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Compact Weighted Composition Operators on Function Algebras

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Dedicated to Professor Junzo Wada on his 60th birthday

Abstract. A weighted endomorphism of an algebra is an endomorphism followed by a multiplier. In [6] and [4], H. Kamowitz characterized compact weighted endomorphisms of $C(X)$ and the disc algebra. In this note we define a weighted composition operator on a function algebra as a generalization of a weighted endomorphism, and characterize compact weighted composition operators on a function algebra satisfying a certain condition [Theorem 2]. This theorem not only includes Kamowitz's results as corollaries, but also has an application to compact weighted composition operators on the Hardy class $H^\infty(D)$.

Introduction.

Let A be a function algebra on a compact Hausdorff space X, that is, a uniformly closed subalgebra of $C(X)$ which contains the constants and separates the points of X. By M_A we denote the maximal ideal space of A and by M_A^∞ the union of M_A and the zero functional θ on A. Then M_A^∞ is considered as a subset of the dual space of A, so M_A^∞ is equipped with the relative topologies induced by the weak* topology and norm topology respectively. We shall understand M_A^∞ is given the weak* topology unless otherwise qualified. For each $f\in A$, we put $\hat{f}(m)=m(f)$ for any $m\in M_A^\infty$, and $\operatorname{supp} f=\{x\in X: f(x)\neq 0\}$. Note that $\operatorname{supp} f$ is open.

A weighted endomorphism of an algebra is defined to be a linear operator which is an endomorphism followed by a multiplier. Thus, if B is an algebra, then T is a weighted endomorphism of B if there are an element u in B and an endomorphism S of B such that

$$Tf=u\cdot Sf \qquad f\in B .$$

Recently, weighted endomorphisms for various algebras were studied by Kamowitz ([4] and [6]) and Kitover ([7]).

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If S is an endomorphism of a function algebra A, then S has the representation;

$$Sf(x)=\hat{f}(\Phi(x)) \qquad x\in X\,,\quad f\in A\,,$$

for some continuous map Φ from X into M_A^∞. In fact, Φ is given by

$$\Phi(x)=S^*(\hat{x}) \qquad x\in X\,,$$

where S^* is the adjoint of S and $\hat{x}$ is the evaluation functional at x, i.e., $\hat{x}(f)=f(x)$ for each $f\in A$. (We note that when $S1=1$, Φ maps X into M_A.) Consequently, a weighted endomorphism T of A has the form;

$$Tf(x)=u(x)\hat{f}(\Phi(x)) \qquad x\in X\,,\quad f\in A\,,$$

for some $u\in A$ and some continuous map Φ from X into M_A^∞. The map T will be denoted by uC_Φ.

Now we define weighted composition operators, which involve weighted endomorphisms.

DEFINITION. Let T be a bounded linear operator from A to A. We call T a weighted composition operator on A if there are an element u in A and a continuous map φ from $\operatorname{supp} u$ into M_A^∞ such that

$$Tf(x)=\begin{cases}u(x)\hat{f}(\varphi(x)) & x\in \operatorname{supp} u\\ 0 & x\in X\setminus \operatorname{supp} u\end{cases}$$

for each $f\in A$. We write uC_φ for T.

In this paper we discuss compact weighted composition operators on a function algebra. A linear operator T on a Banach space B is called compact if, for the unit ball B_0 of B, TB_0 is relatively compact in B.

We begin with the following lemma.

LEMMA 1. *Let uC_φ be a weighted composition operator on A. uC_φ is compact if and only if φ is a continuous map from* $\operatorname{supp} u$ *into M_A^∞ with respect to the norm topology.*

PROOF. Put $A_0=\{f\in A\colon \|f\|\leqq 1\}$. The compactness of uC_φ implies that $uC_\varphi A_0$ is relatively compact in A, and so is in $C(X)$. By the Ascoli-Arzelà theorem, it is equivalent to the fact that $uC_\varphi A_0$ is equicontinuous, that is,

$$(1) \qquad \sup_{f\in A_0}|uC_\varphi f(x_\alpha)-uC_\varphi f(x)|\to 0$$

as $x_\alpha\to x$ in X.

Let uC_φ be a compact operator. For any x, $x_\alpha \in \operatorname{supp} u$, we have

$$\begin{aligned}\|\varphi(x_\alpha)-\varphi(x)\| &= \frac{1}{|u(x)|}\|u(x)\varphi(x_\alpha)-u(x)\varphi(x)\| \\ &\leqq \frac{1}{|u(x)|}(|u(x)-u(x_\alpha)|\ \|\varphi(x_\alpha)\| + \|u(x_\alpha)\varphi(x_\alpha)-u(x)\varphi(x)\|) \\ &\leqq \frac{1}{|u(x)|}(|u(x)-u(x_\alpha)| + \sup_{f\in A_0}|uC_\varphi f(x_\alpha)-uC_\varphi f(x)|)\ .\end{aligned}$$

By the continuity of u and (1), $\|\varphi(x_\alpha)-\varphi(x)\|\to 0$ as $x_\alpha\to x$. This proves the "only if" part of the lemma.

Conversely, assume that φ is a continuous map from $\operatorname{supp} u$ into M_A^∞ with the norm topology. We shall show (1). Suppose $x\in X$ and $\{x_\alpha\}$ is a net with $x_\alpha\to x$. If $x\in\operatorname{supp} u$, we can assume that $\{x_\alpha\}\subset\operatorname{supp} u$, because $\operatorname{supp} u$ is open. Then we have

$$\begin{aligned}\sup_{f\in A_0}|uC_\varphi f(x_\alpha)-uC_\varphi f(x)| &= \|u(x_\alpha)\varphi(x_\alpha)-u(x)\varphi(x)\| \\ &\leqq |u(x_\alpha)|\ \|\varphi(x_\alpha)-\varphi(x)\| + |u(x_\alpha)-u(x)|\ \|\varphi(x)\| \\ &\leqq \|u\|\ \|\varphi(x_\alpha)-\varphi(x)\| + |u(x_\alpha)-u(x)| \to 0\end{aligned}$$

as $x_\alpha\to x$. On the other hand, if $x\notin\operatorname{supp} u$,

$$\begin{aligned}\sup_{f\in A_0}|uC_\varphi f(x_\alpha)-uC_\varphi f(x)| &= \sup_{f\in A_0}|uC_\varphi f(x_\alpha)| \\ &= \begin{cases}|u(x_\alpha)|\ \|\varphi(x_\alpha)\|\leqq|u(x_\alpha)| & \text{when}\quad x_\alpha\in\operatorname{supp} u \\ 0 & \text{when}\quad x_\alpha\notin\operatorname{supp} u\ .\end{cases}\end{aligned}$$

Hence $\sup_{f\in A_0}|uC_\varphi f(x_\alpha)-uC_\varphi f(x)|\to 0$ as $x_\alpha\to x$. Thus the lemma is proved.

§ 1. Relations to Gleason parts.

In this section we investigate relations between compact weighted composition operators and Gleason parts.

It is known that M_A is divided into (Gleason) parts $\{P_\alpha\}$ for A, as follows;

$$M_A=\bigcup_\alpha P_\alpha\ ,\qquad P_\alpha\cap P_\beta=\emptyset\quad(\alpha\neq\beta)\ .$$

The part P containing $m_0\in M_A$ is defined by

$$P=\{m\in M_A\colon \|m-m_0\|<2\}\ .$$

Clearly, each part is open in M_A with the norm topology, and is therefore open in M_A^∞ with the norm topology. Since $\{\theta\}$ is so, we consider $\{\theta\}$ as a part for A. Thus we divide M_A^∞ into parts, and each part is open and closed in M_A^∞ with the norm topology.

THEOREM 1. *Let uC_φ be a weighted composition operator on A. If uC_φ is compact, then for each connected component C of* supp u, *there exist an open set $V\subset$*supp u *and a part P for A such that*

$$C\subset V, \qquad \varphi(V)\subset P.$$

PPOOF. Let C be a connected component of supp u, and fix $x_0\in C$. Then $\varphi(x_0)$ belongs to some part P for A. Put $V=\{x\in \text{supp } u: \varphi(x)\in P\}$. By Lemma 1, φ is a continuous map from supp u into M_A^∞ with the norm topology, and P is open and closed in M_A^∞ with the norm topology. It follows that V is open and closed in supp u. Now suppose $C\not\subset V$. Then the disconnection $C=(C\cap V)\cup(C\cap(\text{supp } u\setminus V))$ induces a contradiction. Hence $C\subset V$, concluding the proof.

Next we consider the converse to Theorem 1. The following lemma is easy.

LEMMA 2. *Let uC_φ be a weighted composition operator on A. Suppose that for each connected component C of* supp u, *there exist an open set $V\subset$*supp u *and an element $m\in M_A^\infty$* such that

$$C\subset V, \qquad \varphi|_V=m. \tag{2}$$

Then uC_φ is compact.

PROOF. Let $x_0\in$ supp u. For the connected component C containing x_0, choose an open set V satisfying (2). Then $x_0\in V$ and $\|\varphi(x)-\varphi(x_0)\|=\|m-m\|=0$ for every $x\in V$. Hence φ is a continuous map from supp u into M_A^∞ with the norm topology. The lemma follows from Lemma 1.

According to this lemma, when each part for A is a one-point part — for example, when $A=C(X)$ —, the converse to Theorem 1 is true. If there exists a non-trivial part, does the converse to Theorem 1 hold?

Let P be a non-trivial part. We say that P satisfies the condition (α) if P has the following property;

> (α) for any $m\in P$, there are some open neighborhood $U(m)$ of m in P and a homeomorphism ρ from a polydisc D^n (a disc if $n=1$, n depends on $U(m)$) onto $U(m)$ such that $\hat{f}\circ\rho$ is an analytic function on D^n for all $f\in A$.

This condition was introduced in Ohno and Wada [8]. See [8] for simple examples.

THEOREM 2. *Suppose that every non-trivial part for A satisfies (α). Let uC_φ be a weighted composition operator on A. Then uC_φ is compact if and only if for each connected component C of* $\operatorname{supp} u$, *there exist an open set $V \subset \operatorname{supp} u$ and a part P for A such that*

$$C \subset V, \qquad \varphi(V) \subset P.$$

PROOF. Since the "only if" part is obvious (Theorem 1), we prove the "if" part. To prove that uC_φ is compact, it suffices to show that φ is a continuous map from $\operatorname{supp} u$ into M_A^∞ with the norm topology.

Let $x_0 \in \operatorname{supp} u$. By hypothesis, we can find an open set $V \subset \operatorname{supp} u$ such that

$$x_0 \in V, \qquad \varphi(V) \subset P,$$

where P is a part for A. If P is a one-point part, we have already proved in Lemma 2 that φ is continuous at x_0 with respect to the norm topology. So, let us suppose P is non-trivial. By the definition of weighted composition operators, φ is a continuous map from $\operatorname{supp} u$ into M_A^∞ with the weak* topology. Hence we only show that the identity map ψ from P with the weak* topology onto P with the norm topology is continuous at $\varphi(x_0)$.

Put $m_0 = \varphi(x_0)$. By (α), there are a neighborhood $U(m_0)$ and a homeomorphism ρ from D^n onto $U(m_0)$ such that $\hat{f} \circ \rho$ is analytic in D^n for all $f \in A$. The Montel theorem says that $\mathscr{G} = \{g : g$ is analytic in D^n and $\|g\|_\infty \leqq 1\}$ is equicontinuous, that is, for any $\varepsilon > 0$, there exists a neighborhood $W (\subset D^n)$ of $\zeta_0 = \rho^{-1}(m_0)$ such that $|g(\zeta) - g(\zeta_0)| < \varepsilon$ for all $\zeta \in W$ and all $g \in \mathscr{G}$. Hence, for each $m = \rho(\zeta) \in \rho(W)$,

$$\begin{aligned} \|\psi(m) - \psi(m_0)\| &= \|m - m_0\| \\ &= \sup\{|m(f) - m_0(f)| : f \in A,\ \|f\| \leqq 1\} \\ &= \sup\{|\hat{f}(\rho(\zeta)) - \hat{f}(\rho(\zeta_0))| : f \in A,\ \|f\| \leqq 1\} \\ &\leqq \sup\{|g(\zeta) - g(\zeta_0)| : g \in \mathscr{G}\} \leqq \varepsilon . \end{aligned}$$

Since $\rho(W)$ is a weak*-neighborhood of m_0, ψ is continuous.

§ 2. Theorems of Kamowitz.

Kamowitz ([6] and [4]) characterized compact weighted endomorphisms of $C(X)$ and the disc algebra. We shall prove two theorems due to Kamowitz as corollaries of Theorem 2. One of them is:

COROLLARY 1 (Kamowitz [6]). *Let uC_φ be a weighted endomorphism* *of* $C(X)$. *Then* uC_Φ *is compact if and only if for each connected component* C *of* supp u, *there exists an open set* $V \supset C$ *such that* Φ *is constant on* V.

PROOF. The statement follows immediately from Theorem 2, since each point of $M_{C(X)}=X$ is a one-point part.

The other theorem deals with compact weighted endomorphisms of the disc algebra. Recall that the disc algebra $A(\bar{D})$ is the algebra of functions analytic in the open unit disc D and continuous on $\bar{D}$. We know that $M_{A(\bar{D})}=\bar{D}$, and that D and each boundary point of D are parts for $A(\bar{D})$. Note that D satisfies (α).

Let uC_Φ be a non-zero weighted endomorphism of $A(\bar{D})$. As we saw in the introduction, Φ is determined by a certain endomorphism S of $A(\bar{D})$. Since S cannot be a zero operator, $S1=1$ holds. Therefore Φ is a map from $\bar{D}$ into $M_{A(\bar{D})}$. Thus Φ is considered as a continuous function from $\bar{D}$ into $\bar{D}$ such that

$$Sf(\zeta)=f(\Phi(\zeta)) \qquad \zeta \in \bar{D}\,, \quad f \in A(\bar{D})\,.$$

By taking f to be the coordinate function, we have $\Phi \in A(\bar{D})$.

COROLLARY 2 (Kamowitz [4]). *Let* uC_Φ *be a non-zero weighted endomorphism of* $A(\bar{D})$. *Then* uC_Φ *is compact if and only if one of the following holds*:

(i) Φ *is constant.*

(ii) $|\Phi(\zeta)|<1$, *whenever* $u(\zeta)\neq 0$.

PROOF. Since $u \in A(\bar{D})$ and $u\not\equiv 0$, the set $\{\zeta \in \bar{D}: u(\zeta)=0\}$ has no accumulation points in D. It follows that supp $u=\bar{D}\setminus\{\zeta \in \bar{D}: u(\zeta)=0\}$ is (arcwise) connected. Thus Theorem 2 implies that uC_Φ is compact if and only if there exists a part P for $A(\bar{D})$ such that

$$\Phi(\text{supp}\, u) \subset P\,. \tag{3}$$

If P in (3) is trivial, that is, a boundary point of D, the fact that $\overline{\text{supp}\, u}=\bar{D}$ and the continuity of Φ show (i). On the other hand, in the case of $P=D$, (3) is equivalent to (ii).

§ 3. Weighted composition operators on $H^\infty(D)$.

Compact composition operators on Hardy class $H^\infty(D)$ were discussed in Swanton [9]. We here consider compact weighted composition operators on $H^\infty(D)$ as an application of § 1.

Let D be the open unit disc in the complex plane C and $H^\infty(D)$ be the algebra of bounded analytic functions on D with the supremum norm. For any $u \in H^\infty(D)$ and any analytic function φ from D into D, the weighted composition operator uC_φ on $H^\infty(D)$ is defined by

$$uC_\varphi f(\zeta)=u(\zeta)f(\varphi(\zeta)) \qquad \zeta \in D\,, \quad f \in H^\infty(D)\,.$$

A weighted composition operator on $H^\infty(D)$ is a bounded linear operator on $H^\infty(D)$.

THEOREM 3. *Let uC_φ be a weighted composition operator on $H^\infty(D)$. Then uC_φ is compact if and only if $\overline{\varphi(E)}\subset D$ whenever $E\subset D$ satisfies*

$$\inf\{|u(\zeta)|: \zeta \in E\}>0\,. \tag{4}$$

Before proving the theorem, we make a few remarks on $H^\infty(D)$. Let M be the maximal ideal space of $H^\infty(D)$, and set $\hat{H}^\infty=\{\hat{f}: f \in H^\infty(D)\}$, where $\hat{f}$ is the Gel'fand transform of f. Then $\hat{H}^\infty$ is a function algebra on the maximal ideal space M of $\hat{H}^\infty$.

For each $\zeta \in D$, denote by $\hat{\zeta}$ the evaluation functional at ζ defined by $\hat{\zeta}(f)=f(\zeta)$ for all $f \in H^\infty(D)$. Put $\mathscr{D}=\{\hat{\zeta}: \zeta \in D\}$. For each $\zeta \in \partial D$, the boundary of D, let $M_\zeta=\{m \in M: m(z)=\zeta\}$ be the fiber over ζ. Here z is the coordinate function. Then we have that

$$M = \mathscr{D} \cup \bigcup_{\zeta \in \partial D} M_\zeta\,.$$

Each fiber M_ζ $(\zeta \in \partial D)$ is a peak set for $\hat{H}^\infty$. In other words, there exists some $f \in H^\infty(D)$ such that $\hat{f}$ is equal to 1 on M_ζ while $|\hat{f}(m)|<1$ for all $m \in M\setminus M_\zeta$. This shows that $\mathscr{D}$ is a part for $\hat{H}^\infty$. On the other hand, the corona theorem [1, p. 34] tells us that $\overline{\mathscr{D}}^{w^*}=M$, where $\overline{}^{w^*}$ denotes the weak*-closure in M.

Now we determine a weighted endomorphism of $\hat{H}^\infty$ corresponding to a weighted composition operator uC_φ on $H^\infty(D)$. Define a continuous map Φ from M into M by

$$\Phi(m)(f)=m(f\circ\varphi) \qquad f \in H^\infty(D)\,, \quad m \in M$$

(note that $f\circ\varphi \in H^\infty(D)$). Then we have

$$\Phi(\hat{\zeta})=\widehat{\varphi(\zeta)} \qquad \zeta \in D\,,$$
$$\hat{f}\circ\Phi=\widehat{f\circ\varphi} \qquad f \in H^\infty(D)\,.$$

Hence we want to determine a weighted endomorphism $\hat{u}C_\Phi$ of $\hat{H}^\infty$ as follows;

$$\hat{u}C_\Phi\hat{f}(m)=\hat{u}(m)\hat{f}(\Phi(m)) \qquad m\in M\,,\quad \hat{f}\in\hat{H}^\infty\,.$$

Of course, $\hat{u}C_\Phi$ is compact if and only if uC_φ is compact.

We return to the proof of Theorem 3.

PROOF. We may assume that $u\not\equiv 0$, otherwise there is nothing to prove. We first observe that $\operatorname{supp}\hat{u}=\{m\in M\colon \hat{u}(m)\neq 0\}$ is connected. If not, $\operatorname{supp}\hat{u}$ has a disconnection $\operatorname{supp}\hat{u}=W_1\cup W_2$. Since $\overline{\mathscr{D}}^{w^*}=M$, this yields another disconnection;

$$\{\hat{\zeta}\in\mathscr{D}\colon \hat{u}(\hat{\zeta})\neq 0\}=(\mathscr{D}\cap W_1)\cup(\mathscr{D}\cap W_2)\,,$$

which implies that $\{\zeta\in D\colon u(\zeta)\neq 0\}$ is not connected. But $\{\zeta\in D\colon u(\zeta)\neq 0\}$ is connected because $\{\zeta\in D\colon u(\zeta)=0\}$ is discrete in D. This contradiction shows that $\operatorname{supp}\hat{u}$ is connected.

Suppose that uC_φ is compact. Since $\hat{u}C_\Phi$ is also compact, we can apply Theorem 1 to $\hat{u}C_\Phi$. Thus we find a part P for $\hat{H}^\infty$ such that $\Phi(\operatorname{supp}\hat{u})\subset P$ (note that $\operatorname{supp}\hat{u}$ is connected). For any $\hat{\zeta}\in\mathscr{D}\cap\operatorname{supp}\hat{u}$, we have $\Phi(\hat{\zeta})=\widehat{\varphi(\zeta)}\in\mathscr{D}$. So P must be $\mathscr{D}$. Hence $\Phi(\operatorname{supp}\hat{u})\subset\mathscr{D}$.

Next assume that $E\subset D$ satisfies (4). Since $\mathscr{E}=\{\hat{\zeta}\colon \zeta\in E\}$ satisfies $\delta=\inf\{|\hat{u}(m)|\colon m\in\mathscr{E}\}>0$, $\min\{|\hat{u}(m)|\colon m\in\overline{\mathscr{E}}^{w^*}\}=\delta>0$ holds. It implies that $\overline{\mathscr{E}}^{w^*}\subset\operatorname{supp}\hat{u}$. Thus we obtain that

$$\Phi(\mathscr{E})\subset\Phi(\overline{\mathscr{E}}^{w^*})\subset\Phi(\operatorname{supp}\hat{u})\subset\mathscr{D}\,.$$

Since $\Phi(\overline{\mathscr{E}}^{w^*})$ is compact, $\overline{\Phi(\mathscr{E})}^{w^*}\subset\mathscr{D}$, that is, $\overline{\varphi(E)}\subset D$.

Conversely assume that $\overline{\varphi(E)}\subset D$ for any $E\subset D$ satisfying (4). We must show that uC_φ, and therefore $\hat{u}C_\Phi$ is compact. By Lemma 1, it suffices to show that Φ is a continuous map from $\operatorname{supp}\hat{u}$ into M with the norm topology.

Suppose $m_0\in\operatorname{supp}\hat{u}$. Since $\overline{\mathscr{D}}^{w^*}=M$, there is a net $\{\zeta_\alpha\}$ in D such that $\hat{\zeta}_\alpha$ converges to m_0 with respect to the weak* topology. Furthermore we can assume that $\inf_\alpha|u(\zeta_\alpha)|>0$, because $\hat{u}(m_0)\neq 0$. Then by the assumption on φ, we have $\overline{\{\varphi(\zeta_\alpha)\}}\subset D$. Hence

$$\Phi(m_0)(z)=m_0(z\circ\varphi)=m_0(\varphi)=\lim_\alpha\hat{\zeta}_\alpha(\varphi)=\lim_\alpha\varphi(\zeta_\alpha)\in D\,.$$

Put $\zeta_0=\Phi(m_0)(z)$, that is, $\hat{\zeta}_0=\Phi(m_0)$. By Montel's theorem, we find a neighborhood W of ζ_0 in D such that $|f(\zeta)-f(\zeta_0)|<\varepsilon$ for all $\zeta\in W$ and $f\in H^\infty(D)$ satisfying $\|f\|\leq 1$. Set $U=\{m\in\operatorname{supp}\hat{u}\colon \Phi(m)(z)\in W\}$. U is a weak*-neighborhood of m_0 in $\operatorname{supp}\hat{u}$, and for each $m\in U$, we have

$$\begin{aligned}&\|\Phi(m)-\Phi(m_0)\|\\&\quad=\sup\{|\Phi(m)(f)-\Phi(m_0)(f)|: f\in H^\infty(D),\ \|f\|\leqq 1\}\\&\quad=\sup\{|\hat{\zeta}(f)-\hat{\zeta}_0(f)|: f\in H^\infty(D),\ \|f\|\leqq 1\}\\&\quad=\sup\{|f(\zeta)-f(\zeta_0)|: f\in H^\infty(D),\ \|f\|\leqq 1\}\leqq\varepsilon\ ,\end{aligned}$$

where $\zeta=\Phi(m)(z)\in W$, i.e., $\hat{\zeta}=\Phi(m)$. Hence Φ is continuous at m_0 as a map from $\operatorname{supp}\hat{u}$ into M with the norm topology. The theorem is proved.

Theorem 3 remains, with the same proof, true for $H^\infty(D)$ on a domain D such that

(i) for each boundary point ζ of D, the fiber over ζ is a peak set for $\hat{H}^\infty$;

(ii) $\mathscr{D}$ is dense in the maximal ideal space of $H^\infty(D)$.

§ 4. A counter-example.

In this section we give a counter-example to the question: does the converse to Theorem 1 hold?

If every part for A satisfies (α), Theorem 2 answered "yes". But, for the general case, the answer is "no". Indeed, there exist a function algebra A and a weighted composition operator uC_φ on A such that

(i) for each connected component C of $\operatorname{supp} u$, there are an open set $V\subset\operatorname{supp} u$ and a part P for A such that

$$C\subset V\ ,\qquad \varphi(V)\subset P\ ;$$

(ii) uC_φ is not compact.

First we construct a function algebra A, according to Garnett [2].

Fix a positive irrational number α, and let A_1 be the function algebra on the torus T^2 generated by the functions $\{z_1^n z_2^m: n,\ m \text{ integers},\ n+m\alpha\geqq 0\}$. Here $z_1^n z_2^m$ is defined by $z_1^n z_2^m(\zeta_1, \zeta_2)=\zeta_1^n\zeta_2^m$ for all $(\zeta_1, \zeta_2)\in T^2$. It is known that $M_{A_1}=\{(\zeta_1, \zeta_2)\in \boldsymbol{C}^2: |\zeta_1|\leqq 1,\ |\zeta_2|=|\zeta_1|^\alpha\}$.

Next recall that $A(\bar{D})$ denotes the disc algebra on the closed unit disc $\bar{D}$. In addition, let $I=[1/2, 1]$ (closed interval), and set

$$A_2=\{h\in C(I\times\bar{D}): h(t,\ \cdot)\in A(\bar{D}) \text{ for each } t\in I\ ,\ h|_{I\times\{0\}} \text{ is constant}\}\ .$$

If we denote by $I\times\bar{D}/\sim$ the quotient space of $I\times\bar{D}$ identifying the points in $I\times\{0\}$, A_2 is a function algebra on $I\times\bar{D}/\sim$, and $M_{A_2}=I\times\bar{D}/\sim$.

Let $A_1\otimes A_2$ be the function algebra on $M_{A_1}\times M_{A_2}$ generated by the functions of the form;

$$g\otimes h(\zeta_1, \zeta_2, t, \zeta)=g(\zeta_1, \zeta_2)h(t, \zeta) \qquad (\zeta_1, \zeta_2, t, \zeta)\in M_{A_1}\times M_{A_2}\ ,$$

where $g\in A_1$ and $h\in A_2$. It is easily seen that $M_{A_1\otimes A_2}=M_{A_1}\times M_{A_2}$.

Set $J=\{(\zeta_1, \zeta_2)\in T^2\colon \operatorname{Re}\zeta_1\leqq 0\}$ and

$$X=\{(\zeta_1, \zeta_2, t, \zeta)\in M_{A_1\otimes A_2}: (\zeta_1, \zeta_2)\in J \text{ or } t=\zeta\}\ .$$

X is a compact subset of $M_{A_1\otimes A_2}$. Define A by the uniform closure on X of $\{f|_X\colon f\in A_1\otimes A_2\}$. Clearly A is a function algebra on X. Furthermore we can show that $M_A=X$ and that

$$Q=\{(0, 0, t, t)\in M_A : 1/2\leqq t<1\}$$

is a part for A. For the details, see [2].

We are now in a position to define a weighted composition operator uC_φ on A satisfying (i) and (ii). Set

$$u(\zeta_1, \zeta_2, t, \zeta)=\zeta\ , \qquad \varphi(\zeta_1, \zeta_2, t, \zeta)=\left(0, 0, \frac{t+1}{3}, \frac{t+1}{3}\right)$$
$$(\zeta_1, \zeta_2, t, \zeta)\in X\ .$$

Clearly, $u\in A$, and φ is a continuous map from X into $X=M_A$. Then u and φ determine a weighted composition operator uC_φ as follows;

$$(5)\qquad \begin{aligned} uC_\varphi f(\zeta_1, \zeta_2, t, \zeta)&=u(\zeta_1, \zeta_2, t, \zeta)f(\varphi(\zeta_1, \zeta_2, t, \zeta))\\ &=\zeta f\left(0, 0, \frac{t+1}{3}, \frac{t+1}{3}\right)\\ &\qquad (\zeta_1, \zeta_2, t, \zeta)\in X\ , \quad f\in A\ . \end{aligned}$$

Note that

$$\varphi(X)=\{(0, 0, t, t)\in X=M_A : 1/2\leqq t\leqq 2/3\}\subset Q\ .$$

If we take $V=\operatorname{supp} u$ and $P=Q$, it follows that

$$C\subset V\ , \qquad \varphi(V)\subset P\ ,$$

for each connected component C of $\operatorname{supp} u$. This implies (i).

Finally we shall show (ii). By the Ascoli-Arzelà theorem, it suffices to show that $uC_\varphi A_0$ is not equicontinuous at some point of X, where A_0 is the unit ball of A. Fix $(\eta_1, \eta_2, s_0, s_0)\in X$. For any $s\in I$ $(s\neq s_0)$, we can construct $F_s\in C([1/2, 2])$ such that

$$\|F_s\|=\frac{1}{4}\ , \qquad F_s(s_0)=0\ , \qquad F_s(s)=\frac{1}{4}\ ,$$

and set

$$f_s(\zeta_1, \zeta_2, t, \zeta)=\frac{\zeta F_s(3t-1)}{t(3t-1)} \qquad (\zeta_1, \zeta_2, t, \zeta)\in X .$$

Then we have $f_s\in A$ and $\|f_s\|\leqq 1$, i.e., $f_s\in A_0$. Moreover, by (5),

$$uC_\varphi f_s(\zeta_1, \zeta_2, t, \zeta)=\frac{\zeta F_s(t)}{t} \qquad (\zeta_1, \zeta_2, t, \zeta)\in X ,$$

so

$$uC_\varphi f_s(\eta_1, \eta_2, s_0, s_0)=F_s(s_0)=0 ,$$
$$uC_\varphi f_s(\eta_1, \eta_2, s, s)=F_s(s)=\frac{1}{4} .$$

By taking (η_1, η_2, s, s) near to $(\eta_1, \eta_2, s_0, s_0)$, we see that $uC_\varphi A_0$ is not equicontinuous at $(\eta_1, \eta_2, s_0, s_0)$.

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