^{\infty}(\lambda\sqrt{z^2}/2)^{-k-(d-1)/2}J_{k+(d-1)/2}(\lambda\sqrt{z^2})\Gamma(k+(d+1)/2)\tilde{S}_{k,k}(z)\ ,$$

where $S_{k,k}$ is the (k,k)-component of F. If $\alpha_\lambda(F)=0$ we have

(2.7) $$F(s)=\sum_{k=0}^{\infty}(\lambda/2)^{-k-(d-1)/2}J_{k+(d-1)/2}(\lambda)\Gamma(k+(d+1)/2)S_{k,k}(s)=0$$

for any $s\in S$. Since $(\lambda/2)^{-k-(d-1)/2}J_{k+(d-1)/2}(\lambda)\Gamma(k+(d+1)/2)\neq 0$ for any $k\in Z_+$, $S_{k,k}=0$ on S by the orthogonality of spherical harmonics. So $\tilde{S}_{k,k}=0$ on C^{d+1} and $F=0$ on C^{d+1} by (2.6). Therefore α_λ is injective.

Next for $f\in\mathcal{O}(\tilde{S})$ we determine the function F as follows:

(2.8) $$F(z)=\sum_{(n,k)\in\Lambda_+}(\sqrt{z^2})^{n-k}\tilde{S}_{n,k}(z)\ ,$$

where

(2.9) $$\tilde{S}_{n,k}(z)=\frac{(\lambda/2)^{n+(d-1)/2}(-1)^{(n-k)/2}\tilde{S}_k(f;z)}{\Gamma((n-k)/2+1)\Gamma((n+k+d+1)/2)J_{k+(d-1)/2}(\lambda)}\ .$$

As $f\in\mathcal{O}(\tilde{S})$, we get

(2.10) $$\limsup_{k\to\infty}\|S_k(f;\ \ )\|_\infty^{1/k}=0$$

by (1.4). For sufficiently large k we have

(2.11) $$|\Gamma(k+(d+1)/2)(\lambda/2)^{-k-(d-1)/2}J_{k+(d-1)/2}(\lambda)|>1/2 .$$

(2.9), (2.10) and (2.11) imply

(2.12) $$\limsup_{k\to\infty}\|S_{k,k}\|_\infty^{1/k}=\limsup_{k\to\infty}\left\|\frac{(\lambda/2)^{k+(d-1)/2}S_k(f;\quad)}{\Gamma(k+(d+1)/2)J_{k+(d-1)/2}(\lambda)}\right\|_\infty^{1/k}$$
$$\leqq\limsup_{k\to\infty}\|S_k(f;\quad)\|_\infty^{1/k}=0 .$$

By Lemma 2.2, (2.9) and (2.12), we see $F\in\mathcal{O}_\lambda(C^{d+1})$. When $z\in\tilde{S}$, we have by (2.8) and (2.9)

(2.13) $$F(z)=\sum_{k=0}^\infty \tilde{S}_k(f;z)=f(z) .$$

Therefore α_λ is bijective.

It is clear that α_λ is continuous. As $\mathcal{O}_\lambda(C^{d+1})$ and $\mathcal{O}(\tilde{S})$ are FS spaces, α_λ^{-1} is also continuous by the closed graph theorem. Q.E.D.

When $(\lambda/2)^{-n-(d-1)/2}J_{n+(d-1)/2}(\lambda)=0$ for some $n\in Z_+$, it is known that λ is real and that $(\lambda/2)^{-k-(d-1)/2}J_{k+(d-1)/2}(\lambda)\neq 0$ for any $k\in Z_+$ with $k\neq n$. So we have

COROLLARY 2.3. *Suppose* $(\lambda/2)^{-n-(d-1)/2}J_{n+(d-1)/2}(\lambda)=0$ *for some* $n\in Z_+$. $F\in\mathcal{O}_\lambda(C^{d+1})$ *belongs to the kernel of* α_λ *if and only if* F *is expressed in the form of*

$$F(z)=(\lambda\sqrt{z^2}/2)^{-n-(d-1)/2}J_{n+(d-1)/2}(\lambda\sqrt{z^2})\tilde{S}_n(z) ,$$

where $S_n\in H_{n,d}$.

$f\in\mathcal{O}(\tilde{S})$ *belongs to the images of* α_λ *if and only if the* n*-th spherical component* $S_n(f;\quad)$ *vanishes.*

For $\rho\in C$ we put

$$M_\rho=\{z\in C^{d+1};\ z^2=\rho^2\} .$$

It is clear that $\tilde{S}=M_1$. We put for $0\leqq|\rho|<r\leqq\infty$

$$M_\rho(r)=M_\rho\cap\tilde{B}(r)$$

and for $0\leqq|\rho|\leqq r<\infty$

$$M_\rho[r]=M_\rho\cap\tilde{B}[r] .$$

$\mathcal{O}(M_\rho(r))$ denotes the space of the holomorphic functions on $M_\rho(r)$ equipped with the topology of uniform convergence on every compact subset of $M_\rho(r)$. We define $\mathcal{O}(M_\rho[r])=\text{ind}\lim_{r'>r}\mathcal{O}(M_\rho(r'))$.

COROLLARY 2.4. *Let* $\lambda, \rho \in \boldsymbol{C}$ *and* $(\lambda\rho/2)^{-n-(d-1)/2}J_{n+(d-1)/2}(\lambda\rho)\neq 0$ *for any* $n \in \boldsymbol{Z}_+$. *Then the following restriction mappings are linear topological isomorphisms*:

$$(2.14) \qquad \alpha_{\lambda,\rho}: \mathcal{O}_\lambda(\tilde{B}(r)) \xrightarrow{\sim} \mathcal{O}(M_\rho(r)) \qquad \textit{for} \quad 0 \leqq |\rho| < r \leqq \infty ,$$

$$(2.15) \qquad \alpha_{\lambda,\rho}: \mathcal{O}_\lambda(\tilde{B}[r]) \xrightarrow{\sim} \mathcal{O}(M_\rho[r]) \qquad \textit{for} \quad 0 \leqq |\rho| \leqq r < \infty .$$

We can prove Corollary 2.4 in the same way as in the proof of Theorem 2.1 if we use Lemmas 1.1 and 2.2.

REMARK. The case $\lambda=0$ and $\rho=1$ is known (see [4] [5]). The case $\rho=0$ is proved in [9].

§ 3. Holomorphic functions of exponential type on $\tilde{S}$.

We put for a norm $N(z)$ on $\boldsymbol{C}^{d+1}$ and $A>0$

$$Y_{A,N}=\{f \in \mathcal{O}(\tilde{S});\ \sup_{z\in\tilde{S}}|f(z)|\exp(-AN(z))<\infty\} .$$

$Y_{A,N}$ is a Banach space with respect to the norm

$$\|f\|_{A,N}=\sup_{z\in\tilde{S}}|f(z)|\exp(-AN(z)) .$$

We define

$$\mathrm{Exp}(\tilde{S}\colon (A\colon N))=\operatorname*{proj\,lim}_{A'>A} Y_{A',N} \qquad \text{for} \quad 0\leqq A<\infty ,$$
$$\mathrm{Exp}(\tilde{S}\colon [A\colon N])=\operatorname*{ind\,lim}_{A'<A} Y_{A',N} \qquad \text{for} \quad 0< A\leqq\infty .$$

The dual Lie norm $L^*(z)$ on $\boldsymbol{C}^{d+1}$ is defined as follows:

$$(3.1) \qquad L^*(z)=\sup\{|z\cdot\zeta|;\ L(\zeta)\leqq 1\} = \left(\frac{\|z\|^2+|z^2|}{2}\right)^{1/2}$$

(see [1]).

The second main theorem in this paper is the following:

THEOREM 3.1. *Suppose* $\lambda\in\boldsymbol{C}$ *and*

$$(\lambda/2)^{-k-(d-1)/2}J_{k+(d-1)/2}(\lambda)\neq 0$$

for any $k\in\boldsymbol{Z}_+$. *Then the following restriction mappings are linear topological isomorphisms*:

$$(3.2) \qquad \alpha_\lambda: \mathrm{Exp}_\lambda(\boldsymbol{C}^{d+1}\colon (A\colon L^*)) \xrightarrow{\sim} \mathrm{Exp}(\tilde{S}\colon (A\colon L^*)) \qquad (|\lambda|\leqq A<\infty) ,$$

$$\alpha_\lambda: \mathrm{Exp}_\lambda(\boldsymbol{C}^{d+1}: [A: L^*]) \xrightarrow{\sim} \mathrm{Exp}(\tilde{S}: [A: L^*]) \qquad (|\lambda|<A\leqq\infty), \tag{3.3}$$

where

$$\mathrm{Exp}_\lambda(\boldsymbol{C}^{d+1}: (A: N))=\mathcal{O}_\lambda(\boldsymbol{C}^{d+1})\cap\mathrm{Exp}(\boldsymbol{C}^{d+1}: (A: N))$$

and

$$\mathrm{Exp}_\lambda(\boldsymbol{C}^{d+1}: [A: N])=\mathcal{O}_\lambda(\boldsymbol{C}^{d+1})\cap\mathrm{Exp}(\boldsymbol{C}^{d+1}: [A: N]) .$$

We put $S[r]=\{z\in\tilde{S}; L^*(z)=r\}=\{x+iy; \|x\|=r, \|y\|=(r^2-1)^{1/2}$ and $x\cdot y=0\}$ for $1<r$, where $x, y\in\boldsymbol{R}^{d+1}$. Remark that $O(d+1)$ acts on $S[r]$ by matrix multiplications and we have $S[r]\simeq O(d+1)/O(d-1)$. In order to prove the theorem we need the following lemmas.

LEMMA 3.2. *For any $\alpha, s\in S$ we have*

$$N(n, d)\int_{S[r]}\overline{P_{n,d}(z\cdot\alpha)}P_{k,d}(z\cdot s)dz = C_n(r)P_{n,d}(\alpha\cdot s)\delta_{n,k}, \tag{3.4}$$

where

$$C_n(r)=P_{n,d}(2r^2-1) \tag{3.5}$$

and dz denotes a unique $O(d+1)$-invariant measure on $S[r]$ with $\int_{S[r]}1dz=1$.

PROOF. We put for $s, \alpha\in S$

$$F(s, \alpha)=N(n, d)\int_{S[r]}P_{k,d}(z\cdot s)\overline{P_{n,d}(z\cdot\alpha)}dz . \tag{3.6}$$

Then for any orthogonal matrix A

$$\begin{aligned}F(As, A\alpha)&=N(n, d)\int_{S[r]}P_{k,d}(z\cdot As)\overline{P_{n,d}(z\cdot A\alpha)}dz\\&=N(n, d)\int_{S[r]}P_{k,d}(A^{-1}z\cdot s)\overline{P_{n,d}(A^{-1}z\cdot\alpha)}dz .\end{aligned}$$

Since dz is $O(d+1)$-invariant we obtain

$$F(As, A\alpha)=F(s, \alpha) \tag{3.7}$$

for any $A\in O(d+1)$. As a function of s, $F(s, \alpha)$ belongs to $H_{k,d}$ and as a function of α, $F(s, \alpha)$ belongs to $H_{n,d}$ because $P_{k,d}(z\cdot s)\in H_{k,d}$ and $\overline{P_{n,d}(z\cdot\alpha)}=P_{n,d}(\bar{z}\cdot\alpha)\in H_{n,d}$ if $z\in\tilde{S}$.

Suppose $n\neq k$. There exists an $A\in O(d+1)$ such that $A\alpha=s$ and $As=\alpha$.

Then (3.7) implies

$$F(\alpha, s)=F(s, \alpha) . \tag{3.8}$$

If we fix α, (3.8) gives that $F(s, \alpha) \in H_{k,d} \cap H_{n,d}=\{0\}$. So we have

$$F(s, \alpha)=0 \quad \text{if} \quad k \neq n . \tag{3.9}$$

Next we assume $n=k$. For any $A \in O(d+1)$ such that $A\alpha=\alpha$ we have from (3.7) $F(As, \alpha)=F(As, A\alpha)=F(s, \alpha)$. Hence we get

$$F(s, \alpha)=C_n(r)P_{n,d}(s\cdot\alpha) , \tag{3.10}$$

where

$$C_n(r)=F(\alpha, \alpha)=N(n, d)\int_{S[r]}|P_{n,d}(z\cdot\alpha)|^2 dz \tag{3.11}$$

(see Müller [6]). Remark that $C_n(r)$ does not depend on α because $F(\alpha, \alpha)=F(A\alpha, A\alpha)$ for any $A \in O(d+1)$ by (3.7). So it is valid that

$$\begin{aligned} C_n(r)&=\int_S C_n(r)ds \\ &=N(n, d)\int_{S[r]}\int_S |P_{n,d}(z\cdot s)|^2 ds dz . \end{aligned} \tag{3.12}$$

As $P_{n,d}(\ \cdot z) \in H_{n,d}$ for any $z \in \tilde{S}$ we have

$$N(n, d)\int_S \tilde{P}_{n,d}(\zeta\cdot s)P_{n,d}(z\cdot s)ds=\tilde{P}_{n,d}(\zeta\cdot z)$$

for any $\zeta \in \boldsymbol{C}^{d+1}$. If $\zeta \in \tilde{S}$, $\tilde{P}_{n,d}(\zeta\cdot s)=P_{n,d}(\zeta\cdot s)$. So we obtain for any $z \in \tilde{S}$

$$N(n, d)\int_S |P_{n,d}(z\cdot s)|^2 ds=\tilde{P}_{n,d}(\bar{z}\cdot z)=P_{n,d}(\|z\|^2) . \tag{3.13}$$

If $z \in S[r]$, $L^*(z)^2=(\|z\|^2+1)/2=r^2$ from (3.1). Therefore we see that $\|z\|^2=2r^2-1$ and we obtain (3.5) from (3.12) and (3.13). (3.4) follows from (3.6), (3.9) and (3.10). Q.E.D.

LEMMA 3.3. *Suppose $f \in \mathcal{O}(\tilde{S})$. Then the n-th spherical harmonic component $S_n(f;\ )$ is represented as follows:*

$$S_n(f; \alpha)=\frac{N(n, d)}{C_n(r)}\int_{S[r]} f(z)\overline{P_{n,d}(z\cdot\alpha)}dz \qquad \textit{for} \quad \alpha \in S . \tag{3.14}$$

PROOF. Since the series $\sum_{k=0}^{\infty} \tilde{S}_k(f; z)$ converges to f uniformly and absolutely on $S[r]$, we have

$$(3.15)\qquad \int_{S[r]} f(z)\overline{P_{n,d}(z\cdot\alpha)}dz = \sum_{k=0}^{\infty}\int_{S[r]} \widetilde{S}_k(f;z)\overline{P_{n,d}(z\cdot\alpha)}dz .$$

It is known that there exists a system of $N(k, d)$ points $\alpha_1, \alpha_2, \cdots, \alpha_{N(k,d)} \in S$ such that $P_{k,d}(\alpha_j\cdot\ )$, $j=1, 2, \cdots, N(k, d)$, is a basis of $H_{k,d}$. Therefore Lemma 3.2 and (3.15) imply (3.14). Q.E.D.

LEMMA 3.4. *Let $F \in \mathcal{O}_\lambda(C^{d+1})$ and $S_{k,k}$ be the (k, k)-component of F. If we have*

$$(3.16)\qquad \limsup_{k\to\infty}(k!\,\|\widetilde{S}_{k,k}\|_{L^*})^{1/k} \leqq A \qquad (|\lambda| \leqq A < \infty) ,$$

(resp.

$$(3.16')\qquad \limsup_{k\to\infty}(k!\,\|\widetilde{S}_{k,k}\|_{L^*})^{1/k} < A \qquad (|\lambda| < A \leqq \infty)) ,$$

then F belongs to $\mathrm{Exp}_\lambda(C^{d+1}: (A: L^*))$ *(resp.* $\mathrm{Exp}_\lambda(C^{d+1}: [A: L^*])$*), where* $\|g\|_{L^*} = \sup\{|g(z)|;\ L^*(z) \leqq 1\}$.

PROOF. If $\lambda=0$ Lemma 3.4 can be proved easily by Lemma 4.2 in [5].

Suppose $\lambda \neq 0$, $F \in \mathcal{O}_\lambda(C^{d+1})$ and $\{S_{k,k}\}_{k\in Z_+}$ satisfies (3.16). Then by (3.16) for any $A' > A$ there is some $C_{A'} > 0$ such that

$$(3.17)\qquad |\widetilde{S}_{k,k}(z)| \leqq C_{A'}(k!)^{-1}A'^k \qquad \text{if} \quad L^*(z) \leqq 1 .$$

From (1.6) and (3.17) we have for $z \in M_0$

$$(3.18)\qquad |F(z)| \leqq \sum_{k=0}^{\infty} |\widetilde{S}_{k,k}(z)| \leqq C_{A'} \sum_{k=0}^{\infty} \frac{1}{k!}(A'L^*(z))^k = C_{A'} \exp(A'L^*(z)) .$$

It is known that for any $f \in \mathcal{O}(C^{d+1})$ such that $\sup_{z\in M_0}|f(z)|\exp(-A'L^*(z)) < \infty$ for any $A' > A$ (resp. some $A' < A$), there exists a unique $g \in \mathrm{Exp}_\lambda(C^{d+1}: (A: L^*))$ (resp. $g \in \mathrm{Exp}_\lambda(C^{d+1}: [A: L^*])$) such that $f=g$ on M_0 if $|\lambda| \leqq A < \infty$ (resp. $|\lambda| < A \leqq \infty$) (cf. [8] Corollary 3.4). Therefore we can see F belongs to $\mathrm{Exp}_\lambda(C^{d+1}: (A: L^*))$ by (3.18). The rest of the proof is a routine argument. Q.E.D.

PROOF OF THEOREM 3.1. It is clear that $\alpha_\lambda(\mathrm{Exp}_\lambda(C^{d+1}: (A: L^*))) \subset \mathrm{Exp}(\widetilde{S}: (A: L^*))$ and α_λ is injective by Theorem 2.1.

Suppose $f \in \mathrm{Exp}(\widetilde{S}: (A: L^*))$. Then for any $A' > A$ there is some $C_{A'} > 0$ such that

$$(3.19)\qquad |f(z)| \leqq C_{A'} \exp(A'L^*(z))$$

on $\tilde{S}$. By Theorem 2.1 there exists a unique $F\in\mathcal{O}_\lambda(C^{d+1})$ such that $\alpha_\lambda(F)=f$ and by (2.6)

$$\tilde{S}_{k,k}(F; z)=\frac{(\lambda/2)^{k+(d-1)/2}}{\Gamma(k+(d+1)/2)J_{k+(d-1)/2}(\lambda)}\tilde{S}_k(f; z)\ . \tag{3.20}$$

It is known

$$P_{n,d}(t)=\frac{\Gamma(d/2)}{\Gamma((d-1)/2)\sqrt{\pi}}\int_{-1}^{1}(t+\sqrt{t^2-1}x)^n(1-x^2)^{(d-3)/2}dx$$

for $t\geqq 1$. Therefore we have for $t\geqq 1$

$$\begin{aligned}|P_{n,d}(t)|&\geqq\frac{\Gamma(d/2)(t^2-1)^{n/2}}{\Gamma((d-1)/2)\sqrt{\pi}}\int_{-1}^{1}(1+x)^n(1-x^2)^{(d-3)/2}dx\\&=\frac{2^{n+d-2}\Gamma(d/2)\Gamma(n+(d-1)/2)}{\Gamma(n+d-1)\sqrt{\pi}}(t^2-1)^{n/2}\\&\geqq 2^nC_d(n+d)^{-d}(t-1)^n\ ,\end{aligned}$$

where C_d is a constant which depends on d. Hence we get for any $r>1$

$$\begin{aligned}C_n(r)=P_{n,d}(2r^2-1)&\geqq 2^nC_d(n+d)^{-d}(2r^2-2)^n\\&\geqq 2^{2n}C_d(n+d)^{-d}(r-1)^{2n}\ .\end{aligned} \tag{3.21}$$

By (3.11), (3.14) and Schwarz' inequality, we have

$$\begin{aligned}\|S_n(f;\ \ )\|_\infty&\leqq\frac{N(n, d)}{C_n(r)}\left\{\int_{S[r]}|f(z)|^2dz\int_{S[r]}|P_{n,d}(z\cdot s)|^2dz\right\}^{1/2}\\&\leqq(N(n, d)/C_n(r))^{1/2}\sup_{z\in S[r]}|f(z)|\ .\end{aligned} \tag{3.22}$$

From (3.19), (3.21) and (3.22) we have

$$\|S_k(f;\ \ )\|_\infty\leqq C'_de^{A'}\frac{\sqrt{N(k, d)(k+d)^d}}{2^k(r-1)^k}e^{A'(r-1)} \tag{3.23}$$

for any $r>1$, where C'_d is a constant. If we put $r=k/A'+1$, (3.23) gives that

$$\|S_k(f;\ \ )\|_\infty\leqq C'_de^{A'}\left(\frac{A'e}{2k}\right)^k\sqrt{N(k, d)(k+d)^d}\ . \tag{3.24}$$

By Stirling's formula we have

$$\limsup_{k\to\infty}\left(\frac{k!e^k}{k^k}\right)^{1/k}=1\ .$$

So (3.24) implies

$$(3.25)\qquad \limsup_{k\to\infty}(k!\,\|S_k(f;\quad)\|_\infty)^{1/k}\leqq A'/2$$

for any $A'>A$. From (3.20), (3.25) and (2.11) we get

$$(3.26)\qquad \limsup_{k\to\infty}(k!\,\|S_{k,k}(F;\quad)\|_\infty)^{1/k}\leqq A/2\ .$$

From Remark 2.2 and (2.21) in [9] it is valid that

$$(3.27)\qquad \limsup_{k\to\infty}(k!\,\|\tilde{S}_{k,k}(F;\quad)\|_{L^*})^{1/k} = 2\limsup_{k\to\infty}(k!\,\|S_{k,k}(F;\quad)\|_\infty)^{1/k}\ .$$

Therefore we have by (3.26) and (3.27)

$$(3.28)\qquad \limsup_{k\to\infty}(k!\,\|\tilde{S}_{k,k}(F;\quad)\|_{L^*})^{1/k}\leqq A\ .$$

Lemma 3.4 and (3.28) imply that $F\in \mathrm{Exp}_\lambda(\boldsymbol{C}^{d+1}\colon (A\colon L^*))$. Hence α_λ is surjective.

Since α_λ is continuous and $\mathrm{Exp}_\lambda(\boldsymbol{C}^{d+1}\colon (A\colon L^*))$ and $\mathrm{Exp}(\tilde{S}\colon (A\colon L^*))$ are FS spaces, α_λ^{-1} is also continuous by the closed graph theorem. So we get (3.2). (3.3) can be proved similarly. Q.E.D.

COROLLARY 3.5. *The following restriction mappings are linear topological isomorphisms*:

$$(3.29)\qquad \alpha_0\colon \mathrm{Exp}_0(\boldsymbol{C}^{d+1}\colon (A\colon L)) \xrightarrow{\sim} \mathrm{Exp}(\tilde{S}\colon (A\colon L)) \qquad (0\leqq A<\infty)\ ,$$

$$(3.30)\qquad \alpha_0\colon \mathrm{Exp}_0(\boldsymbol{C}^{d+1}\colon [A\colon L]) \xrightarrow{\sim} \mathrm{Exp}(\tilde{S}\colon [A\colon L]) \qquad (0<A\leqq\infty)\ .$$

PROOF. It is easy to show that

$$(3.31)\qquad L^*(z)\leqq L(z)\leqq 2L^*(z)$$

for any $z\in\boldsymbol{C}^{d+1}$ and

$$(3.32)\qquad L^*(z)=\frac{L(z)+1/L(z)}{2}\leqq\frac{L(z)+1}{2}$$

for $z\in\tilde{S}$. Hence we get $\mathrm{Exp}(\tilde{S}\colon (A\colon L))=\mathrm{Exp}(\tilde{S}\colon (2A\colon L^*))$ and $\mathrm{Exp}_0(\boldsymbol{C}^{d+1}\colon (A\colon L))\subset \mathrm{Exp}_0(\boldsymbol{C}^{d+1}\colon (2A\colon L^*))$.

Suppose $F\in \mathrm{Exp}_0(\boldsymbol{C}^{d+1}\colon (2A\colon L^*))$. By Lemma 4.2 in [5] we have

$$\limsup_{k\to\infty}(k!\,\|\tilde{S}_{k,k}(F;\quad)\|_{L^*})^{1/k}\leqq 2A\ .$$

Hence by (3.27)

$$(3.33)\qquad \limsup_{k\to\infty}(k!\,\|S_{k,k}(F;\quad)\|_\infty)^{1/k}\leqq A\ .$$

(3.33) and Lemmas 4.2 and 5.5 in [5] give that $F\in \mathrm{Exp}_0(\boldsymbol{C}^{d+1}: (A: L))$. So we get $\mathrm{Exp}_0(\boldsymbol{C}^{d+1}: (2A: L^*))=\mathrm{Exp}_0(\boldsymbol{C}^{d+1}: (A: L))$. Therefore we obtain (3.29) from (3.2). We can prove (3.30) similarly. Q.E.D.

REMARK. If $d=0$ and $F\in \mathcal{O}_\lambda(\boldsymbol{C})$ we have

$$F(z)=a_0\cos\lambda z+a_1\lambda^{-1}\sin\lambda z \qquad (a_0,\ a_1\in \boldsymbol{C})\ .$$

So if $\lambda^{-1}\sin 2\lambda\rho\neq 0$ for any $C,\ C'\in \boldsymbol{C}$ there exists a unique $F\in\mathcal{O}_\lambda(\boldsymbol{C})$ such that $F(\rho)=C$ and $F(-\rho)=C'$ $(\rho\in\boldsymbol{C})$.

When $d=1$, for $z=(z_1,\ z_2)$ we put $u=(iz_1+z_2)/2$ and $v=(iz_1-z_2)/2$. Then we have $M_\rho=\{(u,\ v)\in \boldsymbol{C}^2;\ uv=-\rho^2/4\}$. If $\rho\neq 0$ we identify M_ρ with $\boldsymbol{C}^*=\boldsymbol{C}\setminus\{0\}$ and $\mathcal{O}(M_\rho)$ with $\mathcal{O}(\boldsymbol{C}^*)$ by the mapping $(u,\ v)\to u$. It is known that $L(z)=\max\{|2u|,\ |2v|\}$ and $L^*(z)=|u|+|v|$. Hence we see that $M_\rho(r)=\{u\in\boldsymbol{C};\ |\rho|^2/(2r)<|u|<r/2\}$ $(r>|\rho|>0)$ and $M_\rho[r]=\{u\in\boldsymbol{C};\ |\rho|^2/(2r)\leqq|u|\leqq r/2\}$ $(r\geqq|\rho|>0)$.

$F\in\mathcal{O}_\lambda(\boldsymbol{C}^2)$ is expressed as follows:

$$F(u,\ v)=a_{0,0}J_0(2i\lambda\sqrt{uv}) +\sum_{p=1}^{\infty}(a_{p,0}u^p+a_{0,p}v^p)p!\,(i\lambda\sqrt{uv})^{-p}J_p(2i\lambda\sqrt{uv})\ ,$$

where $a_{i,j}\in\boldsymbol{C}$ (cf. [2] Proposition 3.1). Therefore it is easy to show that Theorem 2.1 and Corollary 2.4 are valid for the case $d=1$ (for the case $\rho=0$ in Corollary 2.4, see [2] Theorem 3.1).

Furthermore, we have

$$Y_{A,L^*}=\left\{f\in\mathcal{O}(\boldsymbol{C}^*);\ \sup_{z\in \boldsymbol{C}^*}|f(u)|\exp\frac{-A}{2}\left(|2u|+\frac{1}{|2u|}\right)<\infty\right\}$$

and we also see that Theorem 3.1 is valid for the case $d=1$.

References

[1] L. DRUŻKOWSKI, Effective formula for the crossnorm in the complexified unitary spaces, Zeszyty Nauk. Uniw. Jagielloń. Prace Mat., **16** (1974), 47–53.
[2] M. MORIMOTO, A generalization of the Fourier-Borel transformation for the analytic functionals with non convex carrier, Tokyo J. Math., **2** (1979), 301–322.
[3] M. MORIMOTO, Analytic functionals on the Lie sphere, Tokyo J. Math., **3** (1980), 1–35.
[4] M. MORIMOTO, *Hyperfunctions on the Sphere*, Sophia Kokyuroku in Math., **12** (1982), Dept. of Math. Sophia Univ., Tokyo (in Japanese).
[5] M. MORIMOTO, Analytic functionals on the sphere and their Fourier-Borel transformations, *Complex Analysis*, Banach Center Publications, **11** (1983), PWN-Polish Scientific Publishers, Warsaw, 223–250.
[6] C. MÜLLER, *Spherical Harmonics*, Lecture Notes in Math., **17** (1966), Springer-Verlag.
[7] R. WADA, The Fourier-Borel transformations of analytic functionals on the complex

sphere, Proc. Japan Acad. Ser. A, **61** (1985), 298-301.
[8] R. WADA, On the Fourier-Borel transformations of analytic functionals on the complex sphere, Tôhoku Math. J., **38** (1986), 417-432.
[9] R. WADA and M. MORIMOTO, A uniqueness set for the differential operator $\Delta_z+\lambda^2$, Tokyo J. Math., **10** (1987), 93-105.
[10] R. WADA, A uniqueness set for linear partial differential operators of the second order, to appear in Funkcial. Ekvac.

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