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# The Involutions of Compact Symmetric Spaces

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Dedicated to Professor Ichiro Satake on his sixtieth birthday

**Introduction.**

The main purpose of this paper is to determine the fixed point sets of the involutions of the compact symmetric spaces. We will explain a few interesting applications to illustrate the significance of the results and the use of our geometric method.

The symmetric spaces $M$ are defined with the point symmetries $s_o$ at the points $o$ in $M$. B. Y. Chen and the author made the local study of the fixed point sets $F(s_o, M)$, [CN-2]; the local structure of each connected component, $M^+$, of $F(s_o, M)$ and its "orthogonal complement", $M^-$, for the selected space (the adjoint space) in every local isomorphism class of the compact symmetric spaces. In this paper we will complete the global study first. We like to point out its theoretical interest. Given a symmetric space $M$, we have the set, $PM$, of the pairs $(M^+, M^-)$ of two symmetric subspaces which is well defined with appropriate identification. Now two (compact and connected) symmetric spaces $M$, $N$ are isomorphic if and only if $PM$ is isomorphic with $PN$ in the obvious sense. And a homomorphism of $M$ into $N$ gives rise to a homomorphism of $PM$ into $PN$. Here a homomorphism of a symmetric space into another means a smooth map which commutes with every point symmetry, (1.2). A local version is found in [CN-2]. Since a homomorphism from a connected space is exactly a totally geodesic mapping, the fact above gives a necessary condition for existence of totally geodesic embeddings, for instance.

Our geometric method as opposed to heavier exploitation of root systems takes the knowledge of $PM$ as the basic information. Indeed $PM$ is closely related to $M$ itself in terms of geometric structure. For example, $M$ is orientable if and only if every $M^+$ has an even dimension; furthermore the Euler number $\chi M$ of $M$ is the sum of the Euler numbers

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of all $M^+$ and $\{o\}$ if $\chi M$ is not zero; see (2.8) through (2.10). We have found the method is powerful in solving certain problems.

After describing $PM$ for every individual $M$, we will proceed to determine all the involutions $t$ of $M$ (See (5.5)) and the fixed point set $F(t, M)$. The description of $F(t, M)$, $t=s_o$ or not, is the main part of this paper. Again the obtained information of $F(t, M)$ makes it quite easy to determine not only the Lefschetz number of $t$ but also the signature of $M$ thanks to a theorem of Atiyah-Singer (See (2.11) and (10.1)).

Next we turn to the second order structure or the curvature, which is virtually the root system, $R(M)$. $R(M)$ completely describes the Jacobi fields along geodesics. On one hand, its study yields a simple rule to find the orthogonal spaces $M^-$ to $M^+$; the rule is similar to the main theorem of [BSi], but the reader might be amused to find the root system monomorphisms: $D_n \to C_n$ and $C_4 \to F_4$, discovered by van der Waerden and denied by Borel-Siebenthal for the Lie algebras, do appear in the category of symmetric spaces. Also interesting is the study of the common features of the spaces which share the same root systems. On the other hand, through study of Jacobi fields gives information on homotopy groups via Morse theory (8.1), e.g.); even the well known Bott periodicity follows from a geometric periodicity (See the diagram in (5.34)).

Problem of understanding the geometric structure of the symmetric space would be fundamental along with determining the morphisms between the symmetric spaces. In this connection, we will briefly mention Chow's work [C] in § 11. Chow defined the arithmetic distance on the (compact) classical Kaehlerian symmetric spaces and showed that an isometry with respect to this distance is necessarily an automorphism of the symmetric spaces of rank $>1$. The arithmetic distance can be defined for more general symmetric spaces and a similar theorem obtains in a milder form (i.e. under the assumption of differentiability). And we find it extremely interesting that the partition of the space by the arithmetic distance from a point $o$ gives a stratification of the space by vector bundles over the connected components of $F(s_o, M)$, namely $\{o\}$ and all the $M^+$'s. Intriguing enough, this stratification is, in a way, dual to that of using the cut locus found by Sakai [Sa] and Takeuchi [T-2]. A few more applications will be explained, including what we call the 2-number $\#_2 M$, of which a more extensive paper will be written in cooperation with B. Y. Chen.

Partly because of the nature of the results, the proof we will present will be sketchy at best, but we hope we will have a chance to give more detailed proofs in the near future. We assume a good knowledge of

Helgason's book [H] for the convenience's sake and follow [B] for the numbering of roots etc. We will use facts from [CN-2].

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## § 1. The category of the compact symmetric spaces.

(1.1) DEFINITIONS. By a *space* we mean a compact symmetric space $M$; thus $M$ is a compact manifold such that every point $x$ of $M$ is an isolated fixed point of an involutive transformation $s_x$ of $M$, called the symmetry at $x$ and that $M$ admits a Riemannian metric $g$ for which all symmetries are isometric.

(1.2) DEFINITIONS. A smooth map $f: M' \to M$ is a *morphism* if $f$ commutes with the symmetries: $f \circ s_{x'} = s_{f(x')} \circ f$ for every point $x'$ of $M'$. In case $M'$ is connected, a smooth map $f$ is a morphism if and only if $f$ is totally geodesic. Hom$((M', N'), (M, N))$ denotes the set of all the morphisms $f: M' \to M$ which carry the subspace $N'$ into $N$. Similarly for the automorphism group Aut$(M, N)$. Inv$(M, N)$ denotes the subset of the involutions in Aut$(M, N)$.

(1.3) DEFINITIONS. If $M$ is a compact Lie group, then G-Aut$(M)$ will denote the group automorphisms, as opposed to the automorphisms Aut$(M)$, of the space $M$ with $s_x(y) = xy^{-1}x$. Similarly for G-Hom$(M', M)$ and so forth.

(1.4) DEFINITIONS. $G = G_M = G(M)$ denotes the transformation group generated by the symmetries of $M$. If $M$ is connected, $G$ is a compact, normal and transitive subgroup of Aut$(M)$ and the isometry group of $(M, g)$ is an intermediate group between $G$ and Aut$(M)$. Generally, $M$ will denote a connected space and $K$ the isotropy subgroup $G \cap \mathrm{Aut}(M, o)$ at a point $o$; $M = G/K$.

(1.5) PROPOSITION. *Every morphism $f: M' \to M$ lifts to a group homomorphism $G(f): G^f_{M'} \to G_M$ with respect to which $f$ is equivariant if $M$ is connected, where $G^f_{M'}$ is the fibre product $G_{M'} \times_{G(f(M'))} G^i_{f(M')}$ for the inclusion $i: f(M') \to M$ and $G^i_{f(M')}$ is the subgroup of $G$ stabilizing $f(M')$. [Note that $G^f_{M'}$ is a covering group of $G_{M'}$. The homomorphism: $G_{M'} \to G_{f(M')}$ carries $s_{x'}$ into $s_{f(x')}$.]*

(1.6) NOTATION. For a subset $B$ of Aut$(M)$, the set (a subspace) of the points fixed by all the members of $B$ is denoted by $F(B, M)$. Some obvious variants of this notation may also be used.

(1.7) DEFINITIONS. For a point $o$ of $M$, the *quadratic transformation* (of E. Cartan) is the map $Q=Q_{(M,o)}: M\to G$ defined by $Q(x)=s_x s_o$. The points $p\neq o$ in $Q^{-1}(1)$ are called the *poles* of $o$ in $M$.

(1.8) PROPOSITION. (i) *$Q$ is an immersion of manifold;* (ii) *$Q$ lifts to a homomorphism* $\mathrm{Aut}(Q)\in$ G-Hom$(\mathrm{Aut}(M), \mathrm{Aut}(G))$ *defined by* $(\mathrm{Aut}(Q)(b))(x)=bx\sigma_o(b)^{-1}$ *for* $b\in\mathrm{Aut}(M)$, $x\in G$ *and* $\sigma_o:=\mathrm{ad}(s_o)$; (iii) *$Q$ is* $\mathrm{Aut}(Q)$*-equivariant;* (iv) *$Q$ is a morphism if* $G\subset\mathrm{Aut}(M)$; (v) *$Q: M\to Q(M)$ is a covering map if $M$ is connected;* and (vi) *$Q(M)$ is isomorphic with $G/F(\sigma_o, G)$.*

(1.9) PROPOSITION (cf. [CN-3]). *The following conditions are equivalent to each other for two distinct points $o$, $p$ in a connected space $M$.* (1) *$p$ is a pole of $o$ in $M$;* (2) $s_p=s_o$; (3) *There is a double covering morphism* $\pi: M\to M''$ *with* $\pi(p)=\pi(o)$; (4) *The singleton $\{p\}$ is open in $F(s_o, M)$;* (5) *$p$ lies in the orbit $F(\sigma_o, G)(o)$;* and (6) *The isotropy subgroup of $G$ at $p$ coincides with $K$, the one at $o$.*

(1.10) PROPOSITION. *If $M$ is a connected space, then* (i) *the kernel of* $\mathrm{Aut}(Q)$ *in* (1.8) *is* $C(G, \mathrm{Aut}(M))$, *the centralizer of $G$ in* $\mathrm{Aut}(M)$; (ii) *the same kernel is the covering transformation group for the covering morphism $Q: M\to Q(M)$;* and (iii) $\mathrm{Aut}(Q)$ *induces an isomorphism of* $\mathrm{Aut}(M, o)$ *onto* G-Aut$(G, K)$.

## §2. The morphisms of O(1).

(2.1) A *finite trivial space* (§12) is a finite set with every $s_x=1$, the identity map. Example. The orthogonal group $O(1)\cong \boldsymbol{Z}_2$ or any elementary abelian 2-group, $(\boldsymbol{Z}_2)^k$.

(2.2) PROPOSITION. *Let $M=G/K$ be a connected space or, in* (i), *a group space. Then* (i) $\mathrm{Hom}((O(1), 1), (M, o))$ *is a space; i.e. this is $K$-equivariantly bijective with $F(s_o, M)$;* and (ii) *every morphism:* $(O(1), 1)\to(M, o)$ *extends to a morphism:* $(U(1), 1)\to(M, o)$.

We will determine $F(s_o, M)$ to complete the local study in [CN-2]. Results, Tables I, II and III in [CN-2] may be summarized as a theorem, (2.5) below, which will be proven in (6.8).

(2.3) DEFINITION. Each component $\neq\{o\}$ of $F(s_o, M)$ is called a *polar* of $o$ in $M$. The *orthogonal* to a subspace $N$ at $p\in N$ is a subspace $N^{\perp}\ni p$

whose tangent space $T_pN^\perp$ is the orthogonal complement of $T_pN$ in $T_pM$. If $N$ is a component $F(t, M)_{(p)}$ through $p$ of $F(t, M)$, $t \in \mathrm{Inv}(M)$, then $F(ts_p, M)_{(p)}$ is clearly the orthogonal to $N$ at $p$.

(2.4) DEFINITION. $M^*$ will denote the space such that $M$ is a covering space of $M^*$ and $M^*$ is that of none non-trivially, called the *bottom* (or *adjoint*) *space*.

(2.5) THEOREM. *Assume $M$ is irreducible. Then* (i) *the orthogonal, $M^-$, to a polar $\neq$ pole in $M$ has the root system $R(M^-)$* (§ 6) *obtained from $R(M)$ in the following way* (cf. [BSi]). *Express the highest root $\tilde{\alpha}$ as a linear combination $\sum n^j\alpha_j$ of the simple roots of $R(M)$. The Dynkin diagram of $R(M^-)$ is obtained either from the extended Dynkin diagram of $R(M)$ by deleting a vertex $\alpha_j$ with $n^j=2$ or from the Dynkin diagram by deleting $\alpha_j$ with $n^j=1$. In the second case $M^-$ is locally the product of the circle $T:=\mathrm{U}(1)$ and a space with that root system.* (ii) *If $M=M^*$, any diagram obtained as above corresponds to some $M^-$. And* (iii) *the multiplicity of the root is preserved in the process.*

(2.6) REMARK. More precisely, the initial tangent to $T$ in the second case is in the direction of the $j$-th fundamental weight $\varpi_j$. Hereafter we will include $T$ in the Dynkin diagram of $M^-$ in that case. We add that the orthogonal $M^-$ to a polar $\neq$ pole is a maximal connected subspace of $M$ except for $M^-=G_p(2p)$ in the Grassmann manifold $G_p(2p+m)$, $m\geqq 1$; see (3.1) for this notation.

(2.7) REMARK. The local structure of a polar is immediately obtained from that of its orthogonal ([CN-2]). In the Introduction we mentioned Theorem 5.1 in [CN-2], in which the isometry may be replaced with the isomorphism unless EI or EI* is involved. Below we give examples to show some relationship between $M$ and its polars, among which (2.9) is an easiest application of the Lefschetz fixed point formula of Atiyah-Singer [AS]. A harder application (2.11) will be proven elsewhere as well as (2.10).

(2.8) PROPOSITION. *A geodesic* $\in \mathrm{Hom}((\boldsymbol{R}, 0), (M, o))$ *is closed if and only if it meets a polar of $o$.*

(2.9) PROPOSITION. *The Lefschetz number* $\mathrm{Lef}(t) = \chi F(t, M)$, *the Euler number, for every* $t \in \mathrm{Inv}(M)$.

(2.9A) COROLLARY. $\chi F(s_o, M)=2^r$ *for a group space of rank $r$.*

(2.9B) REMARK. If $M$ is hermitian of semisimple type and $t$ is holomorphic, then every component of $F(t, M)$ has a positive Euler number; thus (2.9) is particularly powerful in determining the components.

(2.10) PROPOSITION. *$M$ is orientable if and only if every polar has an even dimension.*

(2.11) PROPOSITION. *If $M$ is oriented and its dimension is a multiple of 4 and if $t \in \mathrm{Inv}(M)$ is homotopic to $1_M$, then the signature $\tau(M) = \tau F(t, M)$; where $\tau F(t, M)$ is the sum of the signature of each component of $F(t, M)$ with appropriate orientation. (Ignore the non-orientable components.)*

Consider a double covering morphism $\pi: M \to M''$. All the polars $\neq$ pole of $o$ project to polars of $o'' = \pi(o)$ in $M''$ and poles to those of $o''$. The other polars of $o''$ are contained in the projection of a subspace $C = C_{(o,p)}$, which consists of the midpoints of the geodesic segments from $o$ to a pole $p$. $C_{(o,p)}$ is a subspace of $M$. Thus $F(s_{o''}, M'')$ is the union $\pi(C) \cup \pi(F(s_o, M))$ which is usually a disjoint union with exceptions like $M = \mathrm{AI}(2n)$.

(2.12) DEFINITION and PROPOSITION. We call the above subspace $C_{(o,p)}$ the *centrosome* [CN-3] for the pair $(o, p)$ in $M$. (i) *One has $s_x s_o = s_o s_x$ if and only if $s_o(x) = x$ or $x \in C_{(o,p)}$ for some pole $p$ of $o$*; (ii) $C_{(o,p)} = F(\gamma s_o, M)$ *where $\gamma$ is the covering transformation for the projection $Q: M \to Q(M)$ (and for the above $\pi$) with $\gamma(o) = p$ and* (iii) $\gamma = Q(x) \circ \mathrm{ad}\, Q(x)$ *for any point $x$ in $C_{(o,p)}$*. Here and everywhere else $Q(x)$ or any member $c$ of $G$ acts on $M = G/K$ as an induced left translation carrying a point $bK$ of $M$, $b \in G$, into $cbK$, while $\mathrm{ad}(c)$ on $M$ carries $bK$ into $cbc^{-1}K$ in case $c$ normalizes $K$; these two coincide if $c$ lies in $K$.

## § 3. Determination of the polars. The classical case.

In this section and the next we will determine the polars of every irreducible $M$ and a few others, thereby determining the space $\mathrm{Hom}((O(1), 1), (M, o))$. We find it convenient to introduce a bunch of symbols first.

(3.1) NOTATIONS. $M^\sim$ denotes the universal covering space of $M$. $I_p = I_{p,n-p}$ is the linear involutions of $V = \boldsymbol{R}^n$ or $\boldsymbol{C}^n$ with a fixed basis $(e_i)_{1 \leq i \leq n}$ defined by $I_p(e_i) = -e_i$ for $i \leqq p$ and $= e_i$ for $i > p$. When $n = 2n'$ is even, $J = J_{n'}$ is a linear endomorphism of $V$ defined by $J^2 = -1$ and $J(e_i) = e_{i+n'}$, $i \leqq n'$. $K = K_{n'}$ denotes $I_{n'} J_{n'}$. $P_q$ denotes $I_q \oplus I_q$ on $\boldsymbol{C}^{n'} \oplus \boldsymbol{C}^{n'}$, conjugate with $I_{2q}$. Similarly for $P_J = J \oplus J$. $G_p(V)$ denotes the Grassmann manifold of the $p$-dimensional vector subspaces of $V$. $G_p^0(\boldsymbol{R}^n)$ is that of the oriented subspaces; $G_p^0(\boldsymbol{R}^n)\cong G_p(\boldsymbol{R}^n)^\sim$. $G_p(\boldsymbol{H}^n)$ denotes the quaternion Grassmann manifold. $G_p(n)$ denotes $G_p(V)$ or $G_p(\boldsymbol{H}^n)$ less specifically. If $n=2n'$ is a multiple of 4, then $\pi_1(\mathrm{SO}(n)^*)$ is $\{1, \delta\}\times\{1, \varepsilon\}\cong \boldsymbol{Z}_2\times\boldsymbol{Z}_2$, where $\{1, \delta\}$ is the kernel of the epimorphism: $\mathrm{SO}(n)^\sim\to\mathrm{SO}(n)$. The other group $\mathrm{SO}(n)^\sim/\{1, \varepsilon\}$, the half-spin group, is denoted by $\mathrm{SO}(n)^\sharp$. Similarly for $G_{n'}(\boldsymbol{R}^n)^\sharp$; see (3.15). $\kappa$ denotes the $\boldsymbol{C}$-conjugation of $\boldsymbol{C}^n$ satisfying $\kappa(e_i)=e_i$, $1\leqq i\leqq n$, and also the involution induced on $U(n)$, $\kappa(b)=\bar{b}$ for $b$ in $U(n)$. Finally, less universal notations. $2\times M$ denotes the disjoint union of $M$ with its copy, leaving ambiguous the action of the symmetries on the other components; $\mathrm{OIII}(n):=\mathrm{O}(2n)/U(n)$ is $2\times\mathrm{DIII}(n)$. $M_{(p)}$ denotes the connected component of $M$ through the point $p$. Given free actions of a cyclic group $\boldsymbol{Z}_\mu$ on two spaces $M$ and $N$, the dot product $M\cdot N$ denotes the orbit space $(M\times N)/\boldsymbol{Z}_\mu$. Examples. $\mathrm{U}(n)=T\cdot\mathrm{SU}(n)$ and $\mathrm{SO}(4)=\mathrm{Sp}(1)\cdot\mathrm{Sp}(1)$.

We now enumerate our results for the classical spaces; most proofs are omitted because it takes linear algebra only as illustrated in the proof of (3.3). Our actual proofs, not to be presented here, combined other methods such as use of basic facts in [CN-2] and comparison theorems like (2.9) and (3.16), for quicker results and their cross-checking.

(3.2) The polars of a torus $T^r$ are poles.

(3.3) The polars of $\mathrm{U}(n)/\boldsymbol{Z}_\mu$, $\mu$ positive integer, are $G_p(\boldsymbol{C}^n)$, $0<p\leqq n$, except that $2\times G_p(\boldsymbol{C}^n)^*$ replaces $G_p(\boldsymbol{C}^n)$ in case $2p=n$ and $\mu$ is even.

PROOF. The polars of 1 in a group space are the conjugate classes of the involutive members. A point $x$ of $\mathrm{U}(n)$ projects to an involutive member, $[x]$, of $M=\mathrm{U}(n)/\boldsymbol{Z}_\mu$ if and only if $x^2=\phi 1_n$ for some $\mu$-th root $\phi$ of 1. This is equivalent to $(\theta_1^{-1}x)^2=1_n$ where $\theta_1^2=\phi$. That is, $\theta_1^{-1}x$ is conjugate with some $I_p$; thus $\theta_1^{-1}x\in\mathrm{ad}(\mathrm{U}(n))(I_p)\cong G_p(\boldsymbol{C}^n)$. Another subspace $\theta_2 G_q(\boldsymbol{C}^n)$ projects to the same polar in $M$ if and only if $(\theta_1^{-1}\phi_1\theta_2)I_q$ is conjugate with $I_p$, $\phi_1^\mu=1$. A moment of observing the eigenvalues reveals $\theta_1^{-1}\phi_1\theta_2=1$ or $-1$. Accordingly $\theta_2^\mu=\theta_1^\mu$ (and $p=q$) or $\theta_2^\mu=(-\theta_1)^\mu$ (and $p+q=n$). From this we deduce the following, in which $\theta$ is a fixed primitive $2\mu$-th root of 1 and $[N]$ denotes the projection of a subspace $N$ of $\mathrm{U}(n)$ into $M$. (i) $\theta G_p(\boldsymbol{C}^n)$ and $G_p(\boldsymbol{C}^n)$ project to polars in $M$, $0<p\leqq n$ and vice versa; (ii) if $\mu$ is odd, then $[\theta G_{n-p}(\boldsymbol{C}^n)]=[G_p(\boldsymbol{C}^n)]\neq[\theta G_p(\boldsymbol{C}^n)]=[G_{n-p}(\boldsymbol{C}^n)]$; and (iii) if $\mu$ is even, then $[G_{n-p}(\boldsymbol{C}^n)]=[G_p(\boldsymbol{C}^n)]\neq[\theta G_{n-p}(\boldsymbol{C}^n)]=[\theta G_p(\boldsymbol{C}^n)]$. Comparison of the traces shows that $G_p(\boldsymbol{C}^n)$ and $\theta G_p(\boldsymbol{C}^n)$ project bijectively unless $2p=n$ and $\mu$ is even; in this case the polars are $G_p(\boldsymbol{C}^n)^*$.

