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A Characterization of the Poisson Kernel Associated with $SU(1, n)$

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§1. Introduction.

Let G be a connected non-compact semisimple Lie group with finite center and of real rank one. Let $G=KAN$ be an Iwasawa decomposition of G and M the centralizer of A in K. Then K is a maximal compact subgroup of G and $\dim A=1$; G/K is a non-compact Riemannian symmetric space of rank one and K/M the Furstenberg boundary. Let $\mathfrak{g}=\mathfrak{k}+\mathfrak{a}+\mathfrak{n}$ be the corresponding Iwasawa decomposition of the Lie algebra $\mathfrak{g}$ of G. For $g\in G$ let $H(g)$ denote the unique element in $\mathfrak{a}$ such that $g\in K\exp H(g)N$ and let $\rho(H)=(1/2)\mathrm{tr}(\mathrm{ad}(H)|_{\mathfrak{n}})$ for $H\in\mathfrak{a}$.

The Poisson kernel associated with G is the function on $G/K\times K/M$ defined by

$$P(gK, kM)=\exp(-2\rho H(g^{-1}k)) .$$

Let D be the Laplace-Beltrami operator on G/K (cf. [H1], p. 386). Then the function on G/K defined by $gK\mapsto P^s(gK, kM)$ ($kM\in K/M$ and $s\in \boldsymbol{C}$) is an eigenfunction of D with eigenvalue, say, λ_s. Now we shall consider the converse.

CONJECTURE. Let F be a real valued, C^2 function of G/K satisfying the following three conditions:

(1) $DF=0$,

(2) $D(F^2)=\lambda_2 F^2$,

(3) $F(eK)=1$.

Then there exists an element $kM\in K/M$ such that $F(gK)=P(gK, kM)$ for $gK\in G/K$.

When $G=SO_0(1, n)$, this conjecture was proved in [CET]. In this paper we shall give a proof for the case of $G=SU(1, n)$; in order to state our result more precisely, we shall give the explicit forms of the Poisson kernel and the Laplace-Beltrami operator associated with $G=SU(1, n)$.

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Let C^n be an n-dimensional Hermitian linear space over C and the points of C^n are ordered n-tuples $\zeta=(\zeta_1, \zeta_2, \cdots, \zeta_n)$, where $\zeta_k=\xi_k+i\eta_k \in C$ for $1\leqq k\leqq n$. Let $B(C^n)$ be the open unit ball in C^n and $S(C^n)$ the unit sphere in C^n. $G=SU(1, n)$ is the group of linear transformations on C^{n+1} of determinant one which preserve the form: $-|\zeta_0|^2+|\zeta_1|^2+\cdots+|\zeta_n|^2$. Then this group acts transitively on $B(C^n)$ by the formula:

$$\zeta'_p=\left(g_{p0}+\sum_{q=1}^{n} g_{pq}\zeta_q\right)\left[\left(g_{00}+\sum_{q=1}^{n} g_{0q}\zeta_q\right)\right]^{-1} \qquad (1\leqq p\leqq n) ,$$

where $g=(g_{ij})$ $(0\leqq i, j\leqq n)\in G$, $\zeta=(\zeta_p)\in B(C^n)$ and $\zeta'=(\zeta'_p)=g\cdot\zeta$ $(1\leqq p\leqq n)$; this action gives the identification between $B(C^n)$ (resp. $S(C^n)$) and G/K (resp. K/M), where K and M are given explicitly in § 3.

The Laplace-Beltrami operator D on $B(C^n)$ is invariant under the above action of G and given by

$$D=4(1-|\zeta|^2)\left[\sum_{k=1}^{n} \frac{\partial^2}{\partial\zeta_k\partial\bar{\zeta}_k}-\sum_{k,l=1}^{n} \bar{\zeta}_k\zeta_l\frac{\partial^2}{\partial\bar{\zeta}_k\partial\zeta_l}\right] ,$$

where $\partial/\partial\zeta_k=(1/2)(\partial/\partial\xi_k-i\partial/\partial\eta_k)$, $\partial/\partial\bar{\zeta}_k=(1/2)(\partial/\partial\xi_k+i\partial/\partial\eta_k)$ and $|\zeta|^2=\sum_{k=1}^{n}\zeta_k\bar{\zeta}_k$ for $\zeta=(\zeta_1, \zeta_2, \cdots, \zeta_n)$ and $\zeta_k=\xi_k+i\eta_k$ $(1\leqq k\leqq n)$. The Poisson kernel is also explicitly given by the formula:

$$P(\zeta, u)=\frac{(1-|\zeta|^2)^n}{|1-\langle\zeta, u\rangle|^{2n}} \qquad \text{for} \quad (\zeta, u)\in B(C^n)\times S(C^n) ,$$

where $\langle\zeta, u\rangle=\sum_{k=1}^{n}\zeta_k\bar{u}_k$. Then it satisfies the identity (cf. [R], p. 42):

$$P(g\cdot\zeta, u)=P(\zeta, g^{-1}\cdot u)P(g\cdot 0, u) \tag{1}$$

for $g\in G$ and $(\zeta, u)\in B(C^n)\times S(C^n)$. Moreover, for each $\mu\in C$, as a function of $\zeta\in B(C^n)$, it satisfies the equation:

$$D(P^\mu)=4n^2\mu(\mu-1)P^\mu .$$

Our result can be stated as follows.

THEOREM. *Let F be a real valued, C^2 function on $B(C^n)$ satisfying the following conditions*:

$$DF=0 , \tag{2a}$$

$$D(F^2)=8n^2F^2 . \tag{2b}$$

Then there exist a point $u\in S(C^n)$ and a real constant c such that

$$F(\zeta)=cP(\zeta, u) \qquad (\zeta\in B(C^n)) .$$

§ 2. Simplification by rotation.

If F is identically zero, it satisfies Theorem; we can take $u \in S(C^n)$ arbitrarily and $c=0$. Let us suppose that $F \not\equiv 0$, that is, $F(\zeta_0) \neq 0$ for some ζ_0 in $B(C^n)$. Here we recall that the G-action on $B(C^n)$ is transitive, D is G-invariant and P satisfies (1). Therefore, we can find a $g \in G$ such that $g \cdot \zeta_0 = 0$ and replacing $F(\zeta)$ with $F(\zeta_0)^{-1} F(g^{-1} \cdot \zeta)$, without loss of generality, we may assume that $F(0)=1$.

Computing the identity:

$$D(F^2)=2FDF+8(1-|\zeta|^2)\left[\sum_{k=1}^{n}\left|\frac{\partial F}{\partial \zeta_k}\right|^2-\left|\sum_{k=1}^{n}\zeta_k\frac{\partial F}{\partial \zeta_k}\right|^2\right],$$

we see that the hypotheses (2a) and (2b) are equivalent to

(2a) $$DF=0\,,$$

(2b′) $$(1-|\zeta|^2)\left[\sum_{k=1}^{n}\left|\frac{\partial F}{\partial \zeta_k}\right|^2-\left|\sum_{k=1}^{n}\zeta_k\frac{\partial F}{\partial \zeta_k}\right|^2\right]=n^2F^2\,.$$

Since $F(0)=1$, we have $\sum_{k=1}^{n}|(\partial F/\partial \zeta_k)(0)|^2=n^2$, and we put $\rho_{11}=(1/n)(\partial F/\partial \bar{\zeta}_1)(0)$, $\cdots$, $\rho_{n1}=(1/n)(\partial F/\partial \bar{\zeta}_n)(0)$; let ρ be a matrix in $SU(n)$ whose first column is equal to ${}^t(\rho_{11}, \rho_{12}, \cdots, \rho_{1n})$.

We define the rotation of F by $F_\rho(\zeta)=F(\rho\cdot\zeta)$, where $\zeta'=\rho\cdot\zeta$ is the vector in C^n which the matrix $\rho \in SU(n)$ transforms ζ into. Since D commutes with the action of ρ, the function F_ρ also satisfies the hypotheses (2a) and (2b). Moreover, according to the choice of $\rho \in SU(n)$, we obtain that

$$\begin{bmatrix}\dfrac{\partial (F_\rho)}{\partial \bar{\zeta}_1}(0)\\ \vdots \\ \dfrac{\partial (F_\rho)}{\partial \bar{\zeta}_n}(0)\end{bmatrix}={}^t\bar{\rho}\begin{bmatrix}\dfrac{\partial F}{\partial \bar{\zeta}_1}(0)\\ \vdots \\ \dfrac{\partial F}{\partial \bar{\zeta}_n}(0)\end{bmatrix}=\rho^{-1}\begin{bmatrix}\dfrac{\partial F}{\partial \bar{\zeta}_1}(0)\\ \vdots \\ \dfrac{\partial F}{\partial \bar{\zeta}_n}(0)\end{bmatrix}=ne_1\,,$$

where $e_1={}^t(1, 0, \cdots, 0)$. Therefore, F_ρ satisfies the following condition:

$$\frac{\partial (F_\rho)}{\partial \zeta_1}(0)=n\,; \qquad \frac{\partial (F_\rho)}{\partial \zeta_2}(0)=\cdots=\frac{\partial (F_\rho)}{\partial \zeta_n}(0)=0\,.$$

In what follows we shall denote F_ρ by F; then the desired theorem follows from the next lemma.

LEMMA 1. *Let F be a real valued, C^2 function on $B(C^n)$ which* *satisfies* $F(0)=1$ *and the following conditions*:

(2a) $$DF=0\,,$$

(2b) $$D(F^2)=8n^2F^2\,,$$

(2c) $$\frac{\partial F}{\partial\zeta_1}(0)=n\,;\qquad \frac{\partial F}{\partial\zeta_2}(0)=\cdots=\frac{\partial F}{\partial\zeta_n}(0)=0\,.$$

Then $F(\zeta)=P(\zeta, e_1)$ *for* $\zeta\in B(C^n)$.

In fact, since the Poisson kernel satisfies the functional equation (1), we see from this lemma that the initial F (before rotation) has the form: $F(\zeta)=P(\zeta, \rho\cdot e_1)$ $(\zeta\in B(C^n))$.

§ 3. Coordinate transformation.

Let K, A and N be the subgroups of $G=SU(1, n)$ defined by

$$K=\left\{k_{u,V}=\begin{bmatrix}u & 0\\ 0 & V\end{bmatrix};\quad u\in U(1),\, V\in U(n) \text{ and } u\cdot\det(V)=1\right\}\,,$$

$$A=\left\{a_\tau=\begin{bmatrix}\mathrm{ch}\,\tau & \mathrm{sh}\,\tau & \\ \mathrm{sh}\,\tau & \mathrm{ch}\,\tau & \\ & & I_{n-1}\end{bmatrix};\quad \tau\in \boldsymbol{R}\right\}\,,$$

and

$$N=\left\{n(y, z)=\begin{bmatrix}1+iy+|z|^2/2 & -iy-|z|^2/2 & \bar z_2\cdots\bar z_n\\ iy+|z|^2/2 & 1-iy-|z|^2/2 & \bar z_2\cdots\bar z_n\\ z_2 & z_2 & \\ \vdots & \vdots & I_{n-1}\\ z_n & z_n & \end{bmatrix};\quad \begin{array}{l} y\in \boldsymbol{R},\\ z=(z_2,\cdots, z_n)\in \boldsymbol{C}^{n-1},\\ |z|^2=\sum_{k=2}^n|z_k|^2\end{array}\right\}.$$

Then $G=KAN$ (resp. $G=K\,\mathrm{CL}(A^+)K$, where $\mathrm{CL}(A^+)=\{a_t;\, t\geqq 0\}$) is an Iwasawa (resp. a Cartan) decomposition of G and $M=Z_K(A)$ is given by

$$M=\left\{\begin{bmatrix}b & & \\ & b & \\ & & B\end{bmatrix}\in K\,;\ b\in \boldsymbol{C} \text{ and } |b|^2=1\right\}\,.$$

For each $\zeta=(\zeta_1, \zeta_2, \cdots, \zeta_n)\in B(C^n)$ there exists an element $g\in G$ such that $\zeta=g\cdot 0=n(y, z)a_\tau\cdot 0=k_{u,V}a_t\cdot 0$, where $g\in n(y, z)a_\tau K$ and $k_{u,V}a_tK$. We put $\Lambda=e^{2\tau}+(1+2iy+|z|^2)$ and $c_p=u^{-1}v_{p1}$ for $1\leqq p\leqq n$, where $V=(v_{ij})_{1\leq i,j\leq n}\in$

$U(n)$. Then we see that

$$(3)\quad \begin{cases} \zeta_1=\dfrac{e^{\tau}+(-1+2iy+|z|^2)e^{-\tau}}{e^{\tau}+(1+2iy+|z|^2)e^{-\tau}}=\dfrac{\Lambda-1}{\Lambda}\,, \\ \zeta_p=\dfrac{2z_pe^{-\tau}}{e^{\tau}+(1+2iy+|z|^2)e^{-\tau}}=\dfrac{2z_p}{\Lambda} \qquad (2\leqq p\leqq n)\,, \end{cases}$$

and

$$(4)\quad \zeta_p=c_p\,\mathrm{th}\,t \qquad (1\leqq p\leqq n)\,.$$

Then it follows from (3) and (4) that $c_1\mathrm{th}\,t=(\Lambda-2)/\Lambda=1-2/\Lambda$, and thus,

$$z_p=\frac{c_p\,\mathrm{th}\,t}{1-c_1\,\mathrm{th}\,t} \qquad (2\leqq p\leqq n)\,.$$

Here we put

$$(5)\quad \xi=\frac{c_1\,\mathrm{th}\,t}{1-c_1\,\mathrm{th}\,t}\,.$$

Then $(\tau, y, z)\in \boldsymbol{R}\times\boldsymbol{R}\times\boldsymbol{C}^{n-1}$ and $(\xi, z)\in\boldsymbol{C}\times\boldsymbol{C}^{n-1}$ are the horispheric and the Cartan coordinates of the point $\zeta=(\zeta_1, \zeta_2, \cdots, \zeta_n)\in B(\boldsymbol{C}^n)$ respectively and the relation between these coordinates is given by the formula:

$$(6)\quad \begin{cases} \tau=\dfrac{1}{2}\log(1+\xi+\bar{\xi}-|z|^2)=\dfrac{\xi+\bar{\xi}}{2}+\cdots\,, \\ y=\dfrac{\xi-\bar{\xi}}{2i}\,. \end{cases}$$

§ 4. Reduction by horispheric coordinate.

In this section we shall rewrite Lemma 1 by using horispheric coordinate and obtain a reduced form (see Lemma 3 below). Under this coordinate the Poisson kernel is given by

$$P(n(y, z)a_\tau\cdot 0, u)=\left|\left(\frac{1-\bar{u}_1}{2}\right)e^{\tau}+\left[1-\frac{1-\bar{u}_1}{2}(1-2iy-|z|^2)-\sum_{k=2}^{n} z_k\bar{u}_k\right]e^{-\tau}\right|^{-2n}.$$

In particular, for $u=e_1={}^t(1, 0, \cdots, 0)\in\boldsymbol{C}^n$ it is simply expressed by

$$P(n(y, z)a_\tau\cdot 0, e_1)=e^{2n\tau}\,.$$

On the other hand the Laplace-Beltrami operