Tokyo J. Math.
Vol. 11, No. 1, 1988

2-Type Surfaces of Constant Curvature in S^n

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(Communicated by K. Ogiue)

§0. Introduction.

Let M be a compact C^∞-Riemannian manifold, $C^\infty(M)$ the space of all smooth functions on M, and Δ the Laplacian on M. Then Δ is a self-adjoint elliptic differential operator acting on $C^\infty(M)$, which has an infinite discrete sequence of eigenvalues:

$$\mathrm{Spec}(M)=\{0=\lambda_0<\lambda_1<\lambda_2<\cdots<\lambda_k<\cdots\uparrow\infty\}\ .$$

Let $V_k=V_k(M)$ be the eigenspace of Δ corresponding to the k-th eigenvalue λ_k. Then V_k is finite-dimensional. We define an inner product $(\ ,\ )$ on $C^\infty(M)$ by

$$(f,g)=\int_M fg\,dV\ ,$$

where dV denotes the volume element on M. Then $\sum_{t=0}^\infty V_t$ is dense in $C^\infty(M)$ and the decomposition is orthogonal with respect to the inner product $(\ ,\ )$. Thus we have

$$C^\infty(M)=\sum_{t=0}^{\infty} V_t(M)\qquad (\text{in } L^2\text{-sense})\ .$$

Since M is compact, V_0 is the set of all constant functions which is 1-dimensional.

Let $\tilde{M}$ be another compact C^∞-Riemannian manifold, and assume that M is a submanifold of $\tilde{M}$ which is immersed by an isometric immersion φ. We have the decomposition

$$C^\infty(\tilde{M})=\sum_{s=0}^{\infty} V_s(\tilde{M})\qquad (\text{in } L^2\text{-sense})$$

with respect to the Laplacian $\Delta_{\tilde{M}}$ of $\tilde{M}$. We denote by φ^* the pull-back,

Received June 26, 1987

i.e., φ^* is an $\boldsymbol{R}$-linear map of $C^\infty(\tilde{M})$ into $C^\infty(M)$ such that

$$(\varphi^* F)(p)=F(\varphi(p)) , \qquad p\in M , \quad F\in C^\infty(\tilde{M}) .$$

For each integer s, $\varphi^* V_s(\tilde{M})$ is a subspace of $C^\infty(M)$. Then we have a decomposition

$$\varphi^* V_s(\tilde{M})\subset\sum_{t=0}^{\infty} W_t , \qquad W_t=W_t(M, \tilde{M}, \varphi, s)\subset V_t(M) ,$$

where each W_t is the minimal subspace of $V_t(M)$ such that $\sum_{t=0}^{\infty} W_t$ contains $\varphi^* V_s(\tilde{M})$.

We say that φ (or M) is *of finite-type with respect to* $V_s(\tilde{M})$, if $\sharp\{t\geqq 1 \mid W_t\neq(0)\}$ is finite, and if it is not finite, we say that φ (or M) is *of infinite-type with respect to* $V_s(\tilde{M})$. If $\sharp\{t\geqq 1 \mid W_t\neq(0)\}$ is equal to k, then we say that φ (or M) is *of k-type with respect to* $V_s(\tilde{M})$. Furthermore, we say that φ (or M) is *mass-symmetric with respect to* $V_s(\tilde{M})$ if $W_0=(0)$.

In this paper, we consider the case where $\tilde{M}$ is an n-sphere $S^n(1)$ of constant curvature 1, and $s=1$. So we omit the terms "with respect to $V_1(S^n)$" in conditions for immersions of M into S^n.

These definitions are compatible with those in B. Y. Chen [8, Chap. 6]. In [8], he shows that a mimimal immersion into S^n is mass-symmetric. By T. Takahashi [14]'s result, a mass-symmetric 1-type immersion into $S^n(1)$ is minimal. Moreover, a 1-type immersion into $S^n(1)$ is either a minimal immersion into $S^n(1)$ or a minimal immersion into a small hypersphere of $S^n(1)$.

In this sense, it seems that the next simplest condition for immersions into $S^n(1)$ is "mass-symmetric 2-type". We therefore study mass-symmetric 2-type immersions of compact surfaces into $S^n(1)$ in this paper. First, B. Y. Chen [8, p. 279] shows that the Riemannian product of two plane circles of suitable different radii is the only mass-symmetric 2-type surface in S^3. M. Barros and O. J. Garay [3] show that this result holds without the assumption of mass-symmetry. M. Barros and B. Y. Chen [1] show that there exist no mass-symmetric 2-type surfaces which lie fully in $S^4(1)$. Other results and examples are found in [1] and [8].

The purpose of this paper is to give the classification of mass-symmetric 2-type immersions of surfaces of constant curvature into $S^n(1)$.

The author wishes to thank Professors K. Ogiue, N. Ejiri and Y. Ohnita for many valuable comments and suggestions.

§1. Statement of results.

Let M be an n-dimensional compact C^∞-Riemannian manifold, $C^\infty(M)$ the space of all smooth functions on M, and Δ the Laplacian on M. In a natural manner, Δ can act on $\boldsymbol{R}^m$-valued functions on M. We assume that M is a submanifold of a unit N-sphere $S^N(1)$ centered at the origin, which is immersed by an isometric immersion f. We denote by $\iota: S^N \to E^{N+1}$ the standard imbedding. If f is of k-type, then the E^{N+1}-valued function $F=\iota\circ f$ has the decomposition

$$(1.1)\qquad F=F_0+\sum_{j=1}^{k} F_j\,, \qquad \Delta F_j=\lambda_j F_j\,, \quad F_j\neq 0\,, \quad j=1,\cdots,k\,, \\ 0<\lambda_1<\lambda_2<\cdots<\lambda_k\,,$$

where F_0 is a constant map. In this case, we note that λ_j denotes some positive eigenvalue of Δ which is not necessarily the j-th eigenvalue. f is mass-symmetric if and only if $F_0=0$.

Even if M is not compact, we say that f (or M) is *of k-type* if f has the decomposition (1.1), and that f (or M) is *mass-symmetric* if $F_0=0$ in (1.1).

Let f_1 and f_2 be isometric immersions of M into E^n and $E^{n'}$ respectively. Then the map $f: M\to E^{n+n'}$; $p\mapsto(\alpha f_1(p), \beta f_2(p))$, $\alpha^2+\beta^2=1$ is an isometric immersion. We say that f is a *diagonal sum* of f_1 and f_2. If f_1 and f_2 are minimal immersions of M into $S^n\subset E^{n+1}$ and $S^{n'}\subset E^{n'+1}$ respectively, then a diagonal sum of f_1 and f_2 is a mass-symmetric immersion of M into $S^{n+n'+1}$ which is of 1 or 2-type.

We obtain the following main results.

THEOREM A. *Let $f: M^2(K)\to S^N(1)$ be a mass-symmetric 2-type immersion of a surface $M^2(K)$ of constant positive curvature K into $S^N(1)$. Then f is a diagonal sum of two different standard minimal immersions of $M^2(K)$ into spheres.*

THEOREM B. *There exist no mass-symmetric 2-type immersions of a surface of constant negative curvature into a sphere.*

REMARK. R. L. Bryant [5] shows that there exist no minimal immersions of a surface of constant negative curvature into a sphere.

For the case of $K=0$, we have the following.

THEOREM C. *Let D be a small disk about the origin in the Euclidean plane E^2 and $f: D\to S^N(1)$ be a mass-symmetric 2-type full immersion. Then*

(1) *N is odd,*

(2) *f extends uniquely to a mass-symmetric 2-type immersion of E^2 into $S^N(1)$,*

(3) *f can be written in terms of a suitable complex coordinate z on $C\simeq E^2$ in the form*

$$f(z)=\sqrt{A_1}\sum_{k=1}^{m}\left\{P_k\exp\frac{\sqrt{\lambda_1}}{2}(\mu_k z-\bar{\mu}_k\bar{z})+\bar{P}_k\exp\frac{\sqrt{\lambda_1}}{2}(-\mu_k z+\bar{\mu}_k\bar{z})\right\}$$
$$+\sqrt{A_2}\sum_{j=1}^{m'}\left\{Q_j\exp\frac{\sqrt{\lambda_2}}{2}(\eta_j z-\bar{\eta}_j\bar{z})+\bar{Q}_j\exp\frac{\sqrt{\lambda_2}}{2}(-\eta_j z+\bar{\eta}_j\bar{z})\right\},$$

where λ_1, $\lambda_2\in \boldsymbol{R}$, $A_1=(\lambda_2-2)/(\lambda_2-\lambda_1)$, $A_2=(2-\lambda_1)/(\lambda_2-\lambda_1)$, $\{\pm\mu_k\}_{k=1}^{m}$ (resp. $\{\pm\eta_j\}_{j=1}^{m'}$) are 2m (resp. 2m′) distinct complex numbers of norm 1, $N=2(m+m')-1$, and P_k, $Q_j\in(E^{N+1})^C$ are nonzero vectors satisfying

$$\langle P_k, P_l\rangle=0 \quad \text{for } \forall k, l, \qquad \langle P_k, \bar{P}_l\rangle=0 \quad \text{for } k\neq l, \qquad \sum_{k=1}^{m}\langle P_k, \bar{P}_k\rangle=\frac{1}{2},$$
$$\langle Q_j, Q_i\rangle=0 \quad \text{for } \forall j, i, \qquad \langle Q_j, \bar{Q}_i\rangle=0 \quad \text{for } j\neq i, \qquad \sum_{j=1}^{m'}\langle Q_j, \bar{Q}_j\rangle=\frac{1}{2},$$
$$\langle P_k, Q_j\rangle=\langle P_k, \bar{Q}_j\rangle=0 \quad \text{for } \forall k, j,$$
$$\lambda_1 A_1\sum_{k=1}^{m}\mu_k^2\langle P_k, \bar{P}_k\rangle+\lambda_2 A_2\sum_{j=1}^{m'}\eta_j^2\langle Q_j, \bar{Q}_j\rangle=0.$$

REMARK. This theorem says that $f(M)$ is an orbit of an abelian subgroup of $SO(2(m+m'))$. Let G be the abelian group of parallel displacements of $\boldsymbol{R}^2$. Then f is G-equivariant.

COROLLARY. *Let $f: M^2(K)\to S^n(1)$ be a mass-symmetric 2-type immersion. If the immersion is full, then n is odd.*

Let M be an n-dimensional submanifold of S^N and let σ and H be the second fundamental form and the mean curvature vector of M. We define a normal vector field $\mathscr{B}(H)$ by

$$\mathscr{B}(H)=\sum_{i,j=1}^{n}\langle\sigma(e_i, e_j), H\rangle\sigma(e_i, e_j),$$

where $\{e_1, \cdots, e_n\}$ is an orthonormal basis at each point of M. We put $\mathscr{B}(H)=\alpha H+\mathscr{A}(H)$ where $H\perp\mathscr{A}(H)$ and α is some real number. $\mathscr{A}(H)$ is called the *allied mean curvature vector*. We say that M is a *Chen submanifold* if $\mathscr{A}(H)$ vanishes identically. If f is an isometric immersion of M into S^N which is a diagonal sum of two minimal immersions of M into spheres, then M is a Chen submanifold of S^N. Conversely, for flat surfaces, we obtain the following.

THEOREM D. *Let M be a flat surface and f be a full mass-symmetric 2-type Chen immersion of M into $S^N(1)$. If $N \geqq 9$, then f is a diagonal sum of two different minimal immersions into spheres.*

REMARK. If $N=3$, then M is an open subset of the Riemannian product of two plane circles of different radii (cf. B. Y. Chen [8, p. 279]). For $N=5$ or 7, we classify f later. In the case of $N=7$, there exists a full mass-symmetric 2-type Chen immersion which is not a diagonal sum of minimal immersions (See § 4.)

Regard $\boldsymbol{R}^n$ as an n-dimensional vector space. Let $v_1, \cdots, v_r \in \boldsymbol{R}^n$ be linearly independent. We put $\Lambda = \{\sum_{i=1}^{r} m_i v_i \mid m_i \text{ integers}\}$. Then Λ is a free abelian group generated by $v_1, \cdots, v_r$. Λ is called a discrete lattice of $\boldsymbol{R}^n$ and the integer r is called the rank of Λ. Acting on $\boldsymbol{R}^n$ as translation, Λ acts properly discontinuously and freely on $\boldsymbol{R}^n$. The quotient space $\boldsymbol{R}^n/\Lambda$ with the canonical metric is a flat n-dimensional Riemannian manifold. If the rank of Λ is equal to n, then $\boldsymbol{R}^n/\Lambda$ is compact and is called a flat n-torus.

We obtain a criterion for the existence of mass-symmetric 2-type immersions of flat 2-tori as follows.

PROPOSITION E. *Let $\Lambda \subset \boldsymbol{R}^2$ be a discrete lattice of rank 2 and Λ^* the dual lattice of Λ, i.e., $\Lambda^* = \{u \in \boldsymbol{R}^2 \mid \langle u, v \rangle \equiv 0 \mod 2\pi$ for all $v \in \Lambda\}$. For $r > 0$, let*

$$c(\Lambda, r) = \{(a+ib)^2 \mid a^2+b^2=r^2 \text{ and } (a, b) \in \Lambda^*\} .$$

Then a flat torus $\boldsymbol{R}^2/\Lambda$ admits a mass-symmetric 2-type full immersion $f: \boldsymbol{R}^2/\Lambda \to S^N(1)$ with respect to some $\lambda, \lambda' \in \mathrm{Spec}(\boldsymbol{R}^2/\Lambda)$ if and only if there exist m distinct elements $\{\alpha_k\} \subset c(\Lambda, \sqrt{\lambda})$ and m' distinct elements $\{\beta_j\} \subset c(\Lambda, \sqrt{\lambda'})$, where $N=2(m+m')-1$, satisfying

$$(\lambda'-2)\sum_k \alpha_k R_k + (2-\lambda)\sum_j \beta_j R'_j = 0$$

for some R_k and $R'_j > 0$ with $\sum_k R_k = 1$ and $\sum_j R'_j = 1$.

REMARK. Let $H(\Lambda, r)$ be the convex hull of $c(\Lambda, r)$. $\boldsymbol{R}^2/\Lambda$ admits a minimal immersion into $S^N(1)$ for some N if and only if $0 \in H(\Lambda, \sqrt{2})$ (cf. Bryant [5]).

Let $(S^6(1), J, \tilde{g})$ be a nearly Kaehler manifold with a canonical almost complex structure on S^6. The automorphism group is the compact simple

Lie group G_2. For two imbedded submanifolds M_1 and M_2 of S^6, we say that $M_1 \sim M_2$ if there exists an element φ of G_2 satisfying $\varphi(M_1)=M_2$. Then the relation $\sim$ is an equivalence relation. We denote by $[M_1]$ the equivalence class of M_1.

Let T be a maximal torus of G_2. Since G_2 is of rank 2, any T-orbit in S^6 is a flat surface or a circle or a point. We put

$\mathfrak{F}=\{T$-orbit which is totally real mass-symmetric 2-type$\}$,

$\mathfrak{F}_3=\{T$-orbit which is totally real mass-symmetric 2-type and imbedded fully into a totally geodesic $S^3(1)\}$,

$\mathfrak{F}_5=\{T$-orbit which is totally real mass-symmetric 2-type and imbedded fully into a totally geodesic $S^5(1)\}$.

Then we obtain the following.

Theorem F. (1) *If $M\in\mathfrak{F}$, then M is a Chen surface of S^6.*

(2) *Both $\mathfrak{F}_3/\sim$ and $\mathfrak{F}_5/\sim$ form 1-parameter families. $\mathfrak{F}/\sim$ is a disjoint union of $\mathfrak{F}_3/\sim$ and $\mathfrak{F}_5/\sim$.*

(3) *If $M\in\mathfrak{F}_3$, then M lies fully in a totally real and totally geodesic sphere $S^3(1)$ of $S^6(1)$.*

(4) *Suppose that $M\in\mathfrak{F}$ and denote by H the mean curvature vector field of M in S^6. Then, JH is a normal vector field of M if and only if $M\in\mathfrak{F}_3$, and JH is a tangent vector field of M if and only if $M\in\mathfrak{F}_5$.*

Conversely, we obtain the following.

Theorem G. *Suppose that M is a complete totally real mass-symmetric 2-type Chen surface in S^6 which is imbedded by isometric imbedding f. We denote by H the mean curvature vector of f.*

(1) *If JH is a normal vector field of M, then $M\in\mathfrak{F}_3$, and*

(2) *if JH is a tangent vector field of M, then $M\in\mathfrak{F}_5$.*

The local version of this result also holds.

Remark. If $M\in\mathfrak{F}_3$, then M is the Riemannian product of two plane circles of suitable different radii (cf. B. Y. Chen [8, p. 279]). If $M\in\mathfrak{F}_5$, then we will show in §5 that M is not stationary in S^5 and is not the Riemannian product of circles. This means that M ($M\in\mathfrak{F}_5$) is a new example (See §5).

§2. A lemma.

Let M be an n-dimensional C^∞-manifold, and f a mass-symmetric 2-type immersion of M into an N-sphere $S^N(1)$ of constant curvature 1. Let $\iota\colon S^N(1)\to E^{N+1}$ be the canonical imbedding and put $F=\iota\circ f$. By definition, we have

$$(2.1) \qquad F=F_1+F_2\ , \qquad \Delta F_i=\lambda_i F_i,\ F_i\neq 0,\ i=1,2\ , \quad 0<\lambda_1<\lambda_2\ .$$

We give the following lemma for later use.

LEMMA 2.1.

$$(2.2) \qquad \langle F_1, F_1\rangle=\frac{\lambda_2-n}{\lambda_2-\lambda_1}\ , \qquad \langle F_2, F_2\rangle=\frac{n-\lambda_1}{\lambda_2-\lambda_1}\ , \qquad \langle F_1, F_2\rangle=0\ ,$$

at any point of M, where $\langle\ ,\ \rangle$ denotes the canonical inner product on E^{N+1}.

PROOF. We denote by H (resp. H') the mean curvature vector of M in E^{N+1} (resp. in $S^N(1)$). Then $H=H'-F$, $\Delta F=-nH$. So we have

$$(2.3) \qquad \langle F, F\rangle=1\ , \qquad \langle \Delta F, F\rangle=n\ ,$$

at any point of M. Let D be the Riemannian connection of E^{N+1}, and $\{e_1, \cdots, e_n\}$ an orthonormal frame of M. Then

$$(2.4) \qquad \begin{aligned}\langle \Delta^2 F, F\rangle &= -n\langle \Delta H, F\rangle=-n\langle \Delta H', F\rangle+n\langle \Delta F, F\rangle\\ &= -n\Delta\langle H', F\rangle+n\langle H', \Delta F\rangle-2n\langle DH', DF\rangle+n\langle \Delta F, F\rangle\\ &= n^2\|H'\|^2+n^2\\ &= n^2\|H\|^2=\langle \Delta F, \Delta F\rangle\ ,\end{aligned}$$

where $\langle DH', DF\rangle=\sum_{i=1}^n \langle D_{e_i}H', D_{e_i}F\rangle$. (2.1), (2.3) and (2.4) imply (2.2).
Q.E.D.

§3. A 2-type surface of constant Gaussian curvature.

3.1. In this section, we prove Theorems A, B and C by using Bryant's methods. Let (M^2, g) be an oriented, connected C^∞-surface with a C^∞-metric g, and $\pi: \mathscr{F}\to M$ be the bundle of oriented orthonormal frames. Thus $f\in\mathscr{F}$ is a triplet $f=(x; e_1, e_2)$, where x is a point of M and $e_1, e_2\in T_x(M)$ form an oriented orthonormal basis. The canonical 1-forms ω^1, ω^2 on $\mathscr{F}$ are the unique 1-forms satisfying

$$d\pi=e_1\omega^1+e_2\omega^2\ .$$

It is well-known that there exists a unique 1-form ρ satisfying

$$d\omega^1=-\rho\wedge\omega^2\ , \qquad d\omega^2=\rho\wedge\omega^1\ ,$$

and we have the formula $d\rho=K\omega^1\wedge\omega^2$, where K is the Gaussian curvature of (M, g). From now on, we shall assume that K is constant.

It will be convenient to use a complex form so that we set $\omega=\omega^1+i\omega^2$ and rewrite the structure equations as

$$d\omega=i\rho\wedge\omega\ , \qquad d\rho=\frac{i}{2}K\omega\wedge\bar{\omega}\ ,$$

where $g=\omega\cdot\bar{\omega}$.

Let $\tau\to M$ be the complex line bundle of 1-forms which are multiples of ω and let $\tau^{-1}\to M$ be the complex line bundle of 1-forms which are multiples of $\bar{\omega}$. For $m\geqq 0$, let $\tau^m\to M$ (resp. $\tau^{-m}\to M$) be the m-th tensor product of τ (resp. τ^{-1}) as a complex line bundle. Using the identification $\omega^{-m}=(\bar{\omega})^m$ for all m, we have a canonical pairing $\tau^m\times\tau^k\to\tau^{m+k}$ for all m and k. If σ is any section of $\mathscr{F}$, then we may write $\sigma=s(\omega)^m$ for a unique function s on τ^m. One easily compute that $ds=-mis\rho+s'\omega+s''\bar{\omega}$ for some unique functions s' and s'' on $\mathscr{F}$. Moreover, by differentiating this equation, we deduce that the forms $s'(\omega)^{m+1}=\sigma'$ and $s''(\omega)^{m-1}=\sigma''$ are well-defined sections on τ^{m+1} and τ^{m-1} respectively. This allows us to define operators ∂_m: $C^\infty(\tau^m)\to C^\infty(\tau^{m+1})$ and $\bar{\partial}_m$: $C^\infty(\tau^m)\to C^\infty(\tau^{m-1})$ by $\partial_m\sigma=\sigma'$ and $\bar{\partial}_m\sigma=\sigma''$, where we denote by $C^\infty(\tau^k)$ the space of all sections of τ^k for any $k\in\boldsymbol{Z}$. Let I_m be the identity map of $C^\infty(\tau^m)$. Set $\mathscr{T}=\bigoplus_m C^\infty(\tau^m)$ as a $\boldsymbol{Z}$-graded vector space and define the operators

$$X=\bigoplus_m \partial_m\ ,\quad Y=\bigoplus_m \bar{\partial}_m\ ,\quad H=\bigoplus_m mI_m\ .$$

So Bryant [5] shows the following.

PROPOSITION 3.1.

$$[H, X]=X\ ,\quad [H,Y]=-Y\ ,\quad [X,Y]=-\frac{K}{2}H\ .$$

Moreover, $\Delta=-2(XY+YX)$, *where* Δ: $\mathscr{T}\to\mathscr{T}$ *is the Laplace-Beltrami operator on each graded piece.*

Let V be a real vector space with a Euclidean inner product $\langle\ ,\ \rangle$: $V\times V\to\boldsymbol{R}$. In a natural way, we may set $\mathscr{V}=V\otimes_{\boldsymbol{R}}\mathscr{T}$ and extend the operators X, Y and H to $\mathscr{V}$ and extend the given $\langle\ ,\ \rangle$ to a bi-linear map $\mathscr{V}\times\mathscr{V}\to\mathscr{T}$. If σ is a section of the m-th graded piece of $\mathscr{V}$, then we may write $\sigma=s(\omega)^m$ for a unique V^C-valued function on $\mathscr{F}$, where V^C denotes the complexification of V. We define conjugation in $\mathscr{V}$ by setting $\bar{\sigma}=\bar{s}(\omega)^{-m}$. Then we have $X\bar{\sigma}=\overline{Y\sigma}$, $Y\bar{\sigma}=\overline{X\sigma}$, $H\bar{\sigma}=-\overline{H\sigma}$. So we have

PROPOSITION 3.2 (Bryant [5]). *Let V be a Euclidean vector space of* *dimension* $N+1$ *and* S^N *the unit sphere in* V. *Let* $F: M\to V$ *be a smooth map. In order that* F *be an isometric immersion, it is necessary and sufficient that* $\langle XF, XF\rangle\equiv 0$, $\langle XF, \overline{XF}\rangle\equiv 1/2$. *In addition,* $F(M)\subset S^N$ *if and only if* $\langle F, F\rangle\equiv 1$. *Finally* $F(M)\subset S^N$ *is minimal if and only if* $\Delta F=2F$.

LEMMA 3.3. *Suppose that* $F: M\to V$ *satisfies* $\Delta F=\lambda F$. *Then, for* $m>0$,

$$YX^mF=D_mX^{m-1}F\ ,\qquad XY^mF=D_mY^{m-1}F\ ,$$

where D_m *is a constant depending on* m *and* K *given by*

$$D_m=\frac{1}{2}\left[\binom{m}{2}K-\frac{\lambda}{2}\right].$$

Bryant [5] shows this lemma in the case of $\lambda=2$. We obtain this lemma in the same way as Bryant [5].

In order to prove Theorem A, we assume that $f: M^2(K)\to S^n(1)$ is a mass-symmetric 2-type full immersion and $K>0$. We put $V=E^{n+1}$ and let $\iota: S^N(1)\to V$ be the canonical imbedding. By definition in §1, we have

$$\iota\circ f=F=F_1+F_2\ ,\quad \Delta F_1=\lambda_1F_1\ ,\quad \Delta F_2=\lambda_2F_2\ ,\quad 0<\lambda_1<\lambda_2\ . \tag{3.1}$$

We note F_1 and F_2 are smooth maps of M into V. We put $A_1=(\lambda_2-2)/(\lambda_2-\lambda_1)$ and $A_2=(2-\lambda_1)/(\lambda_2-\lambda_1)$ so that $\langle F_1, F_1\rangle=A_1$ and $\langle F_2, F_2\rangle=A_2$ by Lemma 2.1.

Let $g^*=\rho_1^2g$, $\rho_1=\sqrt{\lambda_1/2}$, be a homothetic change of the metric g. Then the Laplacian Δ^* and the Gaussian curvature K^* of (M, g^*) satisfy

$$\Delta^*=\rho_1^{-2}\Delta\ ,\qquad K^*=\rho_1^{-2}K$$

so that $\Delta^*F_1=2F_1$. On the other hand, by Lemma 2.1, $\langle F_1/\sqrt{A_1}, F_1/\sqrt{A_1}\rangle=1$. By Theorems 1.5 and 1.6 in Bryant [5], there exists some $p\geqq 1$ such that $K^{*-1}=\binom{p+1}{2}$ and $(F_1/\sqrt{A_1})(M)$ lies fully in a $(2p+1)$-dimensional vector space $V'\subset V$ and the map $F_1/\sqrt{A_1}: (M, g^*)\to S^{2p}(1)\subset V'$ is a minimal isometric immersion, i.e., $(F_1/\sqrt{A_1})(M)$ is an open subset of the Boruvka sphere $S^2(K^*)\to S^{2p}(1)$. Moreover, we have $X^kF_1=0$ for any $k>p$, $X^kF_1\neq 0$ for $0\leqq k\leqq p$, $\langle X^kF_1, X^jF_1\rangle=0$ for $0\leqq k, j\leqq p$, $k+j>0$, and $\langle X^kF_1, \bar{X}^jF_1\rangle=0$ for $0\leqq k, j\leqq p$, $k\neq j$.

For $0\leqq k\leqq p$, set $X^kF_1=P_k(\omega)^k$ for some V^C-valued functions P_k. Then $V'^C\subset V^C$ is spanned by $\{P_k, \bar{P}_k\}_{k=1}^p\cup\{P_0\}$. Furthermore $\lambda_1=p(p+1)K$ and $F_1/(\rho_1\sqrt{A_1})$ is a minimal isometric immersion of $(M^2(K), g)$ into $S^{2p}(\rho_1^2)\subset$

$V' \subset V$.

We put $\rho_2 = \sqrt{\lambda_2/2}$ and $X^j F_2 = Q_j(\omega)^j$ for $0 \leqq j \leqq q$. Thus, in the same way, we see that $\lambda_2 = q(q+1)K$ and $F_2/(\rho_2\sqrt{A_2})$ is a minimal isometric immersion of $(M^2(K), g)$ into $S^{2q}(\rho_2^2) \subset V'' \subset V$, where V'' is a $(2q+1)$-dimensional vector space such that $V''^C \subset V^C$ is spanned by $\{Q_j, \bar{Q}_j\}_{j=1}^q \cup \{Q_0\}$.

One notes that $(F_1/(\rho_1\sqrt{A_1}))(M)$ and $(F_2/(\rho_2\sqrt{A_2}))(M)$ are open subsets of the Boruvka spheres $S^2(K) \to S^{2p}(\rho_1^2)$ and $S^2(K) \to S^{2q}(\rho_2^2)$ respectively. Then we can see that these maps extend uniquely to the p-th and q-th standard immersions of $S^2(K)$ respectively. Note that $p < q$.

It is easy to see that $A_1 + A_2 = 1$ and $\rho_1^2 A_1 + \rho_2^2 A_2 = 1$. Then F is a diagonal sum of $F_1/(\rho_1\sqrt{A_1})$ and $F_2/(\rho_2\sqrt{A_2})$ if and only if $V' \perp V''$, i.e.,

$$\langle P_k, Q_j \rangle = \langle P_k, \bar{Q}_j \rangle = 0 \quad \text{for any} \quad 0 \leqq k \leqq p\,, \quad 0 \leqq j \leqq q\,. \tag{3.2}$$

To show the following is sufficient to complete the proof of Theorem A. To be convenient, we set $X^{-h} = \bar{X}^h$ for integer h.

LEMMA 3.4. (1) *For each* $0 \leqq k \leqq p$,

$[k]_1$: $\langle X^k F_1, X^j F_2 \rangle = \langle X^k F_1, \bar{X}^j F_2 \rangle = 0$ *for any* $0 \leqq j \leqq k$.

(2) *For each* $0 \leqq j \leqq q$,

$[j]_2$: $\langle X^k F_1, X^j F_2 \rangle = \langle \bar{X}^k F_1, X^j F_2 \rangle = 0$ *for any* $0 \leqq k \leqq \min(j, p)$.

PROOF. $[0]_1$ and $[0]_2$ is clear by Lemma 2.1. We shall prove (1) for $k \geqq 1$ by induction on k. To be convenient, we put $a_i = \rho_i\sqrt{A_i}$ and $D_{i,m} = \frac{1}{2}\left[\binom{m}{2}K - \frac{\lambda_i}{2}\right]$, $i = 1, 2$. Since F_i/a_i and F are isometric immersions,

$$\langle XF_i, XF_i \rangle = 0 \quad \text{and} \quad \langle XF_i, \bar{X}F_i \rangle = \frac{1}{2}a_i^2\,, \qquad i = 1, 2\,,$$

$$\langle XF, XF \rangle = 0 \quad \text{and} \quad \langle XF, \bar{X}F \rangle = \frac{1}{2}\,,$$

hold by Proposition 3.2 so that we have

$$\langle XF_1, XF_2 \rangle = 0\,. \tag{3.3}$$

Applying X to $\langle F_1, F_2 \rangle = 0$, we get

$$\langle XF_1, F_2 \rangle + \langle F_1, XF_2 \rangle = 0\,.$$

Applying $\bar{X}$ to (3.3) gives

$$D_{1,1}\langle F_1, XF_2 \rangle + D_{2,1}\langle XF_1, F_2 \rangle = 0\,.$$

Since $\lambda_1 < \lambda_2$, we have $\langle XF_1, F_2\rangle = 0$. Applying $\bar{X}$ to this equation, we have $\langle XF_1, \bar{X}F_2\rangle = 0$. So we obtain $[1]_1$.

We assume $[m]_1$ is true for $m \geqq 1$. Applying X to $\langle X^m F_1, \bar{X}^m F_2\rangle = 0$ gives $\langle X^{m+1}F_1, \bar{X}^m F_2\rangle + D_{2,m}\langle X^m F_1, \bar{X}^{m-1}F_2\rangle = 0$. By the assumption of induction, we obtain

$$\langle X^{m+1}F_1, \bar{X}^m F_2\rangle = 0 .$$

Applying $\bar{X}$ to this, we get

$$\langle X^{m+1}F_1, \bar{X}^{m+1}F_2\rangle = 0 .$$

For $0 \leqq h \leqq 2(m+1)$, we put $\sigma_h = \langle X^{m+1}F_1, \bar{X}^{m+1-h}F_2\rangle$ so that σ_h is a section of the complex line bundle τ^h over $S^2(K)$. We get $\sigma_0 = 0$ and $\sigma_1 = 0$ and, for $2 \leqq h \leqq 2(m+1)$,

$$\bar{X}\sigma_h = D_{1,m+1}\langle X^m F_1, \bar{X}^{m+1-h}F_2\rangle + \langle X^{m+1}F_1, \bar{X}(\bar{X}^{m+1-h}F_2)\rangle . \tag{3.4}$$

On the other hand, we see that

$$\begin{aligned} \langle X^m F_1, \bar{X}^{m+1-2(m+1)}F_2\rangle &= \langle X^m F_1, X^{m+1}F_2\rangle \\ &= X\langle X^m F_1, X^m F_2\rangle - \langle X^{m+1}F_1, X^m F_2\rangle \\ &= -\sigma_{2m+1} . \end{aligned} \tag{3.5}$$

By the assumption of induction, (3.4) and (3.5), we obtain

$$\bar{X}\sigma_h = \begin{cases} \sigma_{h-1} & \text{if } 2 \leqq h \leqq m+1 \\ D_{2,h-(m+1)}\sigma_{h-1} & \text{if } m+2 \leqq h \leqq 2m+1 \\ (D_{2,m+1} - D_{1,m+1})\sigma_{2m+1} & \text{if } h = 2(m+1) . \end{cases} \tag{3.6}$$

Since $\sigma_1 = 0$, we get $\bar{X}\sigma_2 = 0$ so that σ_2 is a holomorphic section of τ^2. By Riemann-Roch theorem, σ_2 must be the zero section. Similarly, we see that other sections σ_h are zero by induction, i.e.,

$$\sigma_0 = \sigma_1 = \sigma_2 = \cdots = \sigma_{2(m+1)} = 0 .$$

This implies $[m+1]_1$ so that we obtain (1). (2) is proved similarly. Q.E.D.

3.2. To prove Theorem B, we assume $K < 0$. Let g^* be a homothetic change of the given metric g as in §3.1. Then F_1 is a V-valued C^∞-function of M satisfying $\Delta^* F_1 = 2F_1$ and $\langle F_1, F_1\rangle = A_1$. By Theorem 2.3 in Bryant [5], there is no solution to the above equations. Therefore Theorem B is proved.

3.3. Let D be a small disk about $0 \in D \subset E^2$ with the canonical

Euclidean metric $g=(dx)^2+(dy)^2$, and $f: D\to S^N(1)$ be a mass-symmetric 2-type full immersion. Then we have the decomposition (3.1). To be convenient, we consider $E^2\simeq C$, $z=x+iy\in C$ ($(x, y)\in E^2$) so that $g=dz\cdot d\bar{z}$. Put $\rho_i=\sqrt{\lambda_i/2}$, $g_i^*=\rho_i^2 g$, $i=1, 2$. Each g_i^* is a homothetic change of the metric g. Δ_i^* denotes the Laplacian of (D, g_i^*). It satisfies $\Delta_i^*=\rho_i^{-2}\Delta$. So we get $\Delta_i^* F_i=2F_i$ and $\langle F_i, F_i\rangle=A_i$. Then by Theorem 3.1 in Bryant [5], we obtain the following.

LEMMA 3.5. *We assume $F_i(D)$ is fully contained in a subspace V_i of V. For each i, take a complex coordinate $w=\rho_i z/\sqrt{2}$ so that $g_i^*=\rho_i^2 dz\cdot d\bar{z}=2dw\cdot d\bar{w}$. Then*

(1) *the dimension of V_i is even so that $F_i(D)\subset S^{2m_i-1}(1/A_i)$ fully, where $2m_i=\dim(V_i)$,*

(2) *F_i extends uniquely to a map $C\to S^N(1)$ satisfying $\Delta_i^* F_i=2F_i$ and $\langle F_i, F_i\rangle=A_i$,*

(3) *after rotating w if necessary, F_i can be written in the form*

$$F_i(w)=\sqrt{A_i}\sum_{k=1}^{m_i}\{P_{i,k}\exp(\mu_{i,k}w-\overline{\mu_{i,k}w})+\bar{P}_{i,k}\exp(-\mu_{i,k}w+\overline{\mu_{i,k}w})\} ,$$

where $\{\pm\mu_{i,k}\}_{k=1}^{m_i}$ are $2m_i$ distinct complex numbers of norm 1 and $P_{i,k}\in V'^C\subset V^C$ are nonzero constant vectors satisfying

$$\langle P_{i,k}, P_{i,l}\rangle=0 \quad for\ \forall k, l , \qquad \langle P_{i,k}, \bar{P}_{i,l}\rangle=0 \quad for\ k\neq l ,$$
$$\sum_{k=1}^{m_i}\langle P_{i,k}, \bar{P}_{i,k}\rangle=\frac{1}{2} .$$

We put $m=m_1$, $m'=m_2$, $P_k=P_{1,k}$, $Q_j=P_{2,j}$, $\mu_k=\mu_{1,k}$ and $\eta_j=\mu_{2,j}$. Then after changing a parameter from w to z, F can be written in the form

$$(3.7)\qquad F(z)=F_1(z)+F_2(z) ,$$
$$F_1(z)=\sqrt{A_1}\sum_{k=1}^{m}\{P_k\exp r(\mu_k z-\overline{\mu_k z})+\bar{P}_k\exp r(-\mu_k z+\overline{\mu_k z})\} ,$$
$$F_2(z)=\sqrt{A_2}\sum_{j=1}^{m'}\{Q_j\exp R(\eta_j z-\overline{\eta_j z})+\bar{Q}_j\exp R(-\eta_j z+\overline{\eta_j z})\} ,$$

where $r=\rho_1/\sqrt{2}$ and $R=\rho_2/\sqrt{2}$. By Lemma 2.1, we have $\langle F_1, F_2\rangle=0$ which is equivalent to

$$(3.8)\quad \sum_{k,j}\{\mathrm{Re}(\langle P_k, Q_j\rangle)\cos(2\,\mathrm{Im}(r\mu_k+R\eta_j)z)-\mathrm{Im}(\langle P_k, Q_j\rangle)\sin(2\,\mathrm{Im}(r\mu_k+R\eta_j)z)$$
$$+\mathrm{Re}(\langle P_k, \bar{Q}_j\rangle)\cos(2\,\mathrm{Im}(r\mu_k-R\eta_j)z)$$
$$-\mathrm{Im}(\langle P_k, \bar{Q}_j\rangle)\sin(2\,\mathrm{Im}(r\mu_k-R\eta_j)z)\}=0$$

for all $z\in C$.

On the other hand, F is isometric, i.e., $\langle dF, dF\rangle = dz\cdot d\bar{z}$. This is equivalent to the following

$$(3.9)\qquad \sum_{k,j} \mathrm{Re}(\mu_k\bar{\eta}_j)\{-\mathrm{Re}(\langle P_k, Q_j\rangle)\cos(2\,\mathrm{Im}(r\mu_k+R\eta_j)z) + \mathrm{Im}(\langle P_k, Q_j\rangle)\sin(2\,\mathrm{Im}(r\mu_k+R\eta_j)z) + \mathrm{Re}(\langle P_k, \bar{Q}_j\rangle)\cos(2\,\mathrm{Im}(r\mu_k-R\eta_j)z) - \mathrm{Im}(\langle P_k, \bar{Q}_j\rangle)\sin(2\,\mathrm{Im}(r\mu_k-R\eta_j)z)\} = 0$$

and

$$(3.10)\qquad -\rho_1^2A_1\sum_k \mu_k^2\langle P_k, \bar{P}_k\rangle - \rho_2^2A_2\sum_j \eta_j^2\langle Q_j, \bar{Q}_j\rangle + \sqrt{A_1A_2\lambda_1\lambda_2}\sum_{k,j}\mu_k\eta_j\{\mathrm{Re}(\langle P_k, Q_j\rangle)\cos(2\,\mathrm{Im}(r\mu_k+R\eta_j)z) - \mathrm{Im}(\langle P_k, Q_j\rangle)\sin(2\,\mathrm{Im}(r\mu_k+R\eta_j)z) - \mathrm{Re}(\langle P_k, \bar{Q}_j\rangle)\cos(2\,\mathrm{Im}(r\mu_k-R\eta_j)z) + \mathrm{Im}(\langle P_k, \bar{Q}_j\rangle)\sin(2\,\mathrm{Im}(r\mu_k-R\eta_j)z)\} = 0$$

for all $z \in \boldsymbol{C}$.

Put $\mathscr{A} = \{\pm(r\mu_k+R\eta_j),\ \pm(r\mu_k-R\eta_j) \mid k=1, \cdots, m,\ j=1, \cdots, m'\}$. If $\alpha\in\mathscr{A}$, then α can be written in the form $\alpha=\pm(r\mu_k+R\eta_j)$ or $\pm(r\mu_k-R\eta_j)$ for some μ_k and η_j and hence α has at most two different representations. Suppose that $\alpha\in\mathscr{A}$ has only one representation. For instance, if $\alpha=r\mu_k+R\eta_j$, then the independence of exponentials in (3.8) implies $\langle P_k, Q_j\rangle=0$. In the next place, we assume that $\alpha\in\mathscr{A}$ has two different representations. For instance, if

$$(3.11)\qquad \alpha = r\mu_k+R\eta_j = r\mu_l+R\eta_i\,,\qquad k\neq l,\ j\neq i\,,$$

then the independence of exponentials in (3.8) and (3.10) implies

$$(3.12)\qquad \langle P_k, Q_j\rangle + \langle P_l, Q_i\rangle = 0\,,\qquad \mu_k\eta_j\langle P_k, Q_j\rangle + \mu_l\eta_i\langle P_l, Q_i\rangle = 0\,.$$

Put $\theta=\arg(\alpha)$ so that (3.11) implies $\mu_k=e^{i(\theta-\varphi)}$, $\mu_l=e^{i(\theta+\varphi)}$, $\eta_j=e^{i(\theta+\psi)}$, $\eta_i=e^{i(\theta-\psi)}$ for some φ and ψ. If $\mu_k\eta_j-\mu_l\eta_i=0$, then we see that $\varphi\equiv\psi \pmod{\pi}$, so that (3.11) implies $\mu_k=\mu_l$. This is a contradiction. Thus $\mu_k\eta_j-\mu_l\eta_i\neq 0$. So by (3.12), we have $\langle P_k, Q_j\rangle=\langle P_l, Q_i\rangle=0$. Finally, we get $\langle P_k, Q_j\rangle=\langle P_k, \bar{Q}_j\rangle=0$ for any k and j. It means $V_1\perp V_2$ so that $N=2(m+m')-1$. From (3.10) we have

$$\lambda_1A_1\sum_k \mu_k^2\langle P_k, \bar{P}_k\rangle + \lambda_2A_2\sum_j \eta_j^2\langle Q_j, \bar{Q}_j\rangle = 0\,.$$

The proof of Theorem C is completed.

3.4. Let Λ be a discrete lattice of rank 2 of $\boldsymbol{R}^2$ and Λ^* the dual lattice of Λ, i.e.,

$$\Lambda^* = \{u \in \boldsymbol{R}^2 \mid \langle u, v\rangle \equiv 0 \bmod 2\pi, \text{ for all } v \in \Lambda\},$$

where $\langle\ ,\ \rangle$ denotes the canonical inner product of $\boldsymbol{R}^2$. Denote by T^2 a flat torus $\boldsymbol{R}^2/\Lambda$. Then we have

$$\mathrm{Spec}(T^2) = \{\|u\|^2 \mid u \in \Lambda^*\}.$$

Regarding $\boldsymbol{R}^2 \simeq \boldsymbol{C}$, we obtain

$$\langle z, w\rangle = \mathrm{Re}(z\bar{w}) = \mathrm{Im}(\sqrt{-1}z\bar{w})$$

for $z, w \in \boldsymbol{C}$. For a complex number μ of norm 1 and a positive real number λ, we define a function φ on $\boldsymbol{R}^2$ by

$$\varphi(z) = \exp\frac{\sqrt{\lambda}}{2}(\mu z - \overline{\mu z}) = \exp\sqrt{-1}\langle\sqrt{-\lambda}\bar{\mu}, z\rangle, \qquad z \in \boldsymbol{C}.$$

φ is a function on T^2 if and only if $\sqrt{-\lambda}\bar{\mu} \in \Lambda^*$. In this case, φ is an eigenfunction with respect to the eigenvalue λ.

Putting $\lambda = \lambda_1$, $\lambda' = \lambda_2$, $\alpha_k = -\lambda\bar{\mu}_k^2$, $\beta_j = -\lambda'\bar{\eta}_j^2$, $R_k = 2\langle P_k, \bar{P}_k\rangle$ and $R'_j = 2\langle Q_j, \bar{Q}_j\rangle$ in Theorem C, we easily obtain Proposition E.

§4. Flat 2-type Chen surfaces in S^n.

In this section, we study a flat mass-symmetric 2-type Chen surface in $S^N(1)$ and prove Theorem D.

Let M be a flat mass-symmetric 2-type surface in $S^N(1)$ which is immersed fully by an isometric immersion f and we denote by σ and H the second fundamental form and the mean curvature vector, respectively. By Theorem C, we can see $M \simeq \boldsymbol{R}^2$ and f can be written in the form

$$(4.1) \qquad f(z) = F_1(z) + F_2(z),$$

$$F_1(z) = \sqrt{A_1}\sum_{k=1}^{m} 2\mathrm{Re}\left\{\sqrt{R_k}u_k \exp\frac{\sqrt{\lambda_1}}{2}(\mu_k z - \bar{\mu}_k\bar{z})\right\},$$

$$F_2(z) = \sqrt{A_2}\sum_{j=1}^{m'} 2\mathrm{Re}\left\{\sqrt{R'_j}u_{m+j} \exp\frac{\sqrt{\lambda_2}}{2}(\eta_j z - \bar{\eta}_j\bar{z})\right\},$$

$$u_l = \frac{1}{2}(E_{2l-1} - \sqrt{-1}E_{2l}), \quad l = 1, \cdots, m+m' \ (= (N+1)/2),$$

where $\{E_1, \cdots, E_{N+1}\}$ is an orthonormal basis of E^{N+1}, $A_1 = (\lambda_2 - 2)/(\lambda_2 - \lambda_1)$, $A_2 = (2 - \lambda_1)/(\lambda_2 - \lambda_1)$, $\{\pm\mu_k\}_{k=1}^{m}$ (resp. $\{\pm\eta_j\}_{j=1}^{m'}$) are $2m$ (resp. $2m'$) distinct complex numbers of norm 1, $N=2(m+m')-1$ and R_k and R'_j are positive constants such that $\sum R_k=\sum R'_j=1$ and

$$\lambda_1 A_1 \sum_{k=1}^{m} \mu_k{}^2 R_k+\lambda_2 A_2 \sum_{j=1}^{m'} \eta_j{}^2 R'_j=0 .$$

For convenience, we put $\varphi=\sum \bar{\mu}_k{}^2 R_k$ and $\psi=\sum \bar{\eta}_j{}^2 R'_j$ so that

(4.2) $$\lambda_1 A_1 \varphi+\lambda_2 A_2 \psi=0 .$$

Let $\{e_1, e_2\}$ be an orthonormal basis of M. We can see that $e_1=\partial/\partial x$ and $e_2=\partial/\partial y$ so that $\partial/\partial z=(1/2)\{\partial/\partial x-\sqrt{-1}\partial/\partial y\}$. Let ∇, $\bar{\nabla}$ and D be the Riemannian connections of M, S^N and E^{N+1} respectively. We have

$$\begin{aligned} D_{e_i}D_{e_j}f=D_{e_i}e_j&=\bar{\nabla}_{e_i}e_j-\langle e_i, e_j\rangle f \\ &=\nabla_{e_i}e_j+\sigma(e_i, e_j)-\langle e_i, e_j\rangle f . \end{aligned}$$

Then

$$\sigma(e_i, e_j)=D_{e_i}D_{e_j}f+\langle e_i, e_j\rangle f .$$

So we have

$$\begin{aligned} \sigma(e_1, e_1)&=\frac{\partial^2 f}{\partial z^2}+2\frac{\partial^2 f}{\partial z\partial \bar{z}}+\frac{\partial^2 f}{\partial \bar{z}^2}+f , \\ \sigma(e_2, e_2)&=-\frac{\partial^2 f}{\partial z^2}+2\frac{\partial^2 f}{\partial z\partial \bar{z}}-\frac{\partial^2 f}{\partial \bar{z}^2}+f , \\ \sigma(e_1, e_2)&=\sqrt{-1}\left(\frac{\partial^2 f}{\partial z^2}-\frac{\partial^2 f}{\partial \bar{z}^2}\right) . \end{aligned}$$

Furthermore we obtain after simple computation

$$\begin{aligned} \sigma(e_1, e_1)=&\sqrt{A_1}\sum_{k=1}^{m}\left\{\frac{\lambda_1}{4}(\mu_k{}^2+\bar{\mu}_k{}^2-2)+1\right\} 2\mathrm{Re}\left\{\sqrt{R_k}u_k \exp\frac{\sqrt{\lambda_1}}{2}(\mu_k z-\bar{\mu}_k\bar{z})\right\} \\ &+\sqrt{A_2}\sum_{j=1}^{m'}\left\{\frac{\lambda_2}{4}(\eta_j{}^2+\bar{\eta}_j{}^2-2)+1\right\} 2\mathrm{Re}\left\{\sqrt{R'_j}u_{m+j} \exp\frac{\sqrt{\lambda_2}}{2}(\eta_j z-\bar{\eta}_j\bar{z})\right\} , \\ \sigma(e_2, e_2)=&\sqrt{A_1}\sum_{k=1}^{m}\left\{\frac{\lambda_1}{4}(-\mu_k{}^2-\bar{\mu}_k{}^2-2)+1\right\} 2\mathrm{Re}\left\{\sqrt{R_k}u_k \exp\frac{\sqrt{\lambda_1}}{2}(\mu_k z-\bar{\mu}_k\bar{z})\right\} \\ &+\sqrt{A_2}\sum_{j=1}^{m'}\left\{\frac{\lambda_2}{4}(-\eta_j{}^2-\bar{\eta}_j{}^2-2)+1\right\} 2\mathrm{Re}\left\{\sqrt{R'_j}u_{m+j} \exp\frac{\sqrt{\lambda_2}}{2}(\eta_j z-\bar{\eta}_j\bar{z})\right\} , \\ \sigma(e_1, e_2)=&\sqrt{A_1}\sum_{k=1}^{m}\frac{\lambda_1}{4}\sqrt{-1}(\mu_k{}^2-\bar{\mu}_k{}^2)\, 2\mathrm{Re}\left\{\sqrt{R_k}u_k \exp\frac{\sqrt{\lambda_1}}{2}(\mu_k z-\bar{\mu}_k\bar{z})\right\} \\ &+\sqrt{A_2}\sum_{j=1}^{m'}\frac{\lambda_2}{4}\sqrt{-1}(\eta_j{}^2-\bar{\eta}_j{}^2)\, 2\mathrm{Re}\left\{\sqrt{R'_j}u_{m+j} \exp\frac{\sqrt{\lambda_2}}{2}(\eta_j z-\bar{\eta}_j\bar{z})\right\} . \end{aligned}$$

Moreover, we get

$$
\begin{aligned}
(4.3)\qquad H=&-\sqrt{A_1}\sum_{k=1}^{m}\frac{\lambda_1-2}{2}\,2\mathrm{Re}\left\{\sqrt{R_k}u_k\exp\frac{\sqrt{\lambda_1}}{2}(\mu_k z-\bar{\mu}_k\bar{z})\right\}\\
&-\sqrt{A_2}\sum_{j=1}^{m'}\frac{\lambda_2-2}{2}\,2\mathrm{Re}\left\{\sqrt{R'_j}u_{m+j}\exp\frac{\sqrt{\lambda_2}}{2}(\eta_j z-\bar{\eta}_j\bar{z})\right\}.
\end{aligned}
$$

So we have

$$
\begin{aligned}
(4.4)\qquad \mathscr{B}(H)&=\sum_{h,i=1}^{2}\langle\sigma(e_h,e_i),H\rangle\sigma(e_h,e_i)\\
&=\sqrt{A_1}\sum_{k=1}^{m}\frac{\lambda_1-2}{2}\left[\frac{\lambda_1\lambda_2}{2}\mathrm{Re}(\psi\mu_k^2)-d\right]2\mathrm{Re}\left\{\sqrt{R_k}u_k\exp\frac{\sqrt{\lambda_1}}{2}(\mu_k z-\bar{\mu}_k\bar{z})\right\}\\
&\quad+\sqrt{A_2}\sum_{j=1}^{m'}\frac{\lambda_2-2}{2}\left[\frac{\lambda_1\lambda_2}{2}\mathrm{Re}(\varphi\eta_j^2)-d\right]2\mathrm{Re}\left\{\sqrt{R'_j}u_{m+j}\exp\frac{\sqrt{\lambda_2}}{2}(\eta_j z-\bar{\eta}_j\bar{z})\right\}
\end{aligned}
$$

where

$$
d=\frac{(\lambda_1-2)^2}{2}A_1+\frac{(\lambda_2-2)^2}{2}A_2\,.
$$

In order to prove Theorem D, we assume that M is a Chen surface in $S^N(1)$ so that $\mathscr{B}(H)$ is parallel to H. By (4.3) and (4.4), we have the following.

LEMMA 4.1. *A mass-symmetric 2-type immersion f of $\boldsymbol{R}^2$ into S^N is a Chen immersion if and only if*

$$
(4.5)\qquad \mathrm{Re}(\psi\mu_k^2)=\mathrm{Re}(\varphi\eta_j^2)\qquad \textit{for } \forall k=1,\cdots,m \textit{ and } j=1,\cdots,m'\,.
$$

PROOF OF THEOREM D. Suppose $\varphi\neq0$ so that $\psi\neq0$ by (4.2). By (4.5), $\mathrm{Re}(\psi\mu_k^2)=\mathrm{Re}(\psi\mu_l^2)$. If $k\neq l$, then we get $\psi\mu_k^2=\overline{\psi\mu_l^2}$ so that $\mu_k^2=(\bar{\psi}/\psi)\bar{\mu}_l^2$. This implies $m\leqq2$. Similary we get $m'\leqq2$ so that $N\leqq7$.

Finally, if $N\geqq9$, then we have $\varphi=\psi=0$. We put $\rho_i=\sqrt{\lambda_i/2}$, $i=1,2$. By (4.1) and Theorem 3.1 and Corollary 3.2 in Bryant [5] (or by direct computation), $F_1/(\rho_1\sqrt{A_1})$ (resp. $F_2/(\rho_2\sqrt{A_2})$) is a minimal immersion of M into $S^{2m-1}(\rho_1^2)$ (resp. $S^{2m'-1}(\rho_2^2)$) and f is a diagonal sum of these two minimal immersions. The proof of Theorem D is completed.

For $0<\nu_1<2<\nu_2$, we put $c(\nu_1,\nu_2)=(\nu_2(2-\nu_1))/(\nu_1(\nu_2-2))$ (>0). In the case of $\varphi\neq0$, we obtain the following.

PROPOSITION 4.2. *Let M be a flat surface in $S^5(1)$ which is immersed fully by an isometric immersion f. If M is a mass-symmetric 2-type Chen surface, then we get $c(\lambda_1,\lambda_2)\neq1$ and we have, in (4.1), either*

(1) $R_1=R_2=1/2$, $R_1'=1$ *and* $\cos 2\nu=-c(\lambda_1, \lambda_2)$ (*if* $m=2$ *and* $m'=1$),

or

(2) $R_1=1$, $R_1'=R_2'=1/2$ *and* $\cos 2\nu=-(c(\lambda_1, \lambda_2))^{-1}$
(*if* $m=1$ *and* $m'=2$),

where ν *is the angle between* μ_1 *and* η_1.

PROOF. We assume that $m=2$ and $m'=1$. We have $R_1'=1$, $\psi=\bar{\eta}_1^2$ and $\mu_2^2=(\eta_1^2/\bar{\eta}_1^2)\bar{\mu}_1^2$. By (4.1) and (4.2), we get

$$(4.6)\qquad \left(\frac{\mu_1^2}{\eta_1^2}\right)R_1+\overline{\left(\frac{\mu_1^2}{\eta_1^2}\right)}R_2=-\frac{\lambda_2 A_2}{\lambda_1 A_1}\,,\qquad R_1+R_2=1\,.$$

If $(\mu_1^2/\eta_1^2)-\overline{(\mu_1^2/\eta_1^2)}=0$, then we have $\mu_1^2=\alpha\eta_1^2$ for some real number α so that $\mu_2^2=\mu_1^2$. This is a contradiction. Since R_1 and R_2 are real, we get $R_1=R_2=1/2$ by (4.6). Moreover we get $\mathrm{Re}(\mu_1^2/\eta_1^2)=-(\lambda_2 A_2)/(\lambda_1 A_1)=-c(\lambda_1, \lambda_2)$ and $\mathrm{Im}(\mu_1^2/\eta_1^2)\neq 0$. This implies $\cos 2\nu=-c(\lambda_1, \lambda_2)\neq -1$ where ν is the angle between μ_1 and η_1. Therefore we get (1). (2) is proved similarly. Q.E.D.

PROPOSITION 4.3. *For any two constants* λ_1 *and* λ_2 *such that* $0<\lambda_1<2<\lambda_2$ *and* $\lambda_1\lambda_2-\lambda_1-\lambda_2\neq 0$, *there exists only one mass-symmetric 2-type full Chen immersion* f *of* $\boldsymbol{R}^2$ *into* $S^5(1)$ *with respect to eigenvalues* λ_1 *and* λ_2. $m=2$ *and* $m'=1$ (*resp.* $m=1$ *and* $m'=2$) *in* (4.1) *if and only if* $c(\lambda_1, \lambda_2)<1$ (*resp.* $c(\lambda_1, \lambda_2)>1$). *Moreover* f *is doubly periodic if and only if either*

$$Q=\left(\frac{\lambda_2(\lambda_2-2)}{\lambda_1\lambda_2-\lambda_1-\lambda_2}\right)^{1/2}\ \textit{is rational}\quad (\textit{if } c(\lambda_1, \lambda_2)<1)\,,$$

or

$$Q'=\left(\frac{\lambda_1(\lambda_1-2)}{\lambda_1\lambda_2-\lambda_1-\lambda_2}\right)^{1/2}\ \textit{is rational}\quad (\textit{if } c(\lambda_1, \lambda_2)>1)\,.$$

PROOF. $\lambda_1\lambda_2-\lambda_1-\lambda_2\neq 0$ implies $c(\lambda_1, \lambda_2)\neq 1$. We prove the case of $c(\lambda_1, \lambda_2)<1$. We put $\cos 2\nu=-c(\lambda_1, \lambda_2)$, $\mu_1=1$, $\mu_2=e^{2\nu i}$, $\eta_1=e^{\nu i}$, $R_1=R_2=1/2$ and $R_1'=1$. By Lemma 4.1, the map f in (4.1) is a mass-symmetric 2-type full Chen immersion of $\boldsymbol{R}^2$ into $S^5(1)$ with respect to eigenvalues λ_1 and λ_2. By Proposition 4.2, this immersion is unique up to the action of the isometries of the domain and range. We define

$$\begin{aligned}\Lambda_f&=\{z\in \boldsymbol{R}^2 \mid f(z)=f(0)\}\\ &=\{z \mid \langle i\sqrt{\lambda_1}\bar{\mu}_1, z\rangle\equiv\langle i\sqrt{\lambda_1}\bar{\mu}_2, z\rangle\equiv\langle i\sqrt{\lambda_2}\bar{\eta}_1, z\rangle\equiv 0 \mod 2\pi\}\,.\end{aligned}$$

Clearly $\Lambda_f\subset \boldsymbol{R}^2$ is a discrete lattice. After rotating $\boldsymbol{R}^2$ if necessary, we may put $i\bar{\mu}_1=1$, $i\bar{\mu}_2=e^{2\nu i}$, $i\bar{\eta}_1=e^{\nu i}$ and $0<\nu<\pi/2$. Set

$$x_1=\left(\frac{2\pi}{\sqrt{\lambda_1}},\ \frac{-2\pi\cos 2\nu}{\sqrt{\lambda_1}\sin 2\nu}\right), \qquad x_2=\left(0,\ \frac{2\pi}{\sqrt{\lambda_1}\sin 2\nu}\right).$$

Then we obtain

$$\langle i\sqrt{\lambda_1}\bar{\mu}_1,\ nx_1+mx_2\rangle=2\pi n\ , \qquad \langle i\sqrt{\lambda_1}\bar{\mu}_2,\ nx_1+mx_2\rangle=2\pi m\ ,$$
$$\text{and}\quad \langle i\sqrt{\lambda_2}\bar{\eta}_1,\ nx_1+mx_2\rangle=2\pi\frac{Q}{2}(n+m)\ .$$

So we get

$$\Lambda_f=\left\{nx_1+mx_2\ \middle|\ \frac{Q}{2}(n+m)\in \boldsymbol{Z} \text{ for } n,\ m\in \boldsymbol{Z}\right\}.$$

If f is doubly periodic, then there exists an element x of Λ_f such that $x=nx_1+mx_2$, $n+m\neq 0$. Therefore we see that f is doubly periodic if and only if Q is rational.

The case of $c(\lambda_1, \lambda_2)>1$ is proved similarly. In this case, we have $\cos 2\nu=-c(\lambda_1, \lambda_2)^{-1}$ and

$$\begin{aligned}\Lambda_f&=\{z\in \boldsymbol{R}^2\mid f(z)=f(0)\}\\ &=\left\{nx_1'+mx_2'\ \middle|\ \frac{Q'}{2}(n+m)\in \boldsymbol{Z} \text{ for } n,\ m\in \boldsymbol{Z}\right\}\end{aligned}$$

where

$$x_1'=\left(\frac{2\pi}{\sqrt{\lambda_2}},\ \frac{-2\pi\cos 2\nu}{\sqrt{\lambda_2}\sin 2\nu}\right) \text{ and } x_2'=\left(0,\ \frac{2\pi}{\sqrt{\lambda_2}\sin 2\nu}\right). \qquad \text{Q.E.D.}$$

REMARK. (1) If Q and Q' are not rational, then Λ_f is of rank 1. In this case, f induces a mass-symmetric 2-type full Chen imbedding of a cylinder $\boldsymbol{R}^2/\Lambda_f$ into $S^5(1)$.

(2) If $\lambda_1\lambda_2-\lambda_1-\lambda_2=0$, then there exist no mass-symmetric 2-type full Chen immersions of $\boldsymbol{R}^2$ into $S^5(1)$ with respect to λ_1 and λ_2. But such an immersion into $S^3(1)$ always satisfies $\lambda_1\lambda_2-\lambda_1-\lambda_2=0$.

Let λ_1 and λ_2 be two constants such that $0<\lambda_1<2<\lambda_2$ and $c(\lambda_1, \lambda_2)\neq 1$. We define a discrete lattice $\Lambda(\lambda_1, \lambda_2)$ as follows.

In the case of $c(\lambda_1, \lambda_2)<1$, we put $Q/2=q'/q$, where q and q' are relatively prime, and $q>0$ if Q is rational. We put

$$\begin{aligned}&u=v=x_1-x_2 \quad (\text{if } Q \text{ is irrational})\ ,\\ &u=x_1 \text{ and } v=x_2 \quad (\text{if } q=1)\ ,\\ &u=x_1+x_2 \text{ and } v=x_1-x_2 \quad (\text{if } q=2)\ ,\\ &u=(q-1)x_1+x_2 \text{ and } v=(q-2)x_1+2x_2 \quad (\text{if } q\geqq 3)\ .\end{aligned}$$

In the case of $c(\lambda_1, \lambda_2)>1$, we put $Q'/2=q'/q$, where q and q' are relatively prime, and $q>0$ if Q' is rational. We put

$$u=v=x_1-x_2 \quad (\text{if } Q' \text{ is irrational}),$$
$$u=x_1' \text{ and } v=x_2' \quad (\text{if } q=1),$$
$$u=x_1'+x_2' \text{ and } v=x_1'-x_2' \quad (\text{if } q=2),$$
$$u=(q-1)x_1'+x_2' \text{ and } v=(q-2)x_1'+2x_2' \quad (\text{if } q\geqq 3).$$

We define a discrete lattice $\Lambda(\lambda_1, \lambda_2)$ in $\boldsymbol{R}^2$ by

$$\Lambda(\lambda_1, \lambda_2)=\{ku+lv \mid k, l\in \boldsymbol{Z}\}.$$

Λ_f in Proposition 4.3 is congruent to $\Lambda(\lambda_1, \lambda_2)$.

COROLLARY 4.4. *Let Λ be a discrete lattice of rank 2 in $\boldsymbol{R}^2$ and $T^2\simeq \boldsymbol{R}^2/\Lambda$ a flat torus generated by Λ. Let λ_1 and λ_2 be any eigenvalues such that $\lambda_1<2<\lambda_2$.*

Then T^2 admits a mass-symmetric 2-type full Chen immersion into $S^5(1)$ with respect to λ_1 and λ_2 if and only if $c(\lambda_1, \lambda_2)\neq 1$ and Λ is an abelian subgroup of $\Lambda(\lambda_1, \lambda_2)$.

In particular, T^2 admits such an imbedding if and only if $c(\lambda_1, \lambda_2)\neq 1$ and Λ is congruent to $\Lambda(\lambda_1, \lambda_2)$.

PROOF. Let f_0 be a mass-symmetric 2-type full Chen immersion of T^2 into $S^5(1)$ with respect to λ_1 and λ_2. Then f_0 can be extended uniquely to such an immersion f of the universal covering $\boldsymbol{R}^2$ of T^2. By Proposition 4.2, we have $c(\lambda_1, \lambda_2)\neq 1$, and by the definition, we obtain

$$\Lambda(\lambda_1, \lambda_2)=\{z\in \boldsymbol{R}^2 \mid f(z)=f(0)\}.$$

Since f is the extension of f_0, we see that Λ is an abelian subgroup of $\Lambda(\lambda_1, \lambda_2)$.

Conversely, we assume that $c(\lambda_1, \lambda_2)\neq 1$ and Λ is an abelian subgroup of $\tilde{\Lambda}=\Lambda(\lambda_1, \lambda_2)$ so that $\tilde{\Lambda}$ is of rank 2. By Proposition 4.3, we get a mass-symmetric 2-type full Chen imbedding $\tilde{f}$ of a flat torus $\boldsymbol{R}^2/\tilde{\Lambda}$ into $S^5(1)$ with respect to λ_1 and λ_2. Let π: $T^2\simeq \boldsymbol{R}^2/\Lambda\to \boldsymbol{R}^2/\tilde{\Lambda}$ be a Riemannian covering map and put $f_0=\tilde{f}\circ\pi$. Then f_0 is a mass-symmetric 2-type full Chen immersion of T^2 into $S^5(1)$ with respect to λ_1 and λ_2.

Q.E.D.

For the case $N=7$, we obtain the following.

PROPOSITION 4.5. *Let M be a flat surface in $S^7(1)$ which is immersed* *fully by an isometric immersion f. If M is a mass-symmetric 2-type Chen surface, then we have $m=m'=2$ and $R_1=R_2=R'_1=R'_2=1/2$ in (4.1) and either*

(1) *f is a diagonal sum of two minimal immersions of M into S^3, or*

(2) *$\cos\alpha=-c(\lambda_1, \lambda_2)\cos\beta$, where α (resp. β) is the angle between μ_1^2 (resp. η_1^2) and φ.*

PROOF. Put $N=7$ in (4.1). If $\varphi=0$ in (4.1), then we get (1) in a way similar to the proof of Theorem D. Since F_1 and F_2 are immersions and since $2(m+m')-1=7$, we get $m=m'=2$. By (4.1) and (4.2), we have

$$\mu_1^2R_1+\mu_2^2R_2=0\ , \qquad R_1+R_2=1\ .$$

Since $\mu_2\neq\pm\mu_1$, we obtain $R_1=R_2=1/2$ and $\mu_2^2=-\mu_1^2$. Similarly, we obtain $R'_1=R'_2=1/2$ and $\eta_1^2=-\eta_2^2$.

Assume $\varphi\neq0$. Then in the proof of Theorem D, we get $m=m'=2$ and

$$\mu_2^2=\frac{\bar\psi}{\psi}\bar\mu_1^2\ , \qquad \eta_2^2=\frac{\bar\varphi}{\varphi}\bar\eta_1^2\ .$$

By (4.1) and (4.2), we get

$$\left(\frac{\mu_1^2}{\bar\psi}\right)R_1+\overline{\left(\frac{\mu_1^2}{\bar\psi}\right)}R_2=-\frac{\lambda_2A_2}{\lambda_1A_1}\ , \qquad R_1+R_2=1\ . \tag{4.7}$$

If $(\mu_1^2/\bar\psi)-\overline{(\mu_1^2/\bar\psi)}=0$, then we have $\mu_1^2=\alpha\bar\psi$ for some real number α so that $\mu_2^2=\mu_1^2$. This is a contradiction. Since R_1 and R_2 are real, we get $R_1=R_2=1/2$ by (4.7). Similarly, we get

$$\left(\frac{\eta_1^2}{\bar\varphi}\right)R'_1+\overline{\left(\frac{\eta_1^2}{\bar\varphi}\right)}R'_2=-\frac{\lambda_1A_1}{\lambda_2A_2}\ , \qquad R'_1+R'_2=1\ ,$$

and $R'_1=R'_2=1/2$. After rotating $\boldsymbol{R}^2$ and changing an orthonormal basis of E^8 if necessary, we assume that φ is real, $\mu_1^2=e^{i\alpha}$ and $\eta_1^2=e^{i\beta}$. Then, by (4.2), we see that $\cos\alpha=-c(\lambda_1, \lambda_2)\cos\beta$. Q.E.D.

REMARK. After rotating $\boldsymbol{R}^2$ and changing an orthonormal basis of E^8, we may assume $0<\alpha<\pi/2<\beta<\pi$.

For any three constants λ_1, λ_2 and t such that $0<\lambda_1<2<\lambda_2$ and $0\leqq t\leqq\pi/4$, we define two mass-symmetric 2-type Chen immersions of $\boldsymbol{R}^2$ into $S^7(1)$ with respect to eigenvalues λ_1 and λ_2 as follows.

Put $R_1=R_2=R_1'=R_2'=1/2$, $\mu_1=1$, $\mu_2=e^{i\pi/2}$, $\eta_1=e^{it}$ and $\eta_2=e^{i(t+\pi/2)}$. Then for maps F_1 and F_2 defined in (4.1), we see that $F_1/(\rho_1\sqrt{A_1})$ and $F_2/(\rho_2\sqrt{A_2})$ are minimal immersions of $\boldsymbol{R}^2$ into $S^3(\rho_1^2)$ and $S^3(\rho_2^2)$ respectively, where $\rho_i=\sqrt{\lambda_i/2}$, $i=1, 2$. We put

$$f_1=f_1(\lambda_1, \lambda_2, t)=F_1+F_2 .$$

f_1 is a diagonal sum of these two minimal immersions so that f_1 is a mass-symmetric 2-type full Chen immersion into $S^7(1)$ with respect to eigenvalues λ_1 and λ_2.

Put $c=c(\lambda_1, \lambda_2)$. We define two constants α and β $(0\leqq\alpha\leqq\pi/2\leqq\beta\leqq\pi)$ as follows.

If $c<1$, then we put $\alpha=\cos^{-1}(c\cdot\cos(2t))$ and $\beta=\pi-2t$.

If $c\geqq 1$, then we put $\alpha=2t$ and $\beta=\pi-\cos^{-1}((1/c)\cos(2t))$.

Put $R_1=R_2=R_1'=R_2'=1/2$, $\mu_1=e^{i\alpha/2}$, $\eta_1=e^{i\beta/2}$, $\mu_2=\bar{\mu}_1$ and $\eta_2=\bar{\eta}_1$ and denote by $f_2(\lambda_1, \lambda_2, t)$ (or f_2) a map defined in (4.1). By Lemma 4.1, f_2 is a mass-symmetric 2-type Chen immersion of $\boldsymbol{R}^2$ into $S^7(1)$ with respect to eigenvalues λ_1 and λ_2. Moreover, f_2 is a full immersion into $S^3(1)$ if and only if $c=1$ and $t=0$, f_2 is a full immersion into $S^5(1)$ if and only if $c\neq 1$ and $t=0$, f_2 is a full immersion into $S^7(1)$ if and only if $t>0$ and f_2 is a diagonal sum of two minimal immersions of $\boldsymbol{R}^2$ into S^3 if and only if $t=\pi/4$, i.e., $f_2(\lambda_1, \lambda_2, \pi/4)$ is congruent to $f_1(\lambda_1, \lambda_2, 0)$ up to the action of the isometries of the domain and range. Finally, if $t\neq 0$ or $\pi/4$, then f_2 is a mass-symmetric 2-type full Chen immersion of $\boldsymbol{R}^2$ into $S^7(1)$ which is not a diagonal sum of two minimal immersions of $\boldsymbol{R}^2$ into S^3.

Let f be a mass-symmetric 2-type full Chen immersion of $\boldsymbol{R}^2$ into $S^7(1)$ with respect to eigenvalues λ_1 and λ_2, and assume f is not a diagonal sum of two minimal immersions of $\boldsymbol{R}^2$ into S^3. In the proof of Proposition 4.5, put $t=(\pi-\beta)/2$ (if $c<1$) or $t=\alpha/2$ (if $c\geqq 1$). Thus, we see $0<t<\pi/4$. (See last Remark.) By Proposition 4.5 and the definition of f_2, f is congruent to $f_2(\lambda_1, \lambda_2, t)$ up to the action of the isometries of the domain and range.

Combining the above with Proposition 4.3, we obtain the following.

THEOREM H. *For any constants λ_1, λ_2 such that $0<\lambda_1<2<\lambda_2$, there exists a one-to-one correspondence between mass-symmetric 2-type Chen immersions of $\boldsymbol{R}^2$ into $S^7(1)$ with respect to eigenvalues λ_1 and λ_2 and a family*

$$\{f_1(\lambda_1, \lambda_2, t)\mid 0\leqq t\leqq\pi/4\}\cup\{f_2(\lambda_1, \lambda_2, t)\mid 0\leqq t<\pi/4\}$$

up to the action of isometries of the domain and range.

§5. Totally real 2-type surfaces in S^6.

In this section, we apply our results to some surfaces in S^6 and prove Theorems F and G.

5.0. Totally real submanifolds of S^6. We realize an 8-dimensional Euclidean space E^8 as the underlying vector space of Cayley division algebra $\mathfrak{C}=\{e_0=1, e_i\,(1\leqq i\leqq 7)\}$. The automorphism group of $\mathfrak{C}$ is the compact simple Lie group G_2. Let $\mathfrak{C}_+$ be the subspace of $\mathfrak{C}$ consisting of all pure imaginary Cayley numbers. Then $\mathfrak{C}_+$ is identified with a 7-dimensional Euclidean space E^7 and stable under the action of G_2. A vector cross product for vectors in $\mathfrak{C}_+=E^7$ is defined by

$$x\times y=\langle x, y\rangle e_0+x\cdot y\ ,$$

where $\cdot$ denotes the multiplication as Cayley algebra and $\langle\ ,\ \rangle$ is the canonical Euclidean inner product. The multiplication table is as follows:

(5.1) $e_j\times e_k=$

$j \backslash k$	1	2	3	4	5	6	7
1	0	e_3	$-e_2$	e_5	$-e_4$	e_7	$-e_6$
2	$-e_3$	0	e_1	$-e_6$	e_7	e_4	$-e_5$
3	e_2	$-e_1$	0	e_7	e_6	$-e_5$	$-e_4$
4	$-e_5$	e_6	$-e_7$	0	e_1	$-e_2$	e_3
5	e_4	$-e_7$	$-e_6$	$-e_1$	0	e_3	e_2
6	$-e_7$	$-e_4$	e_5	e_2	$-e_3$	0	e_1
7	e_6	e_5	e_4	$-e_3$	$-e_2$	$-e_1$	0

Regarding $S^6(1)$ as $\{x\in\mathfrak{C}_+\mid\langle x, x\rangle=1\}$, we may define an almost complex structure J on S^6 by

$$JX=x\times X\ ,$$

where $x\in S^6$ and $X\in T_x(S^6)$ (the tangent space of S^6 at x). Let $\tilde{g}$ be the metric on S^6 induced from E^7 so that $\tilde{g}$ is a Hermitian metric of the almost complex manifold (S^6, J). We have

$$(\tilde{\nabla}_X J)Y=X\times Y+\tilde{g}(X, JY)x \tag{5.2}$$

for $x \in S^6$ and $X, Y \in T_x(S^6)$, where $\tilde{\nabla}$ is the Riemannian connection of $(S^6, \tilde{g})$. Thus the almost Hermitian manifold $(S^6, J, \tilde{g})$ is a nearly Kaehlerian manifold, i.e., $(\tilde{\nabla}_X J)X=0$ for any tangent vector X of S^6. We note that the Lie group G_2 is the group of all automorphisms of the nearly Kaehlerian manifold $(S^6, J, \tilde{g})$.

For any vector fields X and Y of S^6, we put $G(X,Y)=(\tilde{\nabla}_X J)Y$ so that G is a skew-symmetric tensor field of type (1, 2) on S^6. We have the following. See Gray [11] and [12].

LEMMA 5.1.
(1) $G(X, JY)=G(JX, Y)=-JG(X, Y)$,
(2) $(\tilde{\nabla}_X G)(Y, Z)=\tilde{g}(Y, JZ)X+\tilde{g}(X, Z)JY-\tilde{g}(X, Y)JZ$,
(3) $\|G(X,Y)\|^2=\|X\|^2\cdot\|Y\|^2-\tilde{g}(X,Y)^2-\tilde{g}(X, JY)^2$,
for any X, Y *and* $Z \in \mathfrak{X}(S^6)$ *(the vector space of all vector fields on* S^6*).*

Let (M, g) be a submanifold of $(S^6, J, \tilde{g})$, and $T_x^\perp(M)$ the normal space of M at a point x of M. From now on, we assume that M is a totally real submanifold of S^6, i.e., $JX \in T_x^\perp(M)$ for any $x \in M$ and any $X \in T_x(M)$. Note that the dimension of M is 2 or 3.

Denote by ∇ the Riemannian connection of M, and by h, A and $\nabla^\perp$ the second fundamental form, the Weingarten map and the normal connection of M in S^6 respectively. We have the Gauss' formula and the Weingarten's formula:

$$\tilde{\nabla}_X Y=\nabla_X Y+h(X,Y)\ , \qquad \tilde{\nabla}_X \xi=-A_\xi X+\nabla_X^\perp \xi\ ,$$

where X, Y and Z are tangent vector fields and ξ is a normal vector field. Moreover, we see

$$g(A_\xi X, Y)=\tilde{g}(h(X,Y), \xi)\ .$$

LEMMA 5.2. $G(X,Y) \in T_x^\perp(M)$ *for any* $X, Y \in T_x(M)$.

Ejiri [9] shows this lemma in the case of $\dim(M)=3$. But the proof in [9] is also true in the case of $\dim(M)=2$. We obtain the following.

LEMMA 5.3. *If* X *and* Y $(\in T_x(M))$ *are linearly independent, then* $JG(X,Y)$ *is perpendicular to both* X *and* Y *in* $T_x(S^6)$.

PROOF. By Lemma 5.1 (1), we have

$$\begin{aligned}(\tilde{\nabla}_X G)(JY, JZ) &= \tilde{\nabla}_X \cdot G(JY, JZ) - G(\tilde{\nabla}_X \cdot JY, JZ) - G(JY, \tilde{\nabla}_X \cdot JZ) \\ &= -\tilde{\nabla}_X \cdot G(Y, Z) - G(G(X, Y), JZ) - G(J \cdot \tilde{\nabla}_X Y, JZ) \\ &\quad - G(JY, G(X, Z)) - G(JY, J \cdot \tilde{\nabla}_X Z) \\ &= -\tilde{\nabla}_X \cdot G(Y, Z) + JG(G(X, Y), Z) + G(\tilde{\nabla}_X Y, Z) \\ &\quad + JG(Y, G(X, Z)) + G(Y, \tilde{\nabla}_X Z) .\end{aligned}$$

Thus, we obtain

$$G(Y, G(Z, X)) + G(Z, G(X, Y)) = J(\tilde{\nabla}_X G)(JY, JZ) + J(\tilde{\nabla}_X G)(Y, Z)$$

for any X, Y and $Z \in \mathfrak{X}(S^6)$. It follows from Lemma 5.1 (2) that

$$\begin{aligned}(5.3) \quad & G(Y, G(Z, X)) + G(Z, G(X, Y)) \\ & = 2\tilde{g}(Y, JZ)JX - \tilde{g}(X, JZ)JY - \tilde{g}(X, Z)Y + \tilde{g}(X, JY)JZ + \tilde{g}(X, Y)Z ,\end{aligned}$$

$$\begin{aligned}(5.4) \quad & G(Z, G(X, Y)) + G(X, G(Y, Z)) \\ & = 2\tilde{g}(Z, JX)JY - \tilde{g}(Y, JX)JZ - \tilde{g}(Y, X)Z + \tilde{g}(Y, JZ)JX + \tilde{g}(Y, Z)X ,\end{aligned}$$

$$\begin{aligned}(5.5) \quad & G(X, G(Y, Z)) + G(Y, G(Z, X)) \\ & = 2\tilde{g}(X, JY)JZ - \tilde{g}(Z, JY)JX - \tilde{g}(Z, Y)X + \tilde{g}(Z, JX)JY + \tilde{g}(Z, X)Y .\end{aligned}$$

Computing $(-(5.3)+(5.4)+(5.5))$, we have

$$(5.6) \quad G(X, G(Y, Z)) = \tilde{g}(X, Z)Y - \tilde{g}(X, Y)Z + \tilde{g}(X, JY)JZ - \tilde{g}(X, JZ)JY ,$$

or

$$G(Z, G(X, Y)) = \tilde{g}(Z, Y)X - \tilde{g}(Z, X)Y + \tilde{g}(Z, JX)JY - \tilde{g}(Z, JY)JX ,$$

for any X, Y and $Z \in \mathfrak{X}(S^6)$. Hence we assume that X, $Y \in T_x(M)$ and $Z = JG(X, Y)$. This equation, together with Lemma 5.1 (1) and 5.2, implies

$$\begin{aligned}(5.7) \quad 0 = JG(Z, Z) &= G(Z, G(X, Y)) \\ &= \tilde{g}(Z, Y)X - \tilde{g}(Z, X)Y + \tilde{g}(G(X, Y), X)JY - \tilde{g}(G(X, Y), Y)JX \\ &= \tilde{g}(Z, Y)X - \tilde{g}(Z, X)Y .\end{aligned}$$

Now, we assume that X and Y are linearly independent. Then (5.7) implies $\tilde{g}(Z, X) = \tilde{g}(Z, Y) = 0$. Q.E.D.

REMARK. This lemma in the case of $\dim(M)=3$ is shown by Ejiri [9].

5.1. Proof of Theorem F. Let T be a maximal torus of G_2. Then, for $\sigma \in G_2$, $\sigma T \sigma^{-1}$ is also a maximal torus. Therefore we may put

$$(5.8)\qquad T=\left\{\left(\begin{array}{ccccccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \cos a & \sin a & 0 & 0 & 0 & 0 \\ 0 & -\sin a & \cos a & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos b & \sin b & 0 & 0 \\ 0 & 0 & 0 & -\sin b & \cos b & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cos c & \sin c \\ 0 & 0 & 0 & 0 & 0 & -\sin c & \cos c \end{array}\right) \middle| \begin{array}{l} a, b, c\in \boldsymbol{R} \\ a+b+c=0 \end{array}\right\}$$

with respect to the basis $\{e_1, \cdots, e_7\}$.

Let $x=\sum_{i=1}^{7} x^i e_i$ be a point of S^6 so that $\sum_{i=1}^{7}(x^i)^2=1$. We assume that an orbit Tx is a flat surface of S^6. We have $|x^i|\neq 1$ for any i, and Tx lies in a hypersphere $S^5=S^5(1/(1-(x^1)^2)$. After changing parameters a and b if necessary, we may assume $x^3=x^5=0$. Thus, we have

$$\begin{aligned} &Tx=\{\varphi(u^1, u^2)\mid u^1, u^2\in \boldsymbol{R}\}\ , \\ &\quad \varphi(u^1, u^2)=x^1e_1+(x^2\cos u^1)e_2-(x^2\sin u^1)e_3+(x^4\cos u^2)e_4-(x^4\sin u^2)e_5 \\ &\qquad +(x^6\cos(u^1+u^2)-x^7\sin(u^1+u^2))e_6+(x^6\sin(u^1+u^2)+x^7\cos(u^1+u^2))e_7\ . \end{aligned}$$

Note that $\varphi(0, 0)=x$. For $i=1, 2$, denote $(\partial\varphi/\partial u^i)(0, 0)$ by φ_i. Then we get

$$\varphi_1=-x^2e_3-x^7e_6+x^6e_7\ ,\qquad \varphi_2=-x^4e_5-x^7e_6+x^6e_7\ .$$

Since Tx is a surface, we see

$$(5.9)\qquad (x^2)^2+(x^4)^2\neq 0\ ,\quad (x^2)^2+(x^6)^2+(x^7)^2\neq 0\ ,\quad \text{and}\quad (x^4)^2+(x^6)^2+(x^7)^2\neq 0\ .$$

Put $g_{ij}=\langle\varphi_i, \varphi_j\rangle$ so that

$$\begin{aligned} g_{11}&=(x^2)^2+(x^6)^2+(x^7)^2\ , \\ g_{22}&=(x^4)^2+(x^6)^2+(x^7)^2\ , \\ g_{12}&=g_{21}=(x^6)^2+(x^7)^2\ , \end{aligned}$$

and put $g=\sum_{i,j=1}^{2} g_{ij}du^i\otimes du^j$. Then φ can be considered as an isometric immersion of $\boldsymbol{R}^2=\{(u^1, u^2)\mid u^1, u^2\in\boldsymbol{R}\}$ with metric tensor g into S^6.

Let Δ be the Laplacian of $(\boldsymbol{R}^2, g)$, and let $(g^{ij})=(g_{ij})^{-1}$. Then we get $\Delta=-\sum_{i,j=1}^{2} g^{ij}(\partial^2/\partial u^i\partial u^j)$. It is easy to show the following.

LEMMA 5.4. *A T-orbit Tx whose dimension is 2 is mass-symmetric in S^5 and at most of 3-type.*

In particular, Tx is mass-symmetric in S^6 if and only if $x^1=0$, and

Tx is of 2-type if and only if Tx is not minimal in S^6 and x satisfies one of the following:

(1) $x^2=0$ *or* $x^4=0$ *or* $x^6=x^7=0$,

(2) $(x^2)^2=(x^4)^2$,

(3) $(x^2)^2=(x^6)^2+(x^7)^2$,

(4) $(x^4)^2=(x^6)^2+(x^7)^2$.

By the definition, we have

$$\begin{aligned}J\varphi_1&=\varphi(0,0)\times\varphi_1\\&=(-(x^2)^2+(x^6)^2+(x^7)^2)e_1+(x^1x^2+x^4x^7)e_2\\&\quad+x^4x^6e_3-2x^2x^7e_4-2x^2x^6e_5-x^1x^6e_6+(x^2x^4-x^1x^7)e_7\,.\end{aligned}$$

Hence we obtain

$$\tilde{g}(J\varphi_1,\varphi_2)=3x^2x^4x^6\,.$$

This implies the following.

LEMMA 5.5. *Tx is totally real if and only if $x^2x^4x^6=0$.*

Denote by h and H the second fundamental form and the mean curvature vector of Tx in S^6. After long but simple computation, we have, at the base point x,

$$h(\varphi_i,\varphi_j)=\frac{\partial^2\varphi}{\partial u^i\partial u^j}(0,0)+g_{ij}\varphi(0,0)$$

or

$$\begin{aligned}h(\varphi_1,\varphi_1)&=-x^2e_2-x^6e_6-x^7e_7+g_{11}x\,,\\h(\varphi_2,\varphi_2)&=-x^4e_4-x^6e_6-x^7e_7+g_{22}x\,,\\h(\varphi_1,\varphi_2)&=-x^6e_6-x^7e_7+g_{12}x\,,\end{aligned}$$

and

$$\begin{aligned}H&=\frac{1}{2}\sum_{i,j=1}^{2}g^{ij}h(\varphi_i,\varphi_j)\\&=\frac{1}{2D}(2Dx^1e_1+(2D-g_{22})x^2e_2+(2D-g_{11})x^4e_4\\&\quad+(2D-g_{11}-g_{22}+2g_{12})x^6e_6+(2D-g_{11}-g_{22}+2g_{12})x^7e_7)\end{aligned}$$

and

$$(5.10)\qquad \begin{aligned} JH&=x\times H\\ &=\frac{1}{2D}((g_{22}-2g_{12})x^4x^6e_2+((-g_{22}+2g_{12})x^4x^7-g_{22}x^1x^2)e_3\\ &\quad+(-g_{11}+2g_{12})x^2x^6e_4+((g_{11}-2g_{12})x^2x^7-g_{11}x^1x^4)e_5\\ &\quad+((g_{11}-g_{22})x^2x^4+(g_{11}+g_{22}-2g_{12})x^1x^7)e_6\\ &\quad+(-g_{11}-g_{22}+2g_{12})x^1x^6e_7)\ , \end{aligned}$$

where $D=\det(g_{ij})$.

From now on, we assume that Tx is a totally real mass-symmetric 2-type surface in S^6. From Lemmas 5.4 and 5.5, we see that $x^1=0$ and $x^2x^4x^6=0$. Hence one of the following four cases occurs:

(case A) $x^2=0$,
(case B) $x^4=0$,
(case C) $x^2\neq 0,\ x^4\neq 0,\ x^6=x^7=0$,
(case D) $x^2\neq 0,\ x^4\neq 0,\ x^6=0,\ x^7\neq 0$.

Lemma 5.6. *Suppose that Tx is totally real and mass-symmetric in S^6, i.e., $x^1=0$ and $x^2x^4x^6=0$. Then*

(1) JH is normal to Tx if and only if

$$x^2=0 \text{ or } x^4=0 \text{ or } x^6=x^7=0$$
$$\text{or } (x^6=0 \text{ and } (x^2)^2=(x^4)^2=(x^7)^2=1/3)\ .$$

In the last case, Tx is minimal in S^6.

(2) JH is tangent to Tx if and only if

$$(5.11)\qquad (x^2)^2=(x^4)^2 \text{ or } (x^2)^2=(x^6)^2+(x^7)^2 \text{ or } (x^4)^2=(x^6)^2+(x^7)^2\ .$$

Proof. (1) From (5.10), we easily see that

$$\begin{aligned} 2D\tilde{g}(JH,\varphi_1)&=(2g_{22}-g_{11}-2g_{12})x^2x^4x^7\\ &=(2(x^4)^2-(x^2)^2-(x^6)^2-(x^7)^2)x^2x^4x^7 \end{aligned}$$

and

$$\begin{aligned} 2D\tilde{g}(JH,\varphi_2)&=-(2g_{11}-g_{22}-2g_{12})x^2x^4x^7\\ &=-(2(x^2)^2-(x^4)^2-(x^6)^2-(x^7)^2)x^2x^4x^7\ . \end{aligned}$$

If JH is normal to Tx, then $\tilde{g}(JH,\varphi_1)=\tilde{g}(JH,\varphi_2)=0$. If $x^2x^4x^7\neq 0$, we have $x^6=0$ and $(x^2)^2=(x^4)^2=(x^7)^2$. Since $(x^2)^2+(x^4)^2+(x^7)^2=1$, we have $(x^2)^2=(x^4)^2=(x^7)^2=1/3$ so that Tx is minimal in S^6. It is easy to see the converse.

(2) Suppose that JH is tangent to Tx. Assume $x^2=0$ so that (5.9) and (5.10) imply

$$JH=\frac{1}{2D}x^4((x^4)^2-(x^6)^2-(x^7)^2)(x^6e_2-x^7e_3) \quad \text{and} \quad x^4\neq 0 .$$

On the other hand, Lemma 5.6 (1) implies that JH is normal so that $JH=0$. If $(x^4)^2\neq(x^6)^2+(x^7)^2$, then we have $x^2=x^6=x^7=0$. This contradicts (5.9). Therefore we have $(x^4)^2=(x^6)^2+(x^7)^2$. Similarly, $x^4=0$ implies $(x^2)^2=(x^6)^2+(x^7)^2$.

Assume $x^2x^4\neq 0$ so that $x^6=0$. From (5.10), we have

$$JH=\frac{1}{2D}(((x^7)^2-(x^4)^2)x^4x^7e_3+((x^2)^2-(x^7)^2)x^2x^7e_5+((x^2)^2-(x^4)^2)x^2x^4e_6) .$$

Define a tangent vector κ by

$$\begin{aligned}\kappa&=-\frac{((x^7)^2-(x^4)^2)x^4x^7}{x^2}\varphi_1-\frac{((x^2)^2-(x^7)^2)x^2x^7}{x^4}\varphi_2\\&=((x^7)^2-(x^4)^2)x^4x^7e_3+((x^2)^2-(x^7)^2)x^2x^7e_5\\&\quad+\frac{(x^7)^2}{x^2x^4}((x^2)^2-(x^4)^2)\{(x^2)^2+(x^4)^2-(x^7)^2\}e_6 .\end{aligned}$$

Since JH is tangent to Tx, we have $JH=(1/(2D))\kappa$ so that

$$\{(x^2)^2-(x^4)^2\}\{(x^4)^2-(x^7)^2\}\{(x^7)^2-(x^2)^2\}=0 .$$

Conversely, we assume (5.11). If $x^2=0$, then, from (5.9) and (5.11), we see $JH=0$. Similarly, if $x^4=0$, then we see $JH=0$. If $x^2x^4\neq 0$, then we see $JH=2D\kappa$ so that JH is tangent to Tx. Q.E.D.

Immediately, Lemmas 5.4 and 5.6 imply the following.

LEMMA 5.7. *Suppose that Tx is totally real and mass-symmetric and is not minimal in S^6. Then Tx is of 2-type if and only if JH is either a normal vector or a tangent vector of S^6. If JH is normal, then Tx is of (case* A) *or (case* B) *or (case* C). *If JH is tangent, then Tx is of (case* D).

We assume that Tx is of (case A) and put

$$\cos\theta=\frac{x^6}{((x^6)^2+(x^7)^2)^{1/2}} , \qquad \sin\theta=\frac{x^7}{((x^6)^2+(x^7)^2)^{1/2}} ,$$

so that $\varphi(-\theta, 0)=x^4e_4+((x^6)^2+(x^7)^2)^{1/2}e_6$ and $\varphi(-\theta-\pi, \pi)=-x^4e_4+((x^6)^2+$

$(x^7)^2)^{1/2}e_6$. Thus we may put $x^4=\alpha$, $x^6=(1-\alpha^2)^{1/2}$ and $x^7=0$ for $0<\alpha<1$. Put $x_\alpha=\alpha e_4+(1-\alpha^2)^{1/2}e_6$.

Define an isometry σ_1 of S^6 by

$$\sigma_1(e_1)=e_1, \quad \sigma_1(e_2)=-e_2, \quad \sigma_1(e_3)=-e_3, \quad \sigma_1(e_4)=e_6,$$
$$\sigma_1(e_5)=e_7, \quad \sigma_1(e_6)=e_4, \quad \sigma_1(e_7)=e_5.$$

From table (5.1), σ_1 is an element of G_2. Therefore, for $0<\alpha<1/\sqrt{2}$, Tx_α and Tx_β, $\beta=(1-\alpha^2)^{1/2}$, are congruent to each other under the action of σ_1. Note that Tx_α, $\alpha=1/\sqrt{2}$, is a minimal surface of S^6.

It is easy to see that Tx_α lies in a totally geodesic $S^3(1)=\{y\in S^6(1)\mid \langle y, e_i\rangle=0,\ i=1, 2, 3\}$. From table (5.1), $S^3(1)$ is totally real in $S^6(1)$. In particular, JH is a normal vector field of Tx_α in S^6. Since Tx_α is a hypersurface of $S^3(1)$, Tx_α is a Chen surface of S^3 (also of S^6).

Define $\sigma_2\in G_2$ by

$$\sigma_2(e_1)=e_1, \quad \sigma_2(e_2)=e_4, \quad \sigma_2(e_3)=e_5, \quad \sigma_2(e_4)=e_2,$$
$$\sigma_2(e_5)=e_3, \quad \sigma_2(e_6)=-e_6, \quad \sigma_2(e_7)=-e_7.$$

It is clear that $\sigma_i^{-1}T\sigma_i=T$, $i=1, 2$. $\sigma_1(x)$ is of (case B) if x is of (case C), and $\sigma_2(x)$ is of (case A) if x is of (case B). Therefore, (case B) and (case C) reduce to (case A).

Let Tx be an orbit of (case D). From Lemma 5.4, we have

$$(x^2)^2=(x^4)^2 \quad \text{or} \quad (x^2)^2=(x^7)^2 \quad \text{or} \quad (x^4)^2=(x^7)^2.$$

Define $\sigma_3\in G_2$ by

$$\sigma_3(e_1)=e_1, \quad \sigma_3(e_2)=e_7, \quad \sigma_3(e_3)=-e_6, \quad \sigma_3(e_4)=-e_4,$$
$$\sigma_3(e_5)=-e_5, \quad \sigma_3(e_6)=-e_3, \quad \sigma_3(e_7)=e_2.$$

Using σ_2 and σ_3, we can assume $(x^2)^2=(x^4)^2$. Since

$$\varphi(0, 0)=x=x^2e_2+x^4e_4+x^7e_7, \qquad \varphi(\pi, 0)=-x^2e_2+x^4e_4-x^7e_7,$$
$$\varphi(0, \pi)=x^2e_2-x^4e_4-x^7e_7, \quad \text{and} \quad \varphi(\pi, \pi)=-x^2e_2-x^4e_4+x^7e_7,$$

we may assume $x^2=x^4>0$. Moreover, applying σ_2, we can assume $x^7>0$. Finally, it is sufficient to study the case where

$$x=y_\beta=\beta e_2+\beta e_4+\gamma e_4, \quad 0<\beta<1/\sqrt{2}, \quad \gamma=(1-2\beta^2)^{1/2}.$$

Note that Ty_β is minimal if $\beta=1/\sqrt{3}$. By Lemma 5.6, JH is tangent to Ty_β. h and H are given by

$$h(\varphi_1, \varphi_1) = -\beta^3 e_2 + (1-\beta^2)\beta e_4 - \beta^2\gamma e_7 ,$$
$$h(\varphi_2, \varphi_2) = (1-\beta^2)\beta e_2 - \beta^3 e_4 - \beta^2\gamma e_7 ,$$
$$h(\varphi_1, \varphi_2) = \beta\gamma^2 e_2 + \beta\gamma^2 e_4 - 2\beta^2\gamma e_7 ,$$

and

$$H = \frac{1-3\beta^2}{2\beta^2(2-3\beta^2)}(-\beta\gamma^2 e_2 - \beta\gamma^2 e_4 + 2\beta^2\gamma e_7) .$$

By direct computation, we see

$$\mathscr{B}(H) = \sum_{i,j,k,l=1}^{2} g^{ik} g^{jl} \tilde{g}(h(\varphi_i, \varphi_j), H) \cdot h(\varphi_k, \varphi_l)$$
$$= \frac{\gamma^2(2-6\beta^2+9\beta^4)}{\beta^2(2-3\beta^2)^2} H .$$

Therefore, Ty_β is a Chen surface of S^6.

Now, we see that

$$\mathfrak{F}_3/\sim = \{Tx_\alpha \mid 0<\alpha<1/\sqrt{2}\} \qquad \text{and}$$
$$\mathfrak{F}_5/\sim = \{Ty_\beta \mid 0<\beta<1/\sqrt{2},\ \beta\neq 1/\sqrt{3}\} .$$

Theorem F is proved completely.

5.2. Proof of Theorem G. Let M be a totally real surface of a nearly Kaehler manifold $(S^6(1), J, \tilde{g})$. Let $\tilde{\nabla}$, ∇, h, A, $\nabla^\perp$ etc. be as in §5.0.

Let $\{e_1, e_2\}$ be a (local) orthonormal frame field of M. Put

(5.12) $$\xi_3 = Je_1 , \quad \xi_4 = Je_2 , \quad \xi_5 = JG(e_1, e_2) , \quad \xi_6 = -G(e_1, e_2) .$$

By Lemmas 5.1, 5.2 and 5.3, $\{\xi_3, \cdots, \xi_6\}$ is a normal orthonormal frame field.

Throughout this section, we use the following convention on the range of indices:

$$A, B, C, \cdots = 1, \cdots, 6 ; \quad i, j, k, \cdots = 1, 2 ; \quad r, s, t, \cdots = 3, \cdots, 6 .$$

Let $\{\omega^1, \omega^2\}$ be a dual frame of $\{e_1, e_2\}$. Define 1-forms $\omega^A{}_B$ by

$$\tilde{\nabla} e_i = \sum_j \omega^j{}_i e_j + \sum_r \omega^r{}_i \xi_r , \qquad \tilde{\nabla} \xi_r = \sum_j \omega^j{}_r e_j + \sum_s \omega^s{}_r \xi_s .$$

It is well-known that the structure equations of M are given by

$$d\omega^i = -\sum_j \omega^i{}_j \wedge \omega^j ,$$

(5.13) $$d\omega^i{}_j = -\sum_A \omega^i{}_A \wedge \omega^A{}_j + \omega^i \wedge \omega^j ,$$

(5.14) $$d\omega^r{}_j = -\sum_A \omega^r{}_A \wedge \omega^A{}_j ,$$

(5.15) $$d\omega^r{}_s = -\sum_A \omega^r{}_A \wedge \omega^A{}_s ,$$

$$\omega^A{}_B + \omega^B{}_A = 0 ,$$

(5.16) $$\omega^r{}_i = \sum_j h^r_{ij} \omega^j , \qquad h^r_{ij} = h^r_{ji} .$$

By the definition, we get

$$h^r_{ij} = \tilde{g}(h(e_i, e_j), \xi_r) .$$

First, we obtain the following lemma.

LEMMA 5.8. *A frame* $\{e_1, e_2, \xi_3, \cdots, \xi_6\}$ *defined by* (5.12) *satisfies*

(5.17) $$\begin{aligned} &\omega^3{}_2 = \omega^4{}_1 , \quad \omega^4{}_3 = \omega^2{}_1 , \quad \omega^5{}_3 = -\omega^6{}_1 , \quad \omega^6{}_3 = \omega^5{}_1 + \omega^2 , \\ &\omega^5{}_4 = -\omega^6{}_2 , \quad \omega^6{}_4 = \omega^5{}_2 - \omega^1 , \quad \omega^6{}_5 = -\omega^3{}_1 - \omega^4{}_2 . \end{aligned}$$

PROOF. From $G(e_1, e_1) = 0$, we see

$$\begin{aligned} 0 = G(e_1, e_1) &= \tilde{\nabla}_{e_1}\xi_3 - J\tilde{\nabla}_{e_1}e_1 \\ &= -A_{\xi_3}e_1 + \nabla^\perp_{e_1}\xi_3 - J\nabla_{e_1}e_1 - Jh(e_1, e_1) \\ &= (-h^3_{12} + h^4_{11})e_2 + \sum \omega^r{}_3(e_1)\xi_r - \omega^2{}_1(e_1)\xi_4 - h^5_{11}\xi_6 + h^6_{11}\xi_5 . \end{aligned}$$

Thus we have

(5.18) $$h^3_{12} = h^4_{11} , \quad \omega^4{}_3(e_1) = \omega^2{}_1(e_1) , \quad \omega^5{}_3(e_1) = -h^6_{11} , \quad \omega^6{}_3(e_1) = h^5_{11} .$$

Similarly, from $G(e_2, e_2) = 0$, we get

(5.19) $$h^3_{22} = h^4_{12} , \quad \omega^4{}_3(e_2) = \omega^2{}_1(e_2) , \quad \omega^5{}_4(e_2) = -h^6_{22} , \quad \omega^6{}_4(e_2) = h^5_{22} .$$

Moreover, from

$$-\xi_6 = G(e_1, e_2) = \tilde{\nabla}_{e_1}\xi_4 - J\tilde{\nabla}_{e_1}e_2$$

and

$$\xi_6 = G(e_2, e_1) = \tilde{\nabla}_{e_2}\xi_3 - J\tilde{\nabla}_{e_2}e_1 ,$$

we have

(5.20)
$$\omega^5{}_4(e_1) = -h^6_{12}, \quad \omega^6{}_4(e_1) = h^5_{12} - 1,$$
$$\omega^5{}_3(e_2) = -h^6_{12}, \quad \omega^6{}_3(e_2) = h^5_{12} + 1.$$

Using Lemma 5.1 (1), (2) and (5.6), we obtain

$$\begin{aligned}\tilde{\nabla}_{e_1}\xi_5 &= \tilde{\nabla}_{e_1}(JG(e_1, e_2)) \\ &= (\tilde{\nabla}_{e_1}J)G(e_1, e_2) + J(\tilde{\nabla}_{e_1}G)(e_1, e_2) \\ &\quad + JG(\tilde{\nabla}_{e_1}e_1, e_2) + JG(e_1, \tilde{\nabla}_{e_1}e_2) \\ &= G(e_1, G(e_1, e_2)) + J(\tilde{\nabla}_{e_1}G)(e_1, e_2) \\ &\quad + JG(\nabla_{e_1}e_1, e_2) + JG(h(e_1, e_1), e_2) \\ &\quad + JG(e_1, \nabla_{e_1}e_2) + JG(e_1, h(e_1, e_2)) \\ &= -h^5_{11}e_1 - h^5_{12}e_2 + h^6_{11}\xi_3 + h^6_{12}\xi_4 + (-h^3_{11} - h^4_{12})e_6.\end{aligned}$$

So we have

(5.21)
$$\omega^6{}_5(e_1) = -h^3_{11} - h^4_{12}.$$

Similarly, computing $\tilde{\nabla}_{e_2}\xi_5$, we have

(5.22)
$$\omega^6{}_5(e_2) = -h^3_{12} - h^4_{22}.$$

From (5.18), (5.19), (5.20), (5.21) and (5.22), we have (5.17). Q.E.D.

For convenience, we put

$$a = h^3_{11}, \quad b = h^3_{12} = h^4_{11}, \quad c = h^3_{22} = h^4_{12}, \quad d = h^4_{22}$$

and

$$A_r = A_{\xi_r},$$

so that, with respect to $\{e_1, e_2\}$, we have

(5.23)
$$A_3 = \begin{pmatrix} a & b \\ b & c \end{pmatrix}, \quad A_4 = \begin{pmatrix} b & c \\ c & d \end{pmatrix} \quad \text{and} \quad A_r = \begin{pmatrix} h^r_{11} & h^r_{12} \\ h^r_{21} & h^r_{22} \end{pmatrix}, \quad r = 5, 6.$$

Let H be the mean curvature vector of M in S^6 and put

$$H = \sum_r \kappa^r \xi_r$$

so that

$$\kappa^3 = \frac{a+c}{2}, \quad \kappa^4 = \frac{b+d}{2} \quad \text{and} \quad \kappa^r = \frac{h^r_{11} + h^r_{22}}{2}, \quad r = 5, 6,$$

and

$$\omega^6{}_5 = -2k^3\omega^1 - 2k^4\omega^2 .$$

Define functions β_1 and β_2 by

$$\omega^2{}_1 = \beta_1\omega^1 + \beta_2\omega^2 .$$

From (5.13), (5.14) and (5.17), we have

(5.24) $$e_2(\beta_1) - e_1(\beta_2) = 1 + \sum_r \det A_r + \beta_1{}^2 + \beta_2{}^2 ,$$

(5.25) $$-e_2(a) + e_1(b) = (-a+2c)\beta_1 - 3b\beta_2 + 2h^5_{11}h^6_{12} - 2h^5_{12}h^6_{11} - h^6_{11} ,$$

(5.26) $$-e_2(b) + e_1(c) = (-2b+d)\beta_1 + (a-2c)\beta_2 + h^5_{11}h^6_{22} - h^5_{22}h^6_{11} - h^6_{12} ,$$

(5.27) $$-e_2(c) + e_1(d) = -3c\beta_1 + (2b-d)\beta_2 + 2h^5_{12}h^6_{22} - 2h^5_{22}h^6_{12} - h^6_{22} ,$$

(5.28) $$\begin{aligned} -e_2(h^5_{11}) + e_1(h^5_{12}) = &-(h^5_{11} - h^5_{22})\beta_1 - 2h^5_{12}\beta_2 + b(h^6_{11} - h^6_{22}) \\ &-(a-c)h^6_{12} - 2k^3h^6_{12} + 2k^4h^6_{11} , \end{aligned}$$

(5.29) $$\begin{aligned} -e_2(h^5_{12}) + e_1(h^5_{22}) = &-2h^5_{12}\beta_1 + (h^5_{11} - h^5_{22})\beta_2 + c(h^6_{11} - h^6_{22}) \\ &-(b-d)h^6_{12} - 2k^3h^6_{22} + 2k^4h^6_{12} , \end{aligned}$$

(5.30) $$\begin{aligned} -e_2(h^6_{11}) + e_1(h^6_{12}) = &-(h^6_{11} - h^6_{22})\beta_1 - 2h^6_{12}\beta_2 - b(h^5_{11} - h^5{}_{22}) \\ &+(a-c)h^5_{12} + 2k^3(h^5_{12}+1) - 2k^4h^5_{11} , \end{aligned}$$

(5.31) $$\begin{aligned} -e_2(h^6_{12}) + e_1(h^6_{22}) = &-2h^6_{12}\beta_1 + (h^6_{11} - h^6_{22})\beta_2 - c(h^5_{11} - h^5_{22}) \\ &+(b-d)h^5_{12} + 2k^3h^5_{22} - 2k^4(h^5_{12}-1) . \end{aligned}$$

From now on, we assume that M is a complete totally real mass-symmetric 2-type Chen surface which is imbedded by an isometric imbedding f. Suppose that f is of 2-type with respect to eigenvalues λ_1 and λ_2, $0<\lambda_1<\lambda_2$. B. Y. Chen shows the following. See Chen [8, p. 274] and [7].

LEMMA 5.9. *The mean curvature α of M in S^6 is constant and given by*

(5.32) $$\alpha^2 = (2-\lambda_1)(\lambda_2-2)/4 ,$$

and the mean curvature vector H satisfies

(5.33) $$\mathrm{tr}(A_{\nabla^\perp H}) = \sum_i (A_{\nabla^\perp_{e_i} H})e_i = 0 ,$$

(5.34) $$\Delta^\perp H + \mathscr{A}(H) + (\|A_\xi\|^2 + 2)H = (\lambda_1 + \lambda_2)H ,$$

where $\Delta^\perp H = \sum_i \{\nabla^\perp_{\nabla_{e_i}e_i} H - \nabla^\perp_{e_i}\nabla^\perp_{e_i} H\}$, $\mathscr{A}(H)$ is the allied mean curvature vector in S^6 and $\xi = H/\alpha$.

Barros and Chen [1] show the following.

LEMMA 5.10. *H satisfies*

$$\|\nabla^{\perp}H\|^2=\alpha^2\{\lambda_1+\lambda_2-\|A_\xi\|^2-2\}\ , \tag{5.35}$$

where $\xi=H/\alpha$, *and for an orthonormal normal frame* $\{\xi_3, \cdots, \xi_6\}$ *such that* $\xi_3=\xi$, *we have*

$$\mathscr{A}(H)=\alpha\sum_{r=4}^{6}\{\mathrm{tr}(\nabla\omega^r{}_3)-\langle\nabla^{\perp}\xi_3,\ \nabla^{\perp}\xi_r\rangle\}\xi_r\ , \tag{5.36}$$

where $\langle\nabla^{\perp}\xi_3, \nabla^{\perp}\xi_r\rangle=\sum_{i=1}^{2}\langle\nabla^{\perp}_{e_i}\xi_3, \nabla^{\perp}_{e_i}\xi_r\rangle$.

On the other hand, by the definition of the allied mean curvature, we obtain

$$\mathscr{A}(H)=\sum_{r=4}^{6}\mathrm{tr}(A_HA_r)\xi_r \tag{5.37}$$

where $\{\xi_r\}$ is an orthonormal normal frame such that $\xi_3=H/\alpha$.

5.2.1. The case that JH is normal. Assume that JH is a normal vector field of M in S^6. Choosing a frame defined by (5.12), we easily see that $\kappa^3=\kappa^4=0$, $c=-a$, $d=-b$ and $\omega^6{}_5=0$. By Lemma 5.8, we see

$$\nabla^{\perp}H=(\kappa^5\omega^6{}_1-\kappa^6(\omega^5{}_1+\omega^2))\xi_3+(\kappa^5\omega^6{}_2-\kappa^6(\omega^5{}_2-\omega^1))\xi_4+d\kappa^5\cdot\xi_5+d\kappa^6\cdot\xi_6\ .$$

Combining this with (5.33), we have

$$\begin{aligned} &e_1(\kappa^5)h^5_{11}+e_1(\kappa^6)h^6_{11}+e_2(\kappa^5)h^5_{12}+e_2(\kappa^6)h^6_{12}\\ &\qquad+a(\kappa^5(h^6_{11}-h^6_{22})-\kappa^6(h^5_{11}-h^5_{22}))+2b(\kappa^5h^6_{12}-\kappa^6h^5_{12})=0 \end{aligned} \tag{5.38}$$

and

$$\begin{aligned} &e_1(\kappa^5)h^5_{12}+e_1(\kappa^6)h^6_{12}+e_2(\kappa^5)h^5_{22}+e_2(\kappa^6)h^6_{22}\\ &\qquad+b(\kappa^5(h^6_{11}-h^6_{22})-\kappa^6(h^5_{11}-h^5_{22}))-2a(\kappa^5h^6_{12}-\kappa^6h^5_{12})=0\ . \end{aligned} \tag{5.39}$$

Put $\eta_5=H/\alpha$ and $\eta_6=(-\kappa^6\xi_5+\kappa^5\xi_6)/\alpha$ so that $\{\xi_3, \xi_4, \eta_5, \eta_6\}$ is an orthonormal normal frame. From (5.37), we see that

$$a(\kappa^5(h^5_{11}-h^5_{22})+\kappa^6(h^6_{11}-h^6_{22}))+2b(\kappa^5h^5_{12}+\kappa^6h^6_{12})=0 \tag{5.40}$$

and

$$b(\kappa^5(h^5_{11}-h^5_{22})+\kappa^6(h^6_{11}-h^6_{22}))-2a(\kappa^5h^5_{12}+\kappa^6h^6_{12})=0\ . \tag{5.41}$$

From (5.25) and (5.27), we get

$$(5.42)\qquad (h^5_{11}-h^5_{22})h^6_{12}-(h^6_{11}-h^6_{22})h^5_{12}=\kappa^6\ .$$

Assume $a^2+b^2\neq 0$. From (5.40) and (5.41), we have

$$(5.43)\qquad \begin{cases}\kappa^5(h^5_{11}-h^5_{22})+\kappa^6(h^6_{11}-h^6_{22})=0\ ,\\ \kappa^5 h^5_{12}+\kappa^6 h^6_{12}=0\ .\end{cases}$$

Combining $(\kappa^5)^2+(\kappa^6)^2=\alpha^2\neq 0$ with (5.42) and (5.43), we see $\kappa^6=0$. Hence we can choose e_1 and e_2 in such a way that $\kappa^5=\alpha$. From (5.43), we have $h^5_{11}=h^5_{22}=\alpha$ and $h^5_{12}=0$. So, from (5.38) and (5.39), we get $h^6_{11}=h^6_{22}=h^6_{12}=0$. Therefore we have $\nabla^\perp H=0$. From (5.32) and (5.35), we obtain $\alpha^2=(2-\lambda_1)(\lambda_2-2)/4$ and $\lambda_1+\lambda_2-2\alpha^2-2=0$ so that $\lambda_1\lambda_2=0$. This is a contradiction. So we obtain $a=b=0$.

Assume $\kappa^6\neq 0$. From (5.25), (5.26) and (5.27), we have

$$(5.44)\qquad 2h^5_{11}h^6_{12}-2h^5_{12}h^6_{11}=h^6_{11}\ ,$$

$$(5.45)\qquad 2h^5_{12}h^6_{22}-2h^5_{22}h^6_{12}=h^6_{22}\ ,$$

$$(5.46)\qquad h^5_{11}h^6_{22}-h^5_{22}h^6_{11}=h^6_{12}\ .$$

From (5.44) and (5.45), we see

$$(5.47)\qquad (h^5_{11}-h^5_{22})h^6_{12}-(h^6_{11}-h^6_{22})h^5_{12}=\kappa^6\ ,$$

$$(5.48)\qquad 4\kappa^5h^6_{12}-4\kappa^6h^5_{12}=h^6_{11}-h^6_{22}\ .$$

Since $\kappa^5e_i(\kappa^5)+\kappa^6e_i(\kappa^6)=0$ $(i=1, 2)$, we get from (5.38), (5.39), (5.46) and (5.48),

$$(5.49)\qquad \begin{cases}-h^6_{12}e_1(\kappa^6)+\dfrac{1}{2}(h^6_{11}-h^6_{22})e_2(\kappa^6)=0\ ,\\[2mm] \dfrac{1}{2}(h^6_{11}-h^6_{22})e_1(\kappa^6)+h^6_{12}e_2(\kappa^6)=0\ .\end{cases}$$

Up to sign, a normal vector field $\xi_6=-G(e_1, e_2)$ is independent of the choice of e_1 and e_2. So we can choose e_1 and e_2 in such a way that $h^5_{12}\geqq 0$ and A_6 is diagonal, i.e., $h^6_{12}=0$. From (5.47) and (5.49), we see that $h^6_{11}-h^6_{22}\neq 0$ and κ^6 is non-zero constant. Thus, from (5.47) and (5.48), we get $4(h^5_{12})^2\kappa^6=\kappa^6$ so that $h^5_{12}=1/2$. Therefore, using (5.44) and (5.46), we obtain

$$(5.50)\qquad \begin{aligned}&h^5_{11}=h^6_{11}=0\ ,\quad h^5_{22}=2\kappa^5=\text{constant}\ ,\quad h^6_{22}=2\kappa^6=\text{constant}\ ,\\ &h^5_{12}=1/2\ ,\quad h^6_{12}=0\ .\end{aligned}$$

Combining this with (5.30) and (5.31), we have

$$\beta_1=\beta_2=0 . \tag{5.51}$$

(5.50) and (5.51) contradict (5.24). Therefore we obtain $\kappa^6=0$.

We can choose e_1 and e_2 in such a way that

$$A_3=A_4=0 , \qquad A_5=\begin{pmatrix} h^5_{11} & 0 \\ 0 & h^5_{22} \end{pmatrix} , \qquad A_6=\begin{pmatrix} h^6_{11} & h^6_{12} \\ h^6_{12} & -h^6_{11} \end{pmatrix} ,$$
$$h^5_{11}+h^5_{22}=2\alpha .$$

From (5.25), (5.26) and (5.27), we have

$$2h^5_{11}h^6_{12}-h^6_{11}=0 , \quad 2h^5_{22}h^6_{12}-h^6_{11}=0 , \quad 2\alpha h^6_{11}+h^6_{12}=0 .$$

These imply $h^6_{11}=h^6_{12}=0$. Therefore the first normal space N is spanned by ξ_5. By Lemma 5.8, N is parallel with respect to the normal connection $\nabla^\perp$. By Erbacher [10], M lies in a totally geodesic $S^3(1)$. By Chen [8, p. 279], M is flat.

Let $p\in M$ so that $S^3=S^6\cap \mathrm{Span}\{p, e_1, e_2, \xi_5\}$. From (5.2), Lemma 5.1 and (5.6), we get

$$\begin{aligned}
&p\times e_1=\xi_3 , && p\times e_2=\xi_4 , && p\times \xi_3=-e_1 , && p\times\xi_4=-e_2 , \\
&p\times \xi_5=\xi_6 , && p\times \xi_6=-\xi_5 , && e_1\times e_2=-\xi_6 , && e_1\times\xi_3=p , \\
&e_1\times \xi_4=-\xi_5 , && e_1\times \xi_5=\xi_4 , && e_1\times \xi_6=e_2 , && e_2\times\xi_3=\xi_5 , \\
&e_2\times \xi_4=p , && e_2\times \xi_5=-\xi_3 , && e_2\times \xi_6=-e_1 , && \xi_3\times\xi_4=\xi_6 , \\
&\xi_3\times \xi_5=e_2 , && \xi_3\times \xi_6=-\xi_4 , && \xi_4\times \xi_5=-e_1 , && \xi_4\times\xi_6=\xi_3 , \\
&\xi_5\times \xi_6=p ,
\end{aligned} \tag{5.52}$$

so that $p\times e_1$, $p\times e_2$, $p\times\xi_5$, $e_1\times e_2$, $e_1\times\xi_5$ and $e_2\times\xi_5$ are contained in $\mathrm{Span}\{\xi_3, \xi_4, \xi_6\}$. This implies that S^3 is totally real in S^6.

By Theorem C, f can be extended to a map of $\boldsymbol{R}^2$ into S^3 and is given by

$$\begin{aligned}
f(x, y)=&\sqrt{1/\lambda_1}\cos(\sqrt{\lambda_1}\,x)E_2+\sqrt{1/\lambda_1}\sin(\sqrt{\lambda_1}\,x)E_3 \\
&+\sqrt{1/\lambda_2}\cos(\sqrt{\lambda_2}\,y)E_4+\sqrt{1/\lambda_2}\sin(\sqrt{\lambda_2}\,y)E_5 , \qquad (x, y)\in \boldsymbol{R}^2 ,
\end{aligned}$$
$$\lambda_1\lambda_2-\lambda_1-\lambda_2=0 ,$$

where $\{E_1, \cdots, E_7\}$ is an orthonormal basis of E^7. Since M is complete and f is an imbedding of M, M is a flat torus $\boldsymbol{R}^2/\Lambda$, where

$$\Lambda=\{(x, y)\mid f(x, y)=f(0, 0)\} .$$

We may assume that $f(0, 0)=p$, $(e_1)_p=(\partial/\partial x)_p$ and $(e_2)_p=(\partial/\partial y)_p$. By direct computation, we have

$$p=\sqrt{1/\lambda_1}E_2+\sqrt{1/\lambda_2}E_4\,, \quad (e_1)_p=E_3\,, \quad (e_2)_p=E_5\,,$$
$$H/\alpha=\sqrt{1/\lambda_2}E_2-\sqrt{1/\lambda_1}E_4\,.$$

Since ξ_5 is parallel to H, after changing the sign of E_3 or E_5, we may assume $(\xi_5)_p=-\sqrt{1/\lambda_2}E_2+\sqrt{1/\lambda_1}E_4$.

Define vectors $\boldsymbol{f}_1, \cdots, \boldsymbol{f}_7$ in E^7 as follows:

$$\boldsymbol{f}_1=-\sqrt{1/\lambda_1}(\xi_3)_p-\sqrt{1/\lambda_2}(\xi_4)_p\,,$$
$$\boldsymbol{f}_2=\sqrt{1/\lambda_1}p-\sqrt{1/\lambda_2}(\xi_5)_p\,,$$
$$\boldsymbol{f}_3=-(e_1)_p\,,$$
$$\boldsymbol{f}_4=\sqrt{1/\lambda_2}p+\sqrt{1/\lambda_1}(\xi_5)_p\,,$$
$$\boldsymbol{f}_5=-(e_2)_p\,,$$
$$\boldsymbol{f}_6=-(\xi_6)_p\,,$$
$$\boldsymbol{f}_7=\sqrt{1/\lambda_2}(\xi_3)_p-\sqrt{1/\lambda_1}(\xi_4)_p\,.$$

Using (5.52), we see that $\{\boldsymbol{f}_1, \cdots, \boldsymbol{f}_7\}$ satisfies Table (5.1). Moreover, we have $E_2=\boldsymbol{f}_2$, $E_3=-\boldsymbol{f}_3$, $E_4=\boldsymbol{f}_4$, $E_5=-\boldsymbol{f}_5$ so that

$$f(x, y)=\sqrt{1/\lambda_1}\cos(\sqrt{\lambda_1}\,x)\boldsymbol{f}_2-\sqrt{1/\lambda_1}\sin(\sqrt{\lambda_1}\,x)\boldsymbol{f}_3$$
$$+\sqrt{1/\lambda_2}\cos(\sqrt{\lambda_2}\,y)\boldsymbol{f}_4-\sqrt{1/\lambda_2}\sin(\sqrt{\lambda_2}\,y)\boldsymbol{f}_5\,.$$

This implies that M $(=f(M))$ is an orbit of a maximal torus of G_2 in S^6. Therefore we see that $M\in\mathfrak{F}_3$.

5.2.2. The case that JH is tangent. Assume that JH is a tangent vector field of M in S^6. We may choose a frame $\{e_1, e_2, \xi_3, \cdots, \xi_6\}$ satisfying

$$e_1=-JH/\alpha\,,$$
$$\xi_3=Je_1\,, \quad \xi_4=Je_2\,, \quad \xi_5=JG(e_1, e_2)\,, \quad \xi_6=-G(e_1, e_2)\,,$$

so that $H=\alpha\xi_3$, $a+c=2\alpha$, $d=-b$, $h^r_{22}=-h^r_{11}$ $(r=5, 6)$. From (5.37), we easily see that $b=d=0$ and

$$(a-c)h^r_{11}=0\,, \qquad r=5, 6\,. \tag{5.53}$$

By Lemma 5.8,

$$\nabla^{\perp}H=\alpha(\omega^2{}_1\xi_4-\omega^6{}_1\xi_5+(\omega^5{}_1+\omega^2)\xi_6)\,.$$

Combining this with (5.33), we have

$$h_{12}^6 = -c\beta_2 , \tag{5.54}$$

$$2h_{11}^5 h_{12}^6 - 2h_{12}^5 h_{11}^6 = -c\beta_1 + h_{11}^6 . \tag{5.55}$$

From (5.36), we have

$$e_1(\beta_1) + e_2(\beta_2) = -2h_{11}^5 , \tag{5.56}$$

$$e_1(h_{11}^6) + e_2(h_{12}^6) = 2\alpha h_{11}^5 , \tag{5.57}$$

$$e_1(h_{11}^5) + e_2(h_{12}^5) = 2\beta_1 - 2\alpha h_{11}^6 . \tag{5.58}$$

From (5.25), (5.27) and (5.55), we get

$$e_2(a) = (a-c)\beta_1 , \qquad e_2(c) = 4c\beta_1 - 2h_{11}^6 .$$

Combining this with $a+c=2\alpha$, we have

$$h_{11}^6 = (a+3c)\beta_1/2 . \tag{5.59}$$

From (5.25), (5.26) and (5.54), we have

$$\begin{cases} e_1(a) = -(a-c)\beta_2 , & e_1(c) = (a-c)\beta_2 , \\ e_2(a) = (a-c)\beta_1 , & e_2(c) = -(a-c)\beta_1 . \end{cases} \tag{5.60}$$

From (5.31), (5.57), (5.54) and (5.59), we have

$$ch_{11}^5 = (a+5c)\beta_1\beta_2/2 . \tag{5.61}$$

From (5.29), (5.58) and (5.59), we obtain

$$\beta_1 h_{12}^5 = \beta_1 + \beta_2 h_{11}^5 + c(a+3c)\beta_1/2 . \tag{5.62}$$

Let ψ be a C^∞-function on M defined by

$$\psi(x) = (\|A_{H/\alpha}\|_x^2 - 2\det(A_{H/\alpha})_x) , \qquad x \in M .$$

Using the above frame, we see that ψ is given by $\psi=(a-c)^2$.

LEMMA 5.11. *$M_0 = \{x \in M \mid \psi(x)=0\}$ does not contain any interior points.*

PROOF. Assume that there exists an open neighborhood U on which $\psi \equiv 0$. From (5.54), (5.59), (5.61) and (5.62), we obtain

$$\begin{cases} a=c=\alpha , \quad h_{11}^6 = 2\alpha\beta_1 , \quad h_{12}^6 = -\alpha\beta_2 , \\ h_{11}^5 = 3\beta_1\beta_2 , \quad \beta_1 h_{12}^5 = \beta_1(1+3\beta_2^2+2\alpha^2) . \end{cases} \tag{5.63}$$

Multiplying (5.55) by β_1 and using (5.63), we obtain $\beta_1=0$ so that $h^5_{11}=h^6_{11}=0$. From (5.24), we get

$$e_1(\beta_2)=(\alpha^2-1)\beta_2{}^2-1+(h^5_{11})^2 .$$

On the other hand, from (5.30) and (5.63), we have

$$e_1(\beta_2)=-2\beta_2{}^2-2h^5_{12}-2 .$$

These two equations imply $(\alpha^2+1)\beta_2{}^2+(h^5_{12}+1)^2=0$ so that $\beta_2=0$, $h^5_{12}=-1$ and $h^6_{12}=0$. Therefore we get $\nabla^\perp H=0$. From (5.32) and (5.35), we obtain

$$\alpha^2=(2-\lambda_1)(\lambda_2-2)/4 , \qquad \lambda_1+\lambda_2-2\alpha^2-2=0$$

so that $\lambda_1\lambda_2=0$. This is a contradiction. Q.E.D.

LEMMA 5.12. *Choose a frame* $\{e_1, e_2, \xi_3, \cdots, \xi_6\}$ *satisfying* (5.12) *with* e_1 *parallel to* JH. *Then* $a=g(A_{\xi_3}e_1, e_1)$ *is constant and* $a\neq 0, \pm 1/\sqrt{2}$. *Moreover, we obtain*

$$\{\lambda_1, \lambda_2\}=\left\{\frac{2(a^2+1)(2a^2+1)}{4a^2+1}, \frac{4(a^2+1)}{4a^2+1}\right\} , \tag{5.64}$$

$$\begin{cases} \alpha=\left|\dfrac{a(2a^2-1)}{4a^2+1}\right| , \\ A_3=\begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix} , \quad A_4=\begin{pmatrix} 0 & c \\ c & 0 \end{pmatrix} , \quad A_5=\begin{pmatrix} 0 & h \\ h & 0 \end{pmatrix} , \quad A_6=0 , \\ \omega^2{}_1=\omega^4{}_3=\omega^5{}_3=\omega^5{}_4=0 , \\ \omega^6{}_3=(h+1)\omega^2 , \quad \omega^6{}_4=(h-1)\omega^1 , \quad \omega^6{}_5=-\dfrac{2a(2a^2-1)}{4a^2+1}\omega^1 , \end{cases} \tag{5.65}$$

where

$$c=-\frac{3a}{4a^2+1} \quad \textit{and} \quad h=-\frac{2a^2-1}{4a^2+1} .$$

PROOF. Let U be an open neighborhood such that $\psi(y)\neq 0$ for any y of U. Choose a frame $\{e_1, e_2, \xi_3, \cdots, \xi_6\}$ on U satisfying (5.12) and $e_1=-JH/\alpha$. $\psi\neq 0$ implies $a\neq c$ on U. From (5.53) and (5.59), we see that $h^5_{11}=h^6_{11}=0$ and $(a+3c)\beta_1=0$ on U.

Assume that $\beta_1\neq 0$ at some point x of U. There exists an open neighborhood V ($\subset U$) of x such that $\beta_1\neq 0$ on V. We get $a+3c=0$ so that $a=3\alpha$ and $c=-\alpha$ on V. By (5.60), we have $\beta_1=0$. This is a contradiction. Therefore $\beta_1=0$ identically on U.

From (5.24), (5.28), (5.30) and (5.54), we get

$$(5.66)\qquad \begin{cases} e_1(\beta_2) = -(\beta_2)^2 - 1 - ac + c^2 + (h^5_{12})^2 + c^2\beta_2{}^2 , \\ e_1(h^5_{12}) = (-2h^5_{12} + 2ac)\beta_2 , \\ e_1(h^6_{12}) = 2c\beta_2{}^2 + 2ah^5_{12} + 2\alpha . \end{cases}$$

On the other hand, from (5.35), we have

$$(5.67)\qquad \lambda_1 + \lambda_2 - 2 = a^2 + c^2 + \beta_2{}^2 + (h^6_{12})^2 + (h^5_{12} + 1)^2 .$$

Differentiating (5.67) by e_1 and combining (5.60) and (5.66), we have

$$\beta_2\{(a-c)^2 + \beta_2{}^2 + (h^5_{12}+1)^2 + c^2\beta_2{}^2\} = 0 .$$

This implies $\beta_2 = 0$ on U.

(5.60) implies that a is constant on U. From (5.54), we see that $h^6_{12} = 0$ on U. The third of (5.66) implies $a \neq 0$ and $h^5_{12} = -\alpha/a$. The first of (5.66) implies $1 + ac - c^2 - (h^5_{12})^2 = 0$. Therefore we get

$$(4a^2+1)c^2 - 2a(2a^2-1)c - 3a^2 = 0 .$$

Since $a \neq c$, we have

$$(5.68)\qquad c = -\frac{3a}{4a^2+1} , \quad \alpha = \frac{a+c}{2} = \frac{a(2a^2-1)}{4a^2+1} , \quad h^5_{12} = -\frac{2a^2-1}{4a^2+1} .$$

Therefore (5.65) is shown for the frame $\{e_1, e_2, \xi_3, \cdots, \xi_6\}$ on U.

Combining (5.67) and (5.32), we have

$$\lambda_1 + \lambda_2 = \frac{2}{4a^2+1}(a^2+1)(2a^2+3) , \qquad \lambda_1\lambda_2 = \frac{8}{(4a^2+1)^2}(a^2+1)^2(2a^2+1) .$$

These imply (5.64).

(5.68) implies $-1/\sqrt{2} < a < 0$ or $1/\sqrt{2} < a$. Choose a frame $\{e'_1, e'_2, \xi'_3, \cdots, \xi'_6\}$ satisfying (5.12) and $e'_1 = JH/\alpha$. Clearly, we see that $e'_2 = e_2$ or $-e_2$. It is easy to show (5.64) and (5.65) for $\{e'_i, \xi'_r\}$ on U. In this case, we get $a \neq 0, \pm 1/\sqrt{2}$. Therefore (5.64) and (5.65) are shown for any frame on $M \setminus M_0$ satisfying (5.12) with e_1 parallel to JH.

By the continuity of ψ and Lemma 5.11, we get Lemma 5.12 on M. Q.E.D.

REMARK. a is determined as

$$|a| = \max\{|\tilde{g}(h(X, X), JX)| \ ; \ X \in T(M), \ \|X\| = 1\} .$$

Lemma 5.12 says that M is flat. Moreover, we obtain the following.

LEMMA 5.13. *M lies fully in a totally geodesic 5-sphere of S^6.*

PROOF. Choose a frame $\{e_i, \xi_r\}$ satisfying (5.12) and $e_1=-JH/\alpha$. Put $r=(c^2+h^2)^{1/2}$, $\eta_4=(1/r)(c\xi_4+h\xi_5)$ and $\eta_5=(1/r)(-h\xi_4+c\xi_5)$. Then $\{\xi_3, \eta_4, \eta_5, \xi_6\}$ is an orthonormal normal frame of M, and we see $A_{\xi_3}\neq 0$, $A_{\eta_4}\neq 0$ and $A_{\eta_5}=A_{\xi_6}=0$. Therefore the first normal space N_1 is spanned by ξ_3 and η_4. By Lemma 5.12, we see

$$\nabla^{\perp}\xi_3=(h+1)\omega^2\xi_6 \ (\neq 0) ,$$
$$\nabla^{\perp}\eta_4=\frac{1}{r}(c(h-1)-2\alpha h)\omega^1\xi_6 \ (\neq 0) .$$

Then the second normal space N_2 is spanned by ξ_6 and

$$\nabla^{\perp}\xi_6=-(h+1)\omega^2\xi_3-\frac{1}{r}(c(h-1)-2\alpha h)\omega^1\eta_4 .$$

Therefore $N_1\oplus N_2$ is parallel with respect to the normal connection $\nabla^{\perp}$. By Erbacher [10], M lies fully in a totally geodesic

$$S^5(1)=\{x\in S^6(1) \mid \langle x, \eta_5\rangle=0\} . \qquad \text{Q.E.D.}$$

From now on, we fix a point p of M and a frame $\{e_i, \xi_r\}$ around p satisfying (5.12) and $e_1=-JH/\alpha$ so that we can assume $-1/\sqrt{2}<a<0$ or $1/\sqrt{2}<a$ in (5.64) and (5.65). Define vectors $\boldsymbol{f}_1, \cdots, \boldsymbol{f}_7$ in E^7 as follows:

$$
\begin{aligned}
&\boldsymbol{f}_1=\sqrt{q/k}\{-h\xi_4+c\xi_5\} ,\\
&\boldsymbol{f}_2=\{\sqrt{1/k}(p-a\xi_3)-\sqrt{q/k}(c\xi_4+h\xi_5)\}/\sqrt{2} ,\\
&\boldsymbol{f}_3=\{-e_1-\sqrt{1/q}(e_2-2a\xi_6)\}/\sqrt{2} ,\\
&\boldsymbol{f}_4=\{\sqrt{1/k}(p-a\xi_3)+\sqrt{q/k}(c\xi_4+h\xi_5)\}/\sqrt{2} ,\\
&\boldsymbol{f}_5=\{e_1-\sqrt{1/q}(e_2-2a\xi_6)\}/\sqrt{2} ,\\
&\boldsymbol{f}_6=\{-2ae_2-\xi_6\}/\sqrt{q} ,\\
&\boldsymbol{f}_7=\{ap+\xi_3\}/\sqrt{k} ,
\end{aligned}
\tag{5.69}
$$

where $k=a^2+1$ and $q=4a^2+1$. Using (5.52), we see that $\{\boldsymbol{f}_1, \cdots, \boldsymbol{f}_7\}$ satisfies Table (5.1).

We put

$$\lambda_{(1)}=\frac{2(a^2+1)(2a^2+1)}{4a^2+1} , \qquad \lambda_{(2)}=\frac{4(a^2+1)}{4a^2+1} ,$$
$$A_{(1)}=\frac{\lambda_{(2)}-2}{\lambda_{(2)}-\lambda_{(1)}}=\frac{1}{k} \quad \text{and} \quad A_{(2)}=\frac{2-\lambda_{(1)}}{\lambda_{(2)}-\lambda_{(1)}}=\frac{a^2}{k} .$$

If $-1/\sqrt{2}<a<0$, then $\lambda_1=\lambda_{(1)}$, $\lambda_2=\lambda_{(2)}$, $c(\lambda_1, \lambda_2)=2a^2/(2a^2+1)<1$ and $Q=(\lambda_2(\lambda_2-2)/(\lambda_1\lambda_2-\lambda_1-\lambda_2))^{1/2}=2$.

If $a>1/\sqrt{2}$, then $\lambda_1=\lambda_{(2)}$, $\lambda_2=\lambda_{(1)}$, $c(\lambda_1, \lambda_2)=(2a^2+1)/(2a^2)>1$ and $Q'=(\lambda_1(\lambda_1-2)/(\lambda_1\lambda_2-\lambda_1-\lambda_2))^{1/2}=2$.

By Theorem C, Proposition 4.2 and Lemma 5.13, f can be extended to a map of $\boldsymbol{R}^2$ into $S^5(1)$ and is given by

$$f(z)=\sqrt{A_{(1)}/2}\sum_{k=1}^{2} 2\mathrm{Re}\left\{u_k \exp \frac{\sqrt{\lambda_{(1)}}}{2}(\mu_k z-\bar{\mu}_k\bar{z})\right\}$$

$$+\sqrt{A_{(2)}}\, 2\mathrm{Re}\left\{u_3 \exp \frac{\sqrt{\lambda_{(2)}}}{2}(\eta z-\bar{\eta}\bar{z})\right\},$$

$$u_j=\frac{1}{2}\{E_{2j}-\sqrt{-1}E_{2j+1}\}, \qquad j=1, 2, 3,$$

where $\{E_1, \cdots, E_7\}$ is some orthonormal basis of E^7 and μ_1, μ_2, η are complex numbers satisfying

$$\eta=e^{i\beta}, \quad \mu_1=e^{i(\alpha+\beta)}, \quad \mu_2=e^{i(-\alpha+\beta)}, \quad \cos 2\alpha=-\frac{2a^2}{2a^2+1}.$$

By Proposition 4.3, $f: \boldsymbol{R}^2\to S^5$ is doubly periodic. Moreover, since M is complete and since f is an imbedding of M, M is a flat torus $\boldsymbol{R}^2/\Lambda$, $\Lambda=\{z\,|\,f(z)=f(0)\}$. We may assume that $f(0)=p$, $(e_1)_p=(\partial/\partial x)_p$, $(e_2)_p=(\partial/\partial y)_p$, $\pi/4<\alpha<\pi/2$ and $-\pi/2<\beta\leqq\pi/2$. So we get

$$\sin 2\alpha=\frac{(4a^2+1)^{1/2}}{2a^2+1}, \quad \cos\alpha=\left(\frac{1}{2(2a^2+1)}\right)^{1/2}, \quad \sin\alpha=\left(\frac{4a^2+1}{2(2a^2+1)}\right)^{1/2}.$$

First, we see

$$p=\sqrt{A_{(1)}/2}\,E_2+\sqrt{A_{(1)}/2}\,E_4+\sqrt{A_{(2)}}\,E_6$$
$$=\frac{1}{\sqrt{2k}}E_2+\frac{1}{\sqrt{2k}}E_4+\frac{|a|}{\sqrt{k}}E_6.$$

By direct computation, we obtain

$$(e_1)_p=\frac{\partial f}{\partial x}(0)=\left(\frac{\partial f}{\partial z}+\frac{\partial f}{\partial \bar{z}}\right)(0)$$
$$=\sqrt{A_{(1)}\lambda_{(1)}/2}\,\sin(\alpha+\beta)E_3+\sqrt{A_{(1)}\lambda_{(1)}/2}\,\sin(-\alpha+\beta)E_5$$
$$+\sqrt{A_{(2)}\lambda_{(2)}}\,\sin(\beta)E_7,$$

$$(e_2)_p=\frac{\partial f}{\partial y}(0)=\sqrt{-1}\Big(\frac{\partial f}{\partial z}-\frac{\partial f}{\partial \bar z}\Big)(0)$$
$$=\sqrt{A_{(1)}\lambda_{(1)}/2}\cos(\alpha+\beta)E_3+\sqrt{A_{(1)}\lambda_{(1)}/2}\cos(-\alpha+\beta)E_5$$
$$+\sqrt{A_{(2)}\lambda_{(2)}}\cos(\beta)E_7 .$$

From (4.3), we have

$$H=-\sqrt{A_{(1)}/2}\,\frac{\lambda_{(1)}-2}{2}\sum_{k=1}^{2}2\mathrm{Re}\Big\{u_k\exp\frac{\sqrt{\lambda_{(1)}}}{2}(\mu_k z-\bar\mu_k\bar z)\Big\}$$
$$-\sqrt{A_{(2)}}\,\frac{\lambda_{(2)}-2}{2}\,2\mathrm{Re}\Big\{u_3\exp\frac{\sqrt{\lambda_{(2)}}}{2}(\eta z-\overline{\eta z})\Big\} ,$$

(5.70) $$\xi_3=H/\alpha=2H/\sqrt{(2-\lambda_{(1)})(\lambda_{(2)}-2)}$$
$$=-\frac{a}{|a|}\sqrt{A_{(2)}/2}\sum_{k=1}^{2}2\mathrm{Re}\Big\{u_k\exp\frac{\sqrt{\lambda_{(1)}}}{2}(\mu_k z-\bar\mu_k\bar z)\Big\}$$
$$+\frac{a}{|a|}\sqrt{A_{(1)}}\,2\mathrm{Re}\Big\{u_3\exp\frac{\sqrt{\lambda_{(2)}}}{2}(\eta z-\overline{\eta z})\Big\} .$$

In particular, we get

(5.71) $$(\xi_3)_p=-\frac{a}{|a|}\sqrt{A_{(2)}/2}\,E_2-\frac{a}{|a|}\sqrt{A_{(2)}/2}\,E_4+\frac{a}{|a|}\sqrt{A_{(1)}}\,E_6$$
$$=-\frac{a}{\sqrt{2k}}E_2-\frac{a}{\sqrt{2k}}E_4+\frac{a}{|a|\sqrt{k}}E_6 .$$

From Lemma 5.12, we have

(5.72) $$\sigma(e_1,e_1)_p=\alpha(\xi_3)_p .$$

On the other hand, we have in §4

$$\sigma(e_1,e_1)_p=\Big(\frac{\partial^2 f}{\partial z^2}+2\frac{\partial^2 f}{\partial z\partial\bar z}+\frac{\partial^2 f}{\partial \bar z^2}+f\Big)(0)$$
$$=\sqrt{A_{(1)}/2}\Big\{\frac{\lambda_{(1)}}{4}(\mu_1^2+\bar\mu_1^2-2)+1\Big\}E_2$$
$$+\sqrt{A_{(1)}/2}\Big\{\frac{\lambda_{(1)}}{4}(\mu_2^2+\bar\mu_2^2-2)+1\Big\}E_4$$
$$+\sqrt{A_{(2)}}\Big\{\frac{\lambda_{(2)}}{4}(\eta^2+\bar\eta^2-2)+1\Big\}E_6 .$$

Combining this with (5.71) and (5.72), we get

$$\sqrt{A_{(2)}}\left\{\frac{\lambda_{(2)}}{4}(\eta^2+\bar{\eta}^2-2)+1\right\}=\frac{a^2}{|a|}\sqrt{A_{(1)}}$$

so that we have $(\eta^2+\bar{\eta}^2)/2=1$, $\cos(2\beta)=1$, $\beta=0$ and $\eta=1$. Therefore we obtain

$$(e_1)_p=\frac{1}{\sqrt{2}}E_3-\frac{1}{\sqrt{2}}E_5\,,\qquad (e_2)_p=\frac{1}{\sqrt{2q}}E_3+\frac{1}{\sqrt{2q}}E_5+\frac{2\,|a|}{\sqrt{q}}E_7\,.$$

By direct computation, we get

$$\begin{aligned}\sigma(e_1, e_2)_p&=\sqrt{-1}\Big(\frac{\partial^2 f}{\partial z^2}-\frac{\partial^2 f}{\partial \bar{z}^2}\Big)(0)\\&=\sqrt{A_{(1)}/2}\frac{\lambda_{(1)}}{4}\sqrt{-1}(\mu_1{}^2-\bar{\mu}_1{}^2)E_2+\sqrt{A_{(1)}/2}\frac{\lambda_{(1)}}{4}\sqrt{-1}(\mu_2{}^2-\bar{\mu}_2{}^2)E_4\\&=-\sqrt{A_{(1)}/2}\frac{\lambda_{(1)}}{4}\sin(2\alpha)E_2+\sqrt{A_{(1)}/2}\frac{\lambda_{(1)}}{2}\sin(2\alpha)E_4\\&=\sqrt{k/q}\left\{-\frac{1}{\sqrt{2}}E_2+\frac{1}{\sqrt{2}}E_4\right\}\,.\end{aligned}$$

From Lemma 5.12, we obtain

$$\begin{aligned}(\eta_4)_p=\sqrt{q/k}(c\xi_4+h\xi_5)_p&=\frac{\sigma(e_1, e_2)_p}{\|\sigma(e_1, e_2)\|}\\&=-\frac{1}{\sqrt{2}}E_2+\frac{1}{\sqrt{2}}E_4\,.\end{aligned}$$

From (5.70), we get

$$\begin{aligned}(\nabla^{\perp}_{e_2}\xi_3)_p&=\Big(\frac{\partial}{\partial y}\xi_3\Big)_p+(A_{\xi_3}e_2)_p\\&=\Big[\sqrt{-1}\Big(\frac{\partial}{\partial z}-\frac{\partial}{\partial \bar{z}}\Big)\xi_3\Big](0)+c(e_2)_p\\&=\sqrt{\lambda_{(1)}/2}\cos(\alpha)\left\{-\frac{a}{|a|}\sqrt{A_{(2)}}+c\sqrt{A_{(1)}}\right\}E_3\\&\quad+\sqrt{\lambda_{(1)}/2}\cos(\alpha)\left\{-\frac{a}{|a|}\sqrt{A_{(2)}}+c\sqrt{A_{(1)}}\right\}E_5\\&\quad+\sqrt{\lambda_{(2)}}\left\{\frac{a}{|a|}\sqrt{A_{(1)}}+c\sqrt{A_{(2)}}\right\}E_7\\&=\frac{2k}{q}\Big(-\frac{2a}{\sqrt{2q}}E_3-\frac{2a}{\sqrt{2q}}E_5+\frac{a}{|a|\sqrt{q}}E_7\Big)\,.\end{aligned}$$

On the other hand, from Lemma 5.12, we see

$$(\nabla^{\perp}_{e_2}\xi_3)_p=(h+1)(\xi_6)_p=\frac{2k}{q}(\xi_6)_p .$$

So we have

$$(\xi_6)_p=-\frac{2a}{\sqrt{2q}}E_3-\frac{2a}{\sqrt{2q}}E_5+\frac{a}{|a|\sqrt{q}}E_7 .$$

From the proof of Lemma 5.13, we may assume

$$E_1=\eta_5=\sqrt{q/k}\{-h\xi_4+c\xi_5\} .$$

From (5.69), we obtain

$$\begin{aligned}&\boldsymbol{f}_1=E_1 , \quad \boldsymbol{f}_2=E_2 , \quad \boldsymbol{f}_3=-E_3 , \quad \boldsymbol{f}_4=E_4 ,\\&\boldsymbol{f}_5=-E_5 , \quad \boldsymbol{f}_6=-\frac{a}{|a|}E_7 , \quad \boldsymbol{f}_7=\frac{a}{|a|}E_6 .\end{aligned} \tag{5.73}$$

Define functions on M as follows:

$$\varphi_j(z)=\langle\sqrt{-\lambda_{(1)}}\bar{\mu}_j, z\rangle=-\sqrt{-1}\frac{\sqrt{\lambda_{(1)}}}{2}(\mu_j z-\bar{\mu}_j\bar{z}) , \qquad j=1, 2 ,$$

$$\varphi_3(z)=\langle\sqrt{-\lambda_{(2)}}, z\rangle=-\sqrt{-1}\frac{\sqrt{\lambda_{(2)}}}{2}(z-\bar{z}) .$$

Then f is given by

$$\begin{aligned}f(z)=\sqrt{A_{(1)}/2}\{&\cos(\varphi_1(z))E_2+\sin(\varphi_1(z))E_3\\&+\cos(\varphi_2(z))E_4+\sin(\varphi_2(z))E_5\}\\&+\sqrt{A_{(2)}}\{\cos(\varphi_3(z))E_6+\sin(\varphi_3(z))E_7\} .\end{aligned}$$

Using the frame (5.73), we have

$$\begin{aligned}f=\frac{1}{\sqrt{2k}}&\{\cos(\varphi_1)\boldsymbol{f}_2-\sin(\varphi_1)\boldsymbol{f}_3+\cos(\varphi_2)\boldsymbol{f}_4-\sin(\varphi_3)\boldsymbol{f}_5\}\\&+\frac{a}{\sqrt{k}}\{-\sin(\varphi_3)\boldsymbol{f}_6+\cos(\varphi_3)\boldsymbol{f}_7\} ,\end{aligned}$$

so that we have

$$p=\frac{1}{\sqrt{2k}}\boldsymbol{f}_2+\frac{1}{\sqrt{2k}}\boldsymbol{f}_4+\frac{a}{\sqrt{k}}\boldsymbol{f}_7 .$$

Put, for any $z\in\boldsymbol{C}$,

$$T(z)=\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \cos\varphi_1 & \sin\varphi_1 & 0 & 0 & 0 & 0 \\ 0 & -\sin\varphi_1 & \cos\varphi_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos\varphi_2 & \sin\varphi_2 & 0 & 0 \\ 0 & 0 & 0 & -\sin\varphi_2 & \cos\varphi_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cos(-\varphi_3) & \sin(-\varphi_3) \\ 0 & 0 & 0 & 0 & 0 & -\sin(-\varphi_3) & \cos(-\varphi_3) \end{pmatrix}$$

with respect to the basis $\{\boldsymbol{f}_1, \cdots, \boldsymbol{f}_7\}$. Hence we easily see that

$$\sqrt{\lambda_{(1)}}(\bar{\mu}_1+\bar{\mu}_2)=2\sqrt{\lambda_{(1)}}\cos(\alpha)=\sqrt{\lambda_{(2)}}$$

so that

$$\varphi_1+\varphi_2-\varphi_3=0 .$$

This implies that $T=\{T(z)\mid z\in \boldsymbol{C}\}$ is a maximal torus of G_2. Since f is given by $f(z)=T(z)p$, $f(M)$ is a T-orbit. Since f is an imbedding of M, $M=Tp$ so that $M\in\mathfrak{F}_5$.

The proof of Theorem G is completed.

5.3. Some remarks. Suppose that M is a flat surface such that $M\in\mathfrak{F}_5$. By Theorem F, M is a Chen surface in S^6. Denote by f an isometric imbedding of M into $S^6(1)$.

In §5.2.2, we see that $Q=2$ (if $c(\lambda_1, \lambda_2)<1$) and $Q'=2$ (if $c(\lambda_1, \lambda_2)>1$). By Theorem C, f can be extended to an isometric immersion of $\boldsymbol{R}^2$ into $S^6(1)$. By Proposition 4.3, we see that $f\colon \boldsymbol{R}^2\to S^6$ is doubly periodic. Put $\Lambda=\{z\mid f(z)=f(0)\}$ so that M is a flat torus $\boldsymbol{R}^2/\Lambda$. Since f is an imbedding of M into S^6, we obtain by Corollary 4.4

$$\Lambda=\Lambda(\lambda_1, \lambda_2) ,$$

where $\Lambda(\lambda_1, \lambda_2)$ is a lattice of rank 2 defined in §4.

In the case of $c(\lambda_1, \lambda_2)<1$, $\Lambda(\lambda_1, \lambda_2)$ is generated by x_1 and x_2 as follows:

$$\Lambda(\lambda_1, \lambda_2)=\{kx_1+lx_2 \mid k, l\in \boldsymbol{Z}\} ,$$

$$x_1=\left(\frac{2\pi}{\sqrt{\lambda_1}}, \frac{-2\pi\cos 2\nu}{\sqrt{\lambda_1}\sin 2\nu}\right), \qquad x_2=\left(0, \frac{2\pi}{\sqrt{\lambda_1}\sin 2\nu}\right),$$

$$\cos 2\nu=-c(\lambda_1, \lambda_2) .$$

It is easy to see that $\langle x_1, x_2\rangle\neq 0$ so that a flat torus $\boldsymbol{R}^2/\Lambda$ is not a Riemannian product of two circles.

We apply similar argument to the case of $c(\lambda_1, \lambda_2)>1$. Therefore we have the following.

PROPOSITION 5.14. *If $M\in\mathfrak{F}_5$, then M is not a Riemannian product of two circles.*

A surface M in $S^n(1)$ is called *stationary* if the mean curvature α of M in S^n satisfies

$$\delta\left(\int_M(\alpha^2+1)dV\right)=0$$

for any δ, where δ is a normal variation. In Barros and Chen [1], we can see many results for stationary 2-type surfaces in S^n. Weiner [15] shows that M is stationary if and only if

$$\Delta^\perp H=-2\alpha^2H+\frac{1}{\alpha^2}\|A_H\|^2H+\mathscr{A}(H)\ . \tag{5.74}$$

(See also Barros and Chen [1].) We obtain the following.

PROPOSITION 5.15. *If $M\in\mathfrak{F}_5$, then M is not stationary.*

PROOF. Assume that M ($\in\mathfrak{F}_5$) is stationary. By Theorem F, M is a Chen surface of S^6, i.e., $\mathscr{A}(H)=0$. Therefore we obtain from (5.74) and Lemma 5.12,

$$\Delta^\perp H=\frac{8a^2(a^2+1)^2}{(4a^2+1)^2}H\ .$$

On the other hand, by Lemma 5.12, we get

$$\Delta^\perp H=\sum_{i=1}^{2}(\nabla^\perp_{\nabla_{e_i}e_i}H-\nabla^\perp_{e_i}\nabla^\perp_{e_i}H)=\frac{4(a^2+1)^2}{(4a^2+1)^2}H\ .$$

Therefore we have $a=\pm 1/\sqrt{2}$. This is a contradiction. Q.E.D.

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