(3.4) $\mathrm{SU}(n)/\boldsymbol{Z}_\mu$, $\mu$ a divisor of $n$, has the polars $G_g(\boldsymbol{C}^n)$, $0<g=\text{even}\leqq n$, if $\mu$ is odd; it has $G_p(\boldsymbol{C}^n)^*$, $0<2p\leqq n$, if $\mu$ is even and $\nu:=n/\mu$ is odd; and it has $G_g(\boldsymbol{C}^n)^*$, $0<g=\text{even}\leqq n$, together with $2\times G_g(\boldsymbol{C}^n)^*$ in case $2g=n$ if both $\mu$ and $\nu$ are even.

(3.5) $\mathrm{O}(n)$ has the polars $G_p(\boldsymbol{R}^n)$, $0<p\leqq n$.

(3.6) $\mathrm{SO}(n)$ has the polars $G_g(\boldsymbol{R}^n)$, $0<g=\text{even}\leqq n$.

(3.7) $\mathrm{SO}(n)^{\sim}=\mathrm{Spin}(n)$, $n>2$, has the polars $G^0_{2g}(\boldsymbol{R}^n)$, $0\leqq g=\text{even}\leqq n/2$, with the understanding that this is a singleton for $g=0$ (i.e. $G^0_0(\boldsymbol{R}^n)$ less $o$) and two points for $g=n/2$.

(3.8) $\mathrm{SO}(2r)^\sharp$, $r$ even, has the polars $\mathrm{OIII}(r)^*$ and $G^0_{2g}(\boldsymbol{R}^{2r})$, $0\leqq g=\text{even}\leqq r/2$, with the same understanding as above for $g=0$ and $G^0_{2g}(\boldsymbol{R}^{2r})^\sharp$ for $g=r/2$.

(3.9) $\mathrm{SO}(2r)^*$ has the polars $G_g(\boldsymbol{R}^{2r})^*$, $0<g=\text{even}\leqq r$, and $\mathrm{DIII}(r)$. Replace $\mathrm{DIII}(r)$ with $\mathrm{OIII}(r)^*$ if $r$ is even.

(3.10) $\mathrm{Sp}(n)$ has the polars $G_p(\boldsymbol{H}^n)$, $0<p\leqq n$.

(3.11) $\mathrm{Sp}(n)^*$ has the polars $G_p(\boldsymbol{H}^n)^*$, $0<2p\leqq n$, and $\mathrm{CI}(n)^*$.

For the Grassmann manifolds $G_p(n)$ we will give a more general proposition, which follows from the fact that $I_h$ fixes a linear subspace $x\in G_p(n)$ if and only if $x=x\cap\ker(I_h-1)\oplus x\cap\ker(-I_h-1)$.

(3.12) PROPOSITION. *$F(I_h, G_p(n))=\amalg_{a+b=p}G_a(h)\times G_b(n-h)$. $G_a(h)\times G_b(n-h)$ is orthogonal to $G_a(n-h+a-b)\times G_b(h+b-a)$. The polars of $G_p(n)$ are $G_a(p)\times G_b(n-p)$, $a+b=p$ and $0\leqq a<p$.*

(3.13) $G_r(2r)^*$ has the polars $G_a(r)\times G_b(r)$, $a+b=r$ with $0<a<r/2$, and, if $r=2r'$ is even, $G_{r'}(r)\cdot G_{r'}(r)$. Add the polar $\mathrm{U}(r)/\boldsymbol{Z}_2$ for $G_r(\boldsymbol{C}^{2r})^*$, $\mathrm{Sp}(r)^*$ for $G_r(\boldsymbol{H}^{2r})^*$, $\mathrm{SO}(r)^*$ for $G_r(\boldsymbol{R}^{2r})^*$ with $r$ odd and $\mathrm{O}(r)^*$ for $G_r(\boldsymbol{R}^{2r})^*$ with $r$ even, to have all the polars.

(3.14) $G^0_p(\boldsymbol{R}^n)$ has the polars $G^0_p(\boldsymbol{R}^{n-p})$, a pole, and $G^0_a(\boldsymbol{R}^p)\cdot G^0_b(\boldsymbol{R}^{n-p})$, $b=p-a$ even and $0<a<p$.

(3.15) $G_r(\boldsymbol{R}^{2r})^\sharp$, $r$ even, has the polars $G^0_g(\boldsymbol{R}^r)\cdot G^0_{r-g}(\boldsymbol{R}^r)$, $0<g=\text{even}<r$, $\mathrm{O}(r)^*$ and if $r'=r/2$ is even, $G_{r'}(\boldsymbol{R}^r)^\sharp\times G_{r'}(\boldsymbol{R}^r)^\sharp$.

PROOF. Embed $G^0_r(\boldsymbol{R}^{2r})$ into the exterior algebra $\wedge^*\boldsymbol{R}^{2r}$. Observe that the covering transformation for the projection onto $G_r(\boldsymbol{R}^{2r})^\sharp$ is the restriction of the Hodge $*$-operator.

The next theorem will help reach the results more easily and gain a deeper insight into the whole situation. Its validity will become obvious later.

(3.16) THEOREM. *Every connected irreducible space $M$ contains a subspace $M'$ with the root system $R(M')\cong R(M)$ of minimal multiplicity ($=1$ if $R(M)\not\cong \mathrm{BC}_r$, and $M'=G_r(\boldsymbol{C}^{2r+1})$ if $R(M)\cong \mathrm{BC}_r$). Also the inclusion induces $\pi_1(M)\cong\pi_1(M')$.*

(3.17) The polars of $\mathrm{AI}(n)=F(\kappa s_1, \mathrm{SU}(n))$ are $G_g(\boldsymbol{R}^n)$, $0<g=\text{even}\leqq n$. Those of $\mathrm{UI}(n)=F(\kappa s_1, \mathrm{U}(n))$ are $G_p(\boldsymbol{R}^n)$, $0<p\leqq n$.

(3.18) REMARK. Notice $R(\mathrm{AI}(n))\cong R(\mathrm{SU}(n))$, the first example of (3.16). The polars of $\mathrm{AI}(n)$ are $F(\kappa s_1$, the polars of $\mathrm{SU}(n))=G_g(\boldsymbol{R}^n)$ by (3.4). The polars of $\mathrm{AI}(n)/\boldsymbol{Z}_\mu$ are easily found from (3.4) and quite similar to (3.4). These and the similar ones in the sequel are omitted. One can countercheck one's results by means of the results in § 5 and others.

(3.19) The polars of $\mathrm{AII}(n)=F(\kappa\,\mathrm{ad}(J)s_1, \mathrm{SU}(2n))_{(1)}$ are $G_g(\boldsymbol{H}^n)$, $0<g=\text{even}\leqq n$. Those of $\mathrm{UII}(n):=F(\kappa\,\mathrm{ad}(J)s_1, \mathrm{U}(2n))_{(1)}$ are $G_p(\boldsymbol{H}^n)$, $0<p\leqq n$.

(3.20) The polars of $\mathrm{DIII}(n)=F(\mathrm{ad}(J)s_1, \mathrm{SO}(2n))_{(1)}$ are $G_g(\boldsymbol{C}^n)$, $0<g=\text{even}\leqq n$, while those of $\mathrm{OIII}(n)=F(s_1\circ\mathrm{ad}\,J, \mathrm{SO}(2n))$ are $G_p(\boldsymbol{C}^n)$, $0<p\leqq n$.

(3.21) The polars of $\mathrm{CI}(n)$ are $G_d(\boldsymbol{C}^n)$, $0<d\leqq n$.

## § 4. Determination of the polars. The exceptional case.

(4.1) NOTATIONS. The standard symbols $\mathrm{E}_6, \cdots, \mathrm{G}_2, \mathrm{EI}, \cdots, \mathrm{GI}$ will denote the 1-connected spaces. The results are easy from [CN-2], since $\pi_1(M)$ does not cause complexities except for $R(M)\cong\mathrm{E}_7$. Given a point $o$ of $M$, $M^+(p)$ denotes the polar of $o$ through the point $p$ in $M$ (i.e., $M^+(p)=F(s_o, M)_{(p)}$) and $M^-(p)$ its orthogonal at $p$.

(4.2) The polars of $\mathrm{E}_8$ are $\mathrm{EVIII}=F(\sigma^{\mathrm{VIII}}s_1, \mathrm{E}_8)\cong\mathrm{E}_8/F(\sigma^{\mathrm{VIII}}, \mathrm{E}_8)$ and $\mathrm{EIX}=F(\sigma^{\mathrm{IX}}s_1, \mathrm{E}_8)$. (See (5.1) for $\sigma^{\mathrm{VIII}}$ and $\sigma^{\mathrm{IX}}$.)

(4.3) The polars of EVIII are $G_8(\boldsymbol{R}^{16})^\#$ and $\mathrm{DIII}(8)^*$.

PROOF. $M:=\mathrm{EVIII}$ has two polars $M^+(p)$ and $M^+(q)$ which are locally isomorphic with $G_8(\boldsymbol{R}^{16})$ and DIII(8) respectively ([CN-2]). The Euler numbers $\chi M^+(p)=\chi G_8(\boldsymbol{R}^{16})^\#$ and $\chi M^+(q)=\chi\mathrm{DIII}(8)^*$ (hence $M^+(q)\cong\mathrm{DIII}(8)^*$) by (2.9). ([T-1] carries a table of the Euler numbers, which is correct except $\chi$EVI$=63$. They and the Lefschetz numbers will be given in § 5.) On the other hand, $M^-(p)\cong G_8(\boldsymbol{R}^{16})^\#$ by (3.16) since $R(M)\cong \mathrm{E}_8$ and so $M^-(p)\subset \mathrm{SO}(16)^\#$. $M^-(p)=F(Q(p), M)_{(o)}$, where ad$(Q(p))$ is conjugate with $\sigma^{\mathrm{VIII}}$ (not $\sigma^{\mathrm{IX}}$) since Aut$(M^-(p))$ is locally SO(16). Therefore $Q(p)$ is conjugate with $s_o$ and we conclude $M^+(p)\cong G_8(\boldsymbol{R}^{16})^\#$.

(4.4) The polars of EIX are EVI and $S^2\cdot$EVII.

(4.5) The polars of $\mathrm{E}_7$ are $2\times$EVI and a pole. Those of $\mathrm{E}_7^*$ are EVI, EV* and EVII*.

(4.6) The polars of EV are $2\times G_4(\boldsymbol{C}^8)^*$ and a pole. Those of EV* are $G_4(\boldsymbol{C}^8)^*$, AI(8)/$\boldsymbol{Z}_4$ and AII(4)*.

(4.7) The polars of EVI are $S^2\cdot$DIII(6) and $G_4^0(\boldsymbol{R}^{12})$.

(4.8) The polars of EVII are $2\times$EIII and a pole, and those of EVII* are EIII and $(T\cdot\mathrm{EIV})/\boldsymbol{Z}_2$.

(4.8A) We explain the space $M^+(p):=(T\cdot\mathrm{EIV})/\boldsymbol{Z}_2$ above, $M=$EVII*, although there is no ambiguity. The dot product $T\cdot$EIV is given by the $\boldsymbol{Z}_3$, see (3.1), which is generated by $(\omega, \omega')$ where $\omega\in T=\mathrm{U}(1)$ is a cubic root of 1 and $\omega'$ generates the covering transformation group for the 3-fold covering morphism: EIV$\to$EIV*. Another explanation. By $R(\mathrm{EIV})\cong R(\mathrm{SU}(3))$, there is an embedding: $\mathrm{U}(3)\to\mathrm{UII}(3)\to T\cdot\mathrm{EIV}$, (5.21), which induces $\pi_1(\mathrm{U}(3))\cong\pi_1(T\cdot\mathrm{EIV})$, cf. (3.16). The above embedding induces $\mathrm{U}(3)/\boldsymbol{Z}_2\to(T\cdot\mathrm{EIV})/\boldsymbol{Z}_2$.

(4.9) $\mathrm{E}_6$ and $\mathrm{E}_6^*$ have both EII and EIII as polars. Those of EI and EI* are both CI(4)* and $G_2(\boldsymbol{H}^4)^*$. Those of EII are $G_2(\boldsymbol{C}^6)$ and $S^2\cdot G_3(\boldsymbol{C}^6)$. EIII has $G_2^0(\boldsymbol{R}^{10})$ and DIII(5). EIV and EIV* have FII. $\mathrm{F}_4$ has FI and FII. FI has $S^2\cdot$CI(3) and $G_1(\boldsymbol{H}^3)$. FII has $S^8$. $\mathrm{G}_2$ has GI, and GI has $S^2\cdot S^2$.

## § 5. The involutions of a space.

We will determine Inv$(M, o)$, its conjugate classes in Aut$(M)$ and $F(t, M)$ for every $t\in$ Inv$(M, o)$. (The result would naturally contain the local classification of the affine symmetric spaces. See S. Kaneyuki [K].)

(5.1) NOTATIONS. Every $t\in$ Aut$(M, o)$ gives rise to $\tau=$ ad$(t)\in$ G-Aut$(G, K)$, and vice versa. Generally we use Roman and corresponding Greek letters this way. Also $\sigma_x:=$ ad$(s_x)$, $\sigma^{\mathrm{IX}}:=$ ad$(s^{\mathrm{IX}})$, where $s^{\mathrm{IX}}=s_o$ for EIX$\cong\mathrm{E}_8/F(\sigma^{\mathrm{IX}}, \mathrm{E}_8)$, etc. $N''$ denotes the space which admits the (unique) double covering space $N$ in (5.22), (5.24), etc.

(5.2) PROPOSITION. *Let* $t \in \mathrm{Inv}(M, o)$ *on a connected* $M$. *Then* (i) *every component of* $F(t, M)$ *meets the given component* $M^{-t} := F(ts_o, M)_{(o)}$; (ii) *every component of the intersection* $F(t, M) \cap F(ts_o, M)$ *is a component of* $F(t, M^+)$ *for some polar* $M^+$ *of* $o$ *in* $M$, *and vice versa.*

PROOF. (i) Take a shortest geodesic $\gamma$ from an arbitrary point of $F(t, M)$ to $F(t, M)_{(o)}$. Let $b \in F(\tau, G)_{(1)}$ carry the endpoint of $\gamma$ into $o$. Then $b\gamma$ is contained in $M^{-t}$ and its initial point lies in the intersection in question. (ii) follows from $ts_o = s_o t$ and $F(t, M) \cap F(ts_o, M) = F(\{t, s_o\}, M) = F(t, F(s_o, M))$, etc.

(5.3) COROLLARY. *If* $\sigma$ *and* $\tau \in \text{G-Inv}(G)$ *belong to a component of* $\text{G-Aut}(G)$, *then* $\sigma$ *commutes with some conjugate of* $\tau$.

(5.4) REMARK. Without the homotopy assumption in (5.3), $\sigma$ commutes with a conjugate of $\tau$ in $H := \text{G-Aut}(G)$ if and only if $F(\mathrm{ad}(\sigma), H)$ meets the polar $H^+(\tau)$.

(5.5) THEOREM. $\mathrm{Inv}(M, o)$ *is bijective with* $F(s_1, F(\mathrm{ad}(\sigma_o), \text{G-Aut}(G))) = F(\mathrm{ad}(\sigma_o), \text{G-Inv}(G))$, *the conjugacy classes in* $\mathrm{Aut}(M)$ *corresponding to* $\{1\}$ *and the polars in* $F(\mathrm{ad}(\sigma_o), \text{G-Aut}(G))$ except for 1-connected $M$ with $R(M) = \mathrm{D}_{2m}$.

PROOF. $\mathrm{Aut}(Q)$ restricts to an isomorphism $\mathrm{Aut}(M, o) \cong \text{G-Aut}(G, K)$, (1.10). Therefore $\mathrm{Inv}(M, o)$ is bijective with $\text{G-Inv}(G, K)$. We will show that every $\tau \in F(\mathrm{ad}(\sigma_o), \text{G-Inv}(G))$ stabilizes $K$. Since $\tau$ stabilizes the Lie algebra $\mathscr{L}(K)$ and hence $F(\sigma_o, G)$ and $Q(M)$ in $G$, (5.2) applies to yield $\tau(K) = K$. As to the conjugacy of the involutions with respect to $\mathrm{Aut}(M)$, one has only to note that $\mathrm{Aut}(Q)(t) = \mathrm{ad}(t)$ for $t \in \mathrm{Inv}(M, o)$ in view of Lemma 2.4 in [CN-2] (whether or not $K$ is connected).

(5.5A) REMARK. (5.5) settles the problem of $\mathrm{Inv}(M, o)$, since $\text{G-Aut}(G)$ is known. But the next (5.6) may be more expedient if an involutive covering transformation is not in $G$.

(5.6) PROPOSITION. *Let* $b \in \mathrm{Inv}(M)$. *Assume* $bo$ *is a pole of* $o$. *Then* (i) $b$ *admits the decomposition* $b = \gamma\delta = \delta\gamma$ *where* $\gamma$ *is a covering morphism and* $\delta$ *is induced by* $\mathrm{ad}(b)$ *with* $\gamma^2 = \delta^2 = 1$; (ii) $\gamma = c\,\mathrm{ad}(c)$ *for some* $c$ *in* $G_{(1)}$, (2.12); *and* (iii) $Gb \cap \mathrm{Inv}(M, o) = F(s_1\,\mathrm{ad}(cb), K)cb$.

(5.7) REMARK. Finding the components of $F(t, M)$ is facilitated by (5.2) and (5.8) or (5.9) below. Also, by (2.9B) in the hermitian case.

(5.2) gives more information on the intersections than the first glance might tell; for instance, $2\dim F(t, F(s_o, M))_{(p)} = \dim M^+(p) - \dim M + \dim F(t, M)_{(p)} + \dim F(ts_o, M)_{(p)}$. Also the components $\neq\{o\}$ of $M^{-t}\cap F(t, M)$ are exactly the polars of $o$ in $M^{-t}$. All these together allow one to make induction arguments on $\dim M$.

(5.8) PROPOSITION. *Let $\sigma$ and $\tau \in$ G-Inv$(G)$ with $\sigma\tau=\tau\sigma$, $G$ a compact Lie group. Let $\sigma$ and $\tau$ act (as $s$ and $t$) on both $M_\sigma := G/G^\sigma$ and $M_\tau := G/G^\tau$, $G^\sigma := F(\sigma, G)$, etc. Then* (i) *$t$ fixes a point $cG^\sigma$ in $M_\sigma$, $c\in G$, if and only if $s$ fixes the point $c^{-1}G^\tau$ in $M_\tau$;* (ii) *this gives a bijection of the set $\{G^\tau$-orbits contained in $F(t, M_\sigma)\}$ onto $\{G^\sigma$-orbits$\subset F(s, M_\tau)\}$;* (iii) *the isotropy subgroup of $G^\tau$ at the point $cG^\sigma \in F(t, M_\sigma)$ is isomorphic with that of $G^\sigma$ at $c^{-1}G^\tau \in F(s, M_\tau)$. (Caution: $G^\sigma$ and $G^\tau$ are not necessarily effective on those orbits.)*

(5.9) COROLLARY. *If $G^\sigma$ and $G^\tau$ in* (5.8) *are connected, then $F(t, M_\sigma)$ and $F(s, M_\tau)$ have an equal number of components.*

(5.10) REMARK. The relationship between $M_\sigma$ and $M_\tau$ as in (5.8) and (11.1), reminiscent of the Radon transform, is called the *Radon duality*, albeit vague admittedly. This is not trivial even if $\sigma$ is conjugate with $\tau$; a case in point is the classical duality between the points and the hypersurfaces in a projective space, which is generalized in (11.1).

Now we enumerate the components of $F(t, M)$ for $t$ chosen from each conjugacy class in Inv$(M, o)$ and $M$ from each local class of connected irreducible spaces $\neq$ groups. If $M$ is a connected simple group space, then Inv$(M, 1)$ is bijective with $\{1, s_1\}\times$G-Inv$(M)$. We omit $t=1$ and $s_o$ but include their Lefschetz numbers.

(5.11) Let $M=\mathrm{AI}(n)\subset\mathrm{SU}(n)$ (See (3.17) for the inclusion. Similarly for the sequel). $F(I_p, M)=T\cdot(\mathrm{AI}(p)\times\mathrm{AI}(n-p))$, $0\leqq 2p<n$, with $\mathrm{Lef}(I_p)=0=\chi M$ and $\mathrm{Lef}(s_o)=2^{[n/2]}$. If $n=2n'$ is even, add $F(J, M)=2\times\mathrm{SU}(n')$, $F(s_oJ, M)=\mathrm{CI}(n')$ and $F(s_o\,\mathrm{ad}(I_1), M)=\amalg_{k=\mathrm{odd}} G_k(\boldsymbol{R}^n)$ with $\mathrm{Lef}(J)=\mathrm{Lef}(s_o\mathrm{ad}(I_1))=0$ and $\mathrm{Lef}(s_oJ)=2^{n'}$.

(5.12) If $M=\mathrm{AII}(n)\subset\mathrm{SU}(2n)$, then $F(\kappa, M)=\mathrm{DIII}(n)$, $F(J, M)=\mathrm{SU}(n)$, $F(P_p, M)=T\cdot(\mathrm{AII}(p)\times\mathrm{AII}(n-p))$, $0\leqq 2p\leqq n$, and $F(s_oP_1, M)=\amalg_{k=\mathrm{odd}} G_k(\boldsymbol{H}^n)$ with $\mathrm{Lef}(\kappa)=2^{n-1}=\mathrm{Lef}(s_o)=\mathrm{Lef}(s_oP_1)$ and $\mathrm{Lef}(J)=0=\mathrm{Lef}(P_p)=\chi M$.

(5.13) Let $M=G_p(\boldsymbol{C}^n)\subset\mathrm{SU}(n)$. $F(\kappa, M)=G_p(\boldsymbol{R}^n)$ and $F(I_h, M)$ in (3.12). Add $F(s^{\mathrm{II}}, M)=G_{p'}(\boldsymbol{H}^{n'})$ if $n=2n'$ and $p=2p'$ are even, where $s^{\mathrm{II}}=\kappa(J_{p'}\oplus J_{n'-p'})$. Add $F(\kappa\,\mathrm{ad}(J), M)=\mathrm{CI}(n')$, $F(s_o\kappa\,\mathrm{ad}(J), M)=\mathrm{OIII}(n')$ and

$F(\mathrm{ad}(J), M)=\mathrm{U}(n')$ if $p=n'=n/2$. $\mathrm{Lef}(I_h)=\chi M=\binom{n}{p}$, $\mathrm{Lef}(\kappa)=\mathrm{Lef}(s^{\mathrm{II}})=\binom{[n/2]}{[p/2]}$, $\mathrm{Lef}(\kappa\,\mathrm{ad}(J))=\mathrm{Lef}(s_o\kappa\,\mathrm{ad}(J))=2^n$ and $\mathrm{Lef}(\mathrm{ad}(J))=0$.

(5.14) Let $M=G_p(\boldsymbol{R}^n)$. See (3.12) for $F(I_h, M)$. Add $F(J_{p'}\oplus J_{n'-p'}, M)=G_{p'}(\boldsymbol{C}^{n'})$ with $\mathrm{Lef}=\chi M$ if $n=2n'$ and $p=2p'$ are even. Add $F(\mathrm{ad}(J), M)=\mathrm{UI}(p)$ and $F(s_o\,\mathrm{ad}(J), M)=\mathrm{O}(p)$ if $n=2p$. $\mathrm{Lef}(I_h)=\mathrm{Lef}(I_1)=2\binom{[n/2]-1}{[p/2]}$ if $n$, $n-h$ and $p$ are all odd and $=\chi M$ otherwise. $\chi M=0$ if $p$ and $n-p$ are odd and $\binom{[n/2]}{[p/2]}$ otherwise. $\mathrm{Lef}(\mathrm{ad}(J))=\mathrm{Lef}(s_o\,\mathrm{ad}(J))=0$.

(5.15) Let $M=G_p(\boldsymbol{H}^n)$. $F(J, M)=G_p(\boldsymbol{C}^n)$ and $F(P_h, M)=\amalg_{a+b=p}G_a(\boldsymbol{H}^h)\times G_b(\boldsymbol{H}^{n-h})$. Add $F(\mathrm{ad}(P_J), M)=\mathrm{UII}(p)$ and $F(s_o\,\mathrm{ad}(J), M)=\mathrm{Sp}(p)$ if $2p=n$. $\mathrm{Lef}(J)=\mathrm{Lef}(P_h, M)=\chi M=\binom{n}{p}$. $\mathrm{Lef}(\mathrm{ad}(J))=\mathrm{Lef}(s_o\,\mathrm{ad}(J))=0$.

(5.16) Let $M=\mathrm{CI}(n)$. $F(P_p, M)=\mathrm{CI}(p)\times\mathrm{CI}(n-p)$, $0<2p\leqq n$, and $F(\mathrm{ad}(I_n), M)=\mathrm{UI}(n)$. Add $F(\mathrm{ad}(P_JK), M)=\mathrm{Sp}(n')$ if $n=2n'$ is even. $\mathrm{Lef}(P_p)=2^n=\chi M$. $\mathrm{Lef}(\mathrm{ad}(I_n))=0=\mathrm{Lef}(\mathrm{ad}(P_JK))$.

(5.17) Let $M=\mathrm{DIII}(n)$. $F(P_h, M)=2\times(\mathrm{DIII}(h)\times\mathrm{DIII}(n-h))$, $0\leqq 2h\leqq n$, and $F(\mathrm{ad}(I_n), M)=\mathrm{SO}(n)$. Add $F(JP_1, M)=\amalg_{k=\mathrm{odd}}G_k(\boldsymbol{C}^n)$ and $F(\mathrm{ad}(P_JI_n), M)=\mathrm{UII}(n')$ if $n=2n'$ is even, $\mathrm{Lef}(P_h)=\mathrm{Lef}(s_o)=\chi M=2^{n-1}=\mathrm{Lef}(JP_1)$. $\mathrm{Lef}(\mathrm{ad}(I_n))=0=\mathrm{Lef}(P_JI_n)$.

(5.18) Let $M=\mathrm{EI}$. $F(s^{\mathrm{II}}, M)=S^2\cdot\mathrm{AI}(6)\amalg\mathrm{AII}(3)$, $F(s^{\mathrm{III}}, M)=T\cdot G_3(\boldsymbol{R}^{10})$ and $F(s^{\mathrm{IV}}, M)=\mathrm{FI}$. $\mathrm{Lef}(s_o)=\mathrm{Lef}(s^{\mathrm{IV}})=12$ and $\mathrm{Lef}(s^{\mathrm{II}})=\mathrm{Lef}(s^{\mathrm{III}})=\chi M=0$.

(5.19) Let $M=\mathrm{EII}$. $F(s^{\mathrm{I}}, M)=\mathrm{CI}(4)^*\amalg G_1(\boldsymbol{H}^4)$, $F(s^{\mathrm{III}}, M)=G_4^0(\boldsymbol{R}^{10})\amalg\mathrm{DIII}(5)$ and $F(s^{\mathrm{IV}}, M)=\mathrm{FI}$. $\mathrm{Lef}(s^{\mathrm{I}})=\mathrm{Lef}(s^{\mathrm{IV}})=12$ and $\mathrm{Lef}(s^{\mathrm{II}})=\mathrm{Lef}(s^{\mathrm{III}})=\chi M=36$.

(5.20) Let $M=\mathrm{EIII}$. $F(s^{\mathrm{I}}, M)=G_2(\boldsymbol{H}^4)^*$, $F(s^{\mathrm{II}}, M)=S^2\times G_1(\boldsymbol{C}^6)\amalg G_2(\boldsymbol{C}^6)$ and $F(s^{\mathrm{IV}}, M)=\mathrm{FII}$. $\mathrm{Lef}(s^{\mathrm{I}})=3=\mathrm{Lef}(s^{\mathrm{IV}})$ and $\mathrm{Lef}(s^{\mathrm{II}})=\mathrm{Lef}(s^{\mathrm{III}})=\chi M=27$.

(5.21) Let $M=\mathrm{EIV}$. $F(s^{\mathrm{I}}, M)=G_1(\boldsymbol{H}^4)$, $F(s^{\mathrm{II}}, M)=\mathrm{AII}(3)$ and $F(s^{\mathrm{III}}, M)=T\cdot S^9$. $\mathrm{Lef}(s^{\mathrm{I}})=4=\mathrm{Lef}(s^{\mathrm{IV}})$ and $\mathrm{Lef}(s^{\mathrm{II}})=\mathrm{Lef}(s^{\mathrm{III}})=0=\chi M$.

(5.22) Let $M=\mathrm{EV}$. $F(s^{\mathrm{VI}}, M)=S^2\cdot G_6^0(\boldsymbol{R}^{12})\amalg\mathrm{DIII}(6)$, the dot product explained in (5.22A). $F(s^{\mathrm{VII}}, M)=2\times\mathrm{EII}$, $F(t^{\mathrm{V}}, M)=\mathrm{AI}(8)''\amalg\mathrm{AII}(4)''$ and $F(t^{\mathrm{VII}}, M)=T\cdot\mathrm{EI}$. $\mathrm{Lef}(s^{\mathrm{VI}})=\mathrm{Lef}(s^{\mathrm{VII}})=\mathrm{Lef}(s_o)=\chi M=72$ and $\mathrm{Lef}(t^{\mathrm{VII}})=\mathrm{Lef}(t^{\mathrm{V}})=0$.

(5.22A) REMARK. In $\mathrm{E}_7$, the two copies of EVI among the polars, (4.5), have common orthogonals $\cong \mathrm{Sp}(1)\cdot\mathrm{SO}(12)^{\sim}$, where the dot product is given by $(-1, \varepsilon)$. The projection: $\mathrm{E}_7 \to \mathrm{E}_7^*$ restricts to an epimorphism: $\mathrm{Sp}(1)\cdot\mathrm{SO}(12)^{\sim} \to \mathrm{Sp}(1)\cdot\mathrm{SO}(12)^{\#}$, whose kernel is $\{1, (1, \varepsilon\delta)\}$. The corresponding dot products in EV and EV* are thus defined in view of (3.16) and $G_6^0(\boldsymbol{R}^{12}) \subset \mathrm{SO}(12)^{\sim}$.

(5.23) Let $M=\mathrm{EVI}$. $F(s^{\mathrm{V}}, M)=G_4(\boldsymbol{C}^8)^* \amalg G_2(\boldsymbol{C}^8)$ and $F(s^{\mathrm{VII}}, M)=\mathrm{EII} \amalg \mathrm{EIII}$ with $\mathrm{Lef}(s_o)=\mathrm{Lef}(s^{\mathrm{V}})=\mathrm{Lef}(s^{\mathrm{VII}})=\chi M=63$.

(5.24) Let $M=\mathrm{EVII}$. $F(s^{\mathrm{V}}, M)=2\times G_2(\boldsymbol{C}^8)$, $F(s^{\mathrm{VI}}, M)=S^2\times G_2^0(\boldsymbol{R}^{12}) \amalg \mathrm{DIII}(6)$, $F(t^{\mathrm{V}}, M)=\mathrm{AII}(4)''$ and $F(t^{\mathrm{VII}}, M)=T\cdot\mathrm{EIV}$. $\mathrm{Lef}(s^{\mathrm{V}})=\mathrm{Lef}(s^{\mathrm{VI}})=\chi M=56=\mathrm{Lef}(s_o, M)$. $\mathrm{Lef}(t^{\mathrm{V}})=0=\mathrm{Lef}(t^{\mathrm{VII}})$.

(5.25) Let $M=\mathrm{EVIII}$. $F(s^{\mathrm{IX}}, M)=\mathrm{EVI} \amalg S^2\cdot\mathrm{EV}$ with $\mathrm{Lef}(s^{\mathrm{IX}})=\mathrm{Lef}(s^{\mathrm{VIII}}, M)=\chi M=135$.

(5.26) Let $M=\mathrm{EIX}$. $F(s^{\mathrm{VIII}}, M)=G_4^0(\boldsymbol{R}^{16}) \amalg \mathrm{DIII}(8)^*$ with $\mathrm{Lef}(s^{\mathrm{VIII}})=\chi M=120$.

(5.27) $F(s^{\mathrm{II}}, \mathrm{FI})=G_4^0(\boldsymbol{R}^9)$ with $\mathrm{Lef}(s^{\mathrm{II}})=\mathrm{Lef}(s^{\mathrm{I}})=12$.

(5.28) $F(s^{\mathrm{I}}, \mathrm{FII})=G_1(\boldsymbol{H}^3)$ with $\mathrm{Lef}(s^{\mathrm{I}})=\chi=3$.

(5.29) GI admits no involution which is not conjugate to 1 or $s_o$. $\mathrm{Lef}(s_o)=\chi=3$.

(5.30) Remark. The above results yield (3.16) and the next (5.31) through (5.33).

(5.31) Theorem. *Two involutions $s$ and $t$ of a connected space $M$, $F(s, M)\neq\varnothing\neq F(t, M)$, are congruent if and only if a component of $F(s, M)$ is isomorphic with some one of $F(t, M)$.*

(5.32) If the isotropy subgroup $K$ is almost effective on a polar $M^+$ in $M=G/K$, then the restriction gives a monomorphism: $\mathrm{Inv}(M, o)/\{1, s_o\} \to \mathrm{Inv}(M^+)$.

(5.33) Proposition. *The monomorphisms: $\mathrm{C}_4 \to \mathrm{F}_4$ and $\mathrm{D}_n \to \mathrm{C}_n$ of root systems are realized by $\mathrm{CI}(4)^* \subset \mathrm{EII}$, $G_4(\boldsymbol{C}^8) \subset \mathrm{EVI}$, $\mathrm{DIII}(8)^* \subset \mathrm{EIX}$ and by $\mathrm{SO}(2n) \subset \mathrm{DIII}(2r)$, $G_n(\boldsymbol{R}^{2n}) \subset G_n(\boldsymbol{C}^{2n})$ and $\mathrm{AII}(4)'' \subset \mathrm{EVII}$, and essentially by these only. Also, the multiplicity of the shorter roots in $R(M')$ is half the one of $R(M)$ in $M' \subset M$ with $R(M') \cong \mathrm{C}_4$ and $R(M) \subsetneqq \mathrm{F}_4$.*

(5.34) Remark. We point out intriguing periodicity in classical spaces consecutively embedded as components of the fixed point sets of involutions. We abbreviate, say, $\mathrm{U}(n)\to G_n(\boldsymbol{C}^{2n})\to \mathrm{U}(2n)$ to $\mathrm{U}\leftrightarrow G_n(\boldsymbol{C}^{2n})$ in the diagram below. This and the counterclockwise outer cycle below $\mathrm{O}\to G_n(\boldsymbol{R}^{2n})\to\cdots\to \mathrm{OIII}\to \mathrm{O}$ induce isomorphisms $\to\pi_i(M')\to\pi_{i+1}(M)\to$ in the stable range, yielding the Bott periodicity. (See the book "Topology of Lie Groups" (in Japanese) by Toda and Mimura or [M].)

$$\begin{tikzcd}
\mathrm{UI} \arrow[r, leftrightarrow] \arrow[d, leftrightarrow] & G_n(\boldsymbol{R}^{2n}) \arrow[r, leftrightarrow] \arrow[d, leftrightarrow] & \mathrm{O} \arrow[d, leftrightarrow] \\
\mathrm{CI} \arrow[r, leftrightarrow] \arrow[d, leftrightarrow] & G_n(\boldsymbol{C}^{2n}) \arrow[r, leftrightarrow] \arrow[d, leftrightarrow] & \mathrm{OIII} \arrow[d, leftrightarrow] \\
\mathrm{Sp} \arrow[r, leftrightarrow] & G_n(\boldsymbol{H}^{2n}) \arrow[r, leftrightarrow] & \mathrm{UII}
\end{tikzcd}$$

(5.35) REMARK. The list of $F(t, M)$ contains all the maximal subspaces of $M$ if $M=G_2^0(\boldsymbol{R}^n)$, [CN-1]. The general case, however, appears formidable to solve, in view of the colossal work of Dynkin [D]. See (7.6).

## § 6. The root system.

(6.1) NOTATION. Given $t\in\mathrm{Inv}(M, o)$, we write $M^t$ for $F(t, M)_{(o)}$ and $M^{-t}$ for $F(ts_o, M)_{(o)}$. Similarly $G^t:=F(\tau, G)_{(1)}$ and $G^{-t}:=F(\tau\sigma_o, G)_{(1)}$. Let $A=A^{-t}$ be a maximal torus in $M^{-t}$, $o\in A$, with the usual group structure which makes $Q\colon A\to G$ a group homomorphism. By its action the Lie algebra $\mathfrak{g}=\mathscr{L}(G)$ is decomposed into the weight spaces: $\mathfrak{g}=\sum_{\rho\in P}\mathfrak{g}_\rho$; $\mathfrak{g}_\rho$ is thus the kernel of $(\mathrm{ad}\,H)^2+\rho(H)^2 1$ for every $H\in\mathfrak{a}=\mathscr{L}(A)$. We use a finer decomposition $\mathfrak{g}=\mathfrak{g}_0+\sum_{\alpha\in R(-t)}\mathfrak{g}_\alpha+\sum_{\lambda\in R(t)}\mathfrak{g}_\lambda$ where $\{0\}\amalg(R(-t)\cup R(t))=P$, $\mathfrak{g}_\alpha\subset\mathfrak{g}^{-t}=\mathscr{L}(G^{-t})$ and $\mathfrak{g}_\lambda\subset\mathfrak{g}^t$. We add $-\alpha$ to $R(-t)$ for every $\alpha\in R(-t)$, to make $R(-t)$ a root system, called that of $M^{-t}$. Thus the root system $R(M)$ of $M$ is $R(-t)$ for $t=s_o$. $\gamma=\gamma_H$ will denote the geodesic $\in\mathrm{Hom}((\boldsymbol{R}, 0), (M, o))$ with $\gamma'(0)=H$, $H\in\mathfrak{a}$.

(6.2) PROPOSITION (Variational Completeness). *Let $J=J(H, M^t)$ denote the vector space of the Jacobi fields $v$ along $\gamma_H$ satisfying $v(0)\in T_o(M^t)$ and $v'(0)\in T_o(M^{-t})$. Take a point $\gamma(u)$, $u\neq 0$, on the geodesic. Then the subspace $\{v\in J\mid v(\gamma(u))=0\}$ is identical with $\{v|\gamma$ or $tv|\gamma$ ; $v\in\mathfrak{g}$ and $v(\gamma(u))=0\}$, where the members of $\mathfrak{g}$ are regarded as vector fields on $M$.*

A point $\gamma(u)$, $u\neq 0$, on the geodesic $\gamma$ is *conjugate* to $M^t$ along $\gamma$ by definition if $v(\gamma(u))=0$ for some $v\neq 0$ in $J$. The next proposition generalizes theorems of T. Sakai [Sa] and M. Takeuchi [T-2] in which $M^t=\{o\}$.

(6.3) PROPOSITION. *The set of all the conjugate points along the* *geodesics which are perpendicular to $M^t$ is the union of the singular $G^t$-orbits in $M$.*

(6.4) PROPOSITION. *Let $p=\gamma(1)\in M^{-t}$. Assume $s_o(p)=p$. Then* (i) *$\rho(2H)\in\pi\boldsymbol{Z}$ for every $\rho\in P$; and* (ii) *if $\gamma|[0, 1]$ is one of the shortest from $M^t$ to $p$, then $|\lambda(2H)|=0$ or $\pi$ for $\lambda$ in $R(t)$ and $|\alpha(2H)|=0$, $\pi$ or $2\pi$ for $\alpha\in R(-t)$.*

PROOF. (i) follows from $\gamma(2)=o$, and (ii) from absence of conjugate points on $\gamma|(0, 1)$.

This proposition has applications such as (2.5) and the next, which we obtain for $t=s_o$ since $A$ is then a maximal torus in $M$. In (6.5) we use the expression $\tilde{\alpha}=\sum n^j\alpha_j$ of the highest root in terms of the simple roots $\alpha_j$.

(6.5) PROPOSITION. *The shortest geodesics from $o$ to an arbitrary polar $M^+$ are $K$-congruent with the geodesic $\gamma_H$, $\gamma_H(1)\in M^+$, such that either $2H=H^j$ for some $n^j\leqq 2$ or $2H=H^i+H^k$ corresponding to $n^i=n^k=1$, where $H^j$ is defined by $\alpha_i(H^j)=\pi\delta_i^j$ or, equivalently, $H^j=(2\pi/\|\alpha_j\|^2)\varpi_j$.*

(6.6) EXAMPLES. The polar $G_{2j}(\boldsymbol{C}^n)$ in $\mathrm{SU}(n)$ is reached by the shortest $\gamma_H$, $2H=H^j+H^{n-j}$, and its projection into $\mathrm{SU}(n)^*$ by $\gamma_H$, $H=H^{2j}$. If $M=\mathrm{E}_6$, $2H=H^1+H^6$ gives the shortest to the polar EIII, but, in $M^*$, $2H=H^1$ does to its projection. If $M=\mathrm{SO}(2r)$, $2H=H^{r-1}+H^r$ gives the shortest to the polar $G_2(\boldsymbol{R}^{2r})$, but $H^r$ does in $M^*$. If $M=\mathrm{E}_7$, the shortest to the polars $\cong$EVI are given by $2H=H^1$ and $H^6$. Since $\|H^6\|^2/\|H^1\|^2=3/2$, one is closer to $o$ than the other EVI. $H^1$ gives the shortest in $M^*$. (Another proof of (4.5) is given this way.) If $M=M^*$ in general, the second case $2H=H^i+H^k$ in (6.5) does not occur, (6.8).

(6.7) PROPOSITION. *Assume $M=M^*$ and $s_o(p)=p$. Choose $A$, a maximal torus in $M$, satisfying $\{0, p\}\subset A\subset M^-(p)$.* Then *$R(M^-(p))$ is the set of the roots $\alpha\in R(M)$ such that the integer $\alpha(2H)/\pi$ is even for every $H\in\mathfrak{a}$ satisfying $\gamma_H(1)=p$.*

PROOF. Apply (6.4) for $t=s_p$ (hence $M^{-t}=M^-(p)$). Observe that $\mathrm{ad}(e^H)$, where $\gamma_H(1)=p$, exchanges $F(\mathrm{ad}(s_p), \mathfrak{g})$ and $F(\mathrm{ad}(s_o), \mathfrak{g})=\mathfrak{k}$, stabilizing $F(\mathrm{ad}(Q(p)), \mathfrak{g})$. Now (6.7) follows since $\{H\in\mathfrak{a}\,|\,\gamma_H(1)=p\}$ spans $\mathfrak{a}$.

(6.8) PROOF OF (2.5). We may work on $M^*$. We use (6.7), from which (iii) is obvious. The second case $2H=H^i+H^k$ in (6.4) cannot occur because $\|H^i+H^k\|>\|H^i-H^k\|$. Thus we have (i). Conversely, let $R^-$

be a root system $\subset R(M)$ in (2.5). Then $R^{-}=\{\alpha \in R(M) \mid \alpha(H^{j})/\pi$ is even$\}$ for some $H^{j}$. $p=\gamma_{H}(1)$ with $H=H^{j}$ belongs to the polar.

## § 7. Curvature and the Helgason sphere.

Let $M$ be a Riemannian manifold in general. The sectional curvature $SK$, restricted to the 2-planes in the tangent space $T_{o}M$ at a point $o$, is critical at $H \wedge X \in G_{2}(T_{o}M)$ if and only if the curvature operator $K(H \wedge X)$ stabilizes the plane, as is easily seen. Back to a compact symmetric space, the above condition is equivalent to saying that ad$[H, X]$ stabilizes $H \wedge X$ where $T_{o}M$ is identified with $\mathfrak{m}:=F(-\sigma_{o}, \mathfrak{g})$; in other words, $H$ and $X$ generate $\mathscr{L}(T^{2})$ or $\mathscr{L}(\mathrm{O}(3))$.

(7.1) PROPOSITION. *The plane $H \wedge X \in G_{2}(T_{o}M)$ is a critical point of the sectional curvature $SK$ restricted to $G_{2}(T_{o}M)$ if and only if $H \wedge X$ is tangent to a subspace of constant curvature $SK(H \wedge X)$.*

(7.2) PROPOSITION. *Let $H \in \mathfrak{a} \subset \mathfrak{m}$ and $X=\sum_{\alpha \in R(M)} X_{\alpha}$, $X_{\alpha} \in \mathfrak{m} \cap \mathfrak{g}_{\alpha}$. Then $H \wedge X \neq 0$ is tangent to a space of constant curvature $>0$ if and only if $[X_{\alpha}, Y_{\beta}]=0$ for any distinct roots $\alpha, \beta \in R^{\prime}:=\{\alpha \in R(M) \mid X_{\alpha} \neq 0, \alpha>0\}$ and there is some positive number $c$ such that $\sum_{\beta \in R^{\prime}}\|X_{\beta}\|^{2}\langle\alpha, \beta\rangle=c^{2}$ for every $\alpha \in R^{\prime}$, where $[H, X_{\alpha}]=\alpha(H) Y_{\alpha} \in \mathfrak{k}$. ($c^{2}=SK(H \wedge X)$ if $H$ and $X$ are orthonormal).*

(7.3) COROLLARY. *If a subset $R^{\prime} \subset R(M)$ is strongly orthogonal, then $H=\sum_{\alpha \in R^{\prime}}\|\alpha\|^{-1} \alpha$ and $X=\sum_{\alpha \in R^{\prime}} X_{\alpha}$, where $\|X_{\alpha}\|=\|\alpha\|^{-1}$ and $X_{\alpha} \in \mathfrak{m} \cap \mathfrak{g}_{\alpha}$, are tangent to a space of constant curvature $>0$.*

(7.4) COROLLARY. *$\boldsymbol{R}\alpha+\mathfrak{m} \cap \mathfrak{g}_{\alpha}$ is the tangent space of a certain subspace, $S(\alpha)$, of constant curvature if $\alpha \in R(M)$ and $2\alpha \notin R(M)$. $S(\alpha)$ is 1-connected unless the rank $r(M)=1$ or $R(M) \cong \mathrm{B}_{r}$ and $\alpha$ is a shorter root.*

(7.5) REMARK AND DEFINITION. If $2\alpha \in R(M)$ and $\alpha \in R(M)$, then $\boldsymbol{R}\alpha+\mathfrak{m} \cap(\mathfrak{g}_{\alpha}+\mathfrak{g}_{2\alpha})$ is tangent to a subspace of rank 1.

S. Helgason ([H] § VII-11) studied $S(\tilde{\alpha})$ thoroughly, $\tilde{\alpha}$ being the highest root. We call it and all the $G$-congruents $bS(\tilde{\alpha})$, $b \in G$, the *Helgason spheres* in $M$ if $M$ is irreducible. $S(\tilde{\alpha})$ is a maximal subspace of rank 1 only when $M=$AI, CI or Sp. And $\dim S(\tilde{\alpha})>2$ if and only if the homotopy group $\pi_{2}M=0$. (In general the embedding induces an epimorphism: $\pi_{2}S(\tilde{\alpha}) \rightarrow \pi_{2}(M)$.)

(7.6) PROPOSITION. *Let $H$ be as in* (6.4) (ii) *with $t=s_{p}$ for the space $S^{2}$ [resp. $G_{1}(\boldsymbol{R}^{2})$]. Then $f^{*}\alpha_{j}(2H) \in\{0, \pi, 2\pi\}$ [resp. $\{0, \pi/2, \pi\}$] for every* *embedding $f$ of the space into another space $M$ and the suitable simple roots $\alpha_j$ of $R(M)$.*

PROOF. The proof of Prop. 5 in § VII-11 of [B] or Theorem 8.3 of [D] is sufficient with a slight modification.

(7.7) REMARK. We do not have the definition of the appropriate sphere $S(H)\subset M$, $H$ the initial tangent to a circle, other than a root vector, but it might be suggested by [W], in which J. A. Wolf determined the maximal sphere in $G_n(2n)$ which contains the shortest circles containing the origin $o$ and its pole in $G_n(2n)$.

## § 8. Homotopy groups.

In proving the Bott periodicity for $U(n)$, Milnor ([M] § 23) shows the centrosome $C_{(1,-1)}=G_r(\boldsymbol{C}^{2r})$ in $SU(2r)$ has the homotopy groups $\pi_i(G_r(\boldsymbol{C}^{2r}))\cong\pi_{i+1}(SU(2r))$ for $i\leqq 2r$. Translating his arguments into the language of roots, one gets $\pi_i(C_{(o,p)})\cong\pi_{i+1}(M)$ for $i\leqq m(r+1)-2$ and every 1-connected space $M$ with $R(M)\cong R(SU(2r))$ and multiplicity $m$.

One has similar isomorphism with $R(SU(2r))$ replaced by $C_r$ in the range $i\leqq m(r-1)+2(n-1)$, where $m$ and $n$ are the multiplicities of the shorter and the longer roots; for instance, $\pi_{i+1}(\mathrm{EVII})\cong\pi_i(T\cdot\mathrm{EIV})$, $i\leqq 16$. This is a part of J. Burns' thesis (Notre Dame, 1985). We will state a variant in case $R(M)\cong F_4$. We use the centrosome $C_{(o,p)}$ in $M^-(p)$, where $R(M^-(p))\cong B_4$.

(8.1) THEOREM. *In the above notation for $R(M)\cong F_4$, one has $\pi_{i+1}(M)\cong\pi_i(C_{(o,p)})$, $i\leqq 2m+3n-1$, where $m$, $n$ are the multiplicities of the shorter and the longer roots of* $F_4$. *That is,* (i) $\pi_i(\mathrm{EIX})\cong\pi_{i-1}(S^3\cdot S^{11})$, $i\leqq 18$; (ii) $\pi_i(\mathrm{EVI})\cong\pi_{i-1}(S^3\cdot S^7)$, $i\leqq 10$; (iii) $\pi_i(\mathrm{EII})\cong\pi_{i-1}(S^3\cdot S^5)$, $i\leqq 6$; (iv) $\pi_i(\mathrm{FI})\cong S^3\cdot S^2$, $i\leqq 4$; *and* (v) $\pi_i(F_4)\cong\pi_{i-1}(G_2^0(\boldsymbol{R}^9))$, $i\leqq 9$.

## § 9. Graded Lie algebras.

We will explain how $F(t, M)$, $t\in\mathrm{Inv}(M, o)$ is related to a simple graded Lie algebra $\mathfrak{l}=\sum_{p=-2}^{2}\mathfrak{l}^{(p)}$, $\dim\mathfrak{l}^{(2)}\leqq 1$, $\mathfrak{l}^{(1)}\neq 0$. The classification in the case $\mathfrak{l}^{(2)}=\{0\}$ was done in [KN] and [N], correcting E. Cartan's classification of the primitive transitive Lie algebras $(\mathfrak{l}, \mathfrak{p})$, $\mathfrak{p}:=\sum_{q\geqq 0}\mathfrak{l}^{(q)}$. It turns out that (i) $M=L/P$ is a compact space, $\mathscr{L}(L)=\mathfrak{l}$ and $\mathscr{L}(p)=\mathfrak{p}$; (ii) $M$ is a totally real subspace of a hermitian space $M^L$ with appropriate choice of $L$ and $P$; and (iii) the action of $L_{(1)}$ extends to the holomorphic transformation group of $M^L$; here "totally real" means that $T_oM$ is carried onto its orthogonal complement in $T_o M^L$ by the complex structure. The converse is true too.

(9.1) PROPOSITION. *Every hermitian space admits involutions $t$ such that a component of $F(t, M)$ is totally real. Actually $F(t, M)$ is connected.*

(9.2) PROPOSITION. *A space $M$ is locally isomorphic with a totally real subspace of a hermitian symmetric space if and only if $R(M)$ is classical, which means $R(M)$ is the direct sum of root system $\cong R(\mathrm{U}(n))$, $\mathrm{B}_r$, $\mathrm{D}_r$, $\mathrm{C}_r$ or $\mathrm{BC}_r$.*

A geometric meaning is hinted by the fact that if a space $M=G/K$ admits a larger Lie transformation group $L\supset G$, so $M=L/P$, then $\mathscr{L}(L)$ is a graded algebra with $P$ as above (See [N] for a precise statement).

A typical example of $L/P$ with $\dim \mathfrak{l}^{(2)}=1$ is the CR-automorphism group $L$ acting on an odd-dimensional sphere, [MN]. This case of $\dim \mathfrak{l}^{(2)}=1$ was studied by J. H. Cheng (thesis, Notre Dame, 1983; [Ch]). $L/P$ is a circle bundle over a hermitian space $M$ which is immersed into an $\boldsymbol{H}$-kaehlerian space $M^L$ as a "totally complex" subspace and $L/P$ is the fixed point set of a complex-conjugation $k$ of the pullback (to $M$) of a "canonical" $S^2$-bundle which is a (generalized) complex flag manifold such that $k$ induces an "$\boldsymbol{H}$-conjugation" $k''$ on $M^L$ and the immersed $M$ is open in $F(k'', M^L)$. Since the converse is true, the classification of $\mathfrak{l}$, $\dim \mathfrak{l}^{(2)}=1$, is reduced to that of $k''\in \mathrm{Inv}(M^L)$ such that a component of $F(k'', M^L)$ is "totally complex" in $M^L$. The irreducible $\boldsymbol{H}$-kaehlerian spaces are $G_4^0(\boldsymbol{R}^n)$, $G_2(\boldsymbol{C}^n)$, $G_1(\boldsymbol{H}^n)$, GI and every $M\neq \mathrm{F}_4$ with $R(M)\cong \mathrm{F}_4$. ($M$ is necessarily 1-connected if it is connected.)

(9.3) Every $\boldsymbol{H}$-kaehlerian space $M^L$ admits "$\boldsymbol{H}$-conjugation" $k''\in \mathrm{Inv}(M^L)$. Exactly one component, $M$, of $F(k'', M^L)$ is hermitian locally and the others are $\boldsymbol{H}$-kaehlerian.

(9.4) The topological space $\mathrm{Aut}(M^L)/\mathrm{Aut}(M)$ is characterized by $\pi_2\neq 0$.

## § 10. Signature.

We will determine the signature $\tau(M)$ as another application. We use a theorem of Atiyah-Singer [AS] to the effect that the $g$-signature of $M$ is the signature of the self-intersection $(F(g, M))^2$ of $F(g, M)$ for every orientation-preserving involution $g$ of $M$. We thank B. Y. Chen for his cooperation in an early stage.

(10.1) PROPOSITION. *If $\tau(M)\neq 0$ and $M$ is 1-connected and irreducible, then $M$ is one of the following spaces with the indicated signature:*

$$(1/2)\tau(G_{2p}^0(\boldsymbol{R}^{2n}))=\tau(G_{2p}(\boldsymbol{R}^{2n}))=\tau(G_p(\boldsymbol{C}^n))=\tau(G_p(\boldsymbol{H}^n))=\chi G_p(\boldsymbol{R}^n)\ ,$$

$\tau(\mathrm{EII})=4$, $\tau(\mathrm{EIII})=3$, $\tau(\mathrm{EVI})=\tau(\mathrm{EVIII})=7$, $\tau(\mathrm{EIX})=8$ *and* $\tau(\mathrm{FII})=\tau(\mathrm{GI})=1$.

PROOF. We explain how to find $\tau(M)$ only for $M=\mathrm{EVI}$, a more involved case, without using (2.11) but inductively assuming that $\tau(N)$ is known for every space $N$ with $\dim N<\dim M$. Since $s^{\mathrm{VII}}$ is homotopic with $1_M$ and $F(s^{\mathrm{VII}}, M)=\mathrm{EII}\amalg\mathrm{EIII}$, (5.23), we will find the self-intersection in $(\mathrm{EII})^2$ and $(\mathrm{EIII})^2$ in $M$. It is easy to see that the orthogonal EIII′ to EIII at $o$ is isomorphic with EIII. Hence $F(s^{\mathrm{VII}}\circ s_o, M)=\mathrm{EIII}'\amalg\mathrm{EII}'$, $\mathrm{EII}'\cong\mathrm{EII}$ and $s^{\mathrm{VIII}}\circ s_o\cong s^{\mathrm{VII}}$. By matching the polars, one sees $\mathrm{EIII}\cap\mathrm{EIII}'=\{o\}\amalg G_2^0(\boldsymbol{R}^{10})$. Thus $\mathrm{EII}\cap\mathrm{EII}'=G_4^0(\boldsymbol{R}^{10})$ by (5.19). We may assume $(\mathrm{EIII})^2\subset\{o\}\amalg G_4^0(\boldsymbol{R}^{10})$ and $(\mathrm{EII})^2\subset G_4^0(\boldsymbol{R}^{10})$. On the other hand, $G_2^0(\boldsymbol{R}^{10})\amalg G_4^0(\boldsymbol{R}^{10})$ is contained in the polar $G_4^0(\boldsymbol{R}^{12})$ of $o$ in $M$; see (5.2). Therefore the contributions of $(\mathrm{EII})^2$ and $(\mathrm{EIII})^2\cap G_2^0(\boldsymbol{R}^{10})$ to $\tau(M)$ are $\tau(G_4^0(\boldsymbol{R}^{10}))=4$ and $\tau(G_2^0(\boldsymbol{R}^{10}))=2$ respectively (with the correct signs). (We thus evade the question of orientation by applying that theorem of Atiyah-Singer and the induction assumption repeatedly.) Similarly we find the contribution from $(\mathrm{EIII})^2$ is $1+2=3$, completing the computation. The result for this $\boldsymbol{H}$-kaehlerian manifold naturally agrees with [NT], because the Betti number $b_{32}(\mathrm{EVI})=7$.

(10.2) PROPOSITION. *If $t$ is a complex conjugation of a hermitian space $M$, then $\tau(t, M)=\chi F(t, M)$.*

PROOF. Obvious if one notes that the normal bundle to $F(t, M)$ in $M$ is isomorphic with the tangent bundle.

## § 11. Chow's theorem and Radon duality.

W. L. Chow [C] defined the "arithmetic distance" $d$ on every classical hermitian symmetric space $M$ and proved that the $d$-preserving transformations of $M$ are holomorphic (or anti-holomorphic), provided the rank $>1$. In case $M=G_p(\boldsymbol{C}^n)$, $d(x, y)=\dim_C x/(x\cap y)$ where the operations in the right hand side are applied to vector subspaces $x$, $y$ of $\boldsymbol{C}^n$. Now S. Peterson generalized this by dropping "hermitian" (Thesis, Notre Dame, 1985; [P]). In this general case, $d(x, y)\leqq j$ by definition if two points $x$, $y$ in $M$ are joined with each other by a chain of $j$ Helgason spheres, (7.5). If $M$ is hermitian and irreducible, the holomorphic transformations permute the Helgason spheres and hence they preserve the new arithmetic distance obviously. Also it equals Chow's $d$ if $M=G_p(\boldsymbol{C}^n)$. The tools were the fundamental theorem on projective geometry and another version of the Radon duality, which we are about to explain. In the setting of (5.8), consider the collection $O_\tau$ of the $G$-congruents $bG^\tau(o)$, $b\in G$, of the orbit $G^\tau(o)\subset M_\sigma$ through $o$. The members of $O_\tau$ are objects of our interest. Let $L_\tau$ denote the group of all the transformations (=bijections) of $M_\sigma$ which permute the members of $O_\tau$. Exchanging $\sigma$ and $\tau$, we have $O_\sigma$ and $L_\sigma$. The next proposition (11.1) explains a geometric meaning of $M_\tau$ in relation to $M_\sigma$ and is easily seen, since one has $b\in G^\tau$ if $bG^\tau G^\sigma=G^\tau G^\sigma$, $b\in G$, by an argument in the proof of (5.5).

(11.1) PROPOSITION. *In the above notations,* (i) *there exists a $G$-equivariant bijection $F_\tau$: $M_\tau\to O_\tau$ such that $x\in F_\sigma(y)$ if and only if $F_\tau(x)\ni y$, where $x\in M_\tau$ and $y\in M_\sigma$. Hence* (ii) $L_\tau\cong L_\sigma$.

(11.2) REMARK. In passing we like to point out an intriguing fact. Let $M=F(t, M^L)$ be a totally real subspace of an irreducible hermitian space $M^L$, (9.1). A larger group $L$ is acting on $M=G/K=L/P$ (§ 9). The $P$-orbits $M_j$, $0\leqq j\leqq r$, give the partition of $M$ into the subsets of the points of the arithmetic distance $j$ =const. from $o$. On the other hand, the map $s_1$: $G\to G$ in the Radon duality (5.8) (ii) with $M=M_\sigma=M_\tau$ extends to $\alpha$: $L\to L$ in a certain way and the non-open $\alpha(P)$-orbits give a stratification of the cut-locus of $o$ (See [T-2]). Finally the next theorem on geometry may convey some flavor of Peterson's work as well as the proofs.

(11.3) THEOREM. *Let $M_\sigma=$*FI. *Let $L$ be the group of smooth transformations of $M_\sigma$ which permute the members of $O_\tau$, $\tau=\sigma^{\mathrm{II}}$, as defined above. Then* $L=\mathrm{Aut}(M_\sigma)$.

PROOF. We will show $L\subset\mathrm{Aut}(M)$. The members of $O_\tau$ are isomorphic with $G_4^0(\boldsymbol{R}^9)$ and those of $O_\sigma$ with $G_1(\boldsymbol{H}^3)$. Every member of $O_\tau$ is orthogonal to another at any point $o$ on it. The intersection of these two spaces is $\{o\}$ plus the common polar $\cong S^4$, which is a subspace of the polar $\cong S^8$ of $o$ in $M_\tau=$FII. It follows that this $S^8$ is stabilized by the isotropy subgroup $P$ of $L$ at $o$, $L$ acting on FII by (11.1). Hence $L$ is locally compact by the fundamental theorem of projective geometry (See [S] e.g.), basically. Therefore $L$ is a Lie group by the theorem of Bochner and Montgomery. But FI does not admit a larger group than $\mathrm{Aut}(M_\sigma)$ by (9.2); hence $L=\mathrm{Aut}(M_\sigma)$.

## § 12. The 2-numbers, $\#_2 M$.

This, $\#_2 M$, is the maximal cardinality of a finite trivial space $\Sigma$, (2.1), in the space $M$. If $M$ is a connected group, then the maximal $\Sigma$ is necessarily a maximal elementary abelian 2-group [CN-3]; thus $\#_2 M$ generalizes the 2-rank [BSe]. $\#_2 M$ was determined in [CN-3] as another application of our method. An obvious, but important fact, is this: if $M'$ is a subspace of $M$ then $\#_2 M' \leqq \#_2 M$. Thus $\#_2 M$ gives a new obstruction to embeddings.

(12.1) EXAMPLE. $G_r(C^{2r})$ cannot be embedded into $\mathrm{Sp}(r)$, $r>1$, because $\#_2 G_r(C^{2r}) = \binom{2r}{r} > 2^r = \#_2 \mathrm{Sp}(r)$. Notice that there is a monomorphism $R(G_r(C^{2r})) \subset R(\mathrm{Sp}(r))$ with multiplicity increasing.

(12.2) EXAMPLE. $\mathrm{EV}^*$ cannot be embedded into $\mathrm{E}_7$, although EV can.

(12.3) REMARK. The full significance of $\#_2 M$ is yet to know, but we do know that $\chi M \leqq \#_2 M$ and $\chi M \equiv \#_2 M$ modulo 2. If $M$ is 1-connected, $\#_2 M \geqq$ the sum of the Betti numbers. The equality can occur for both inequalities.

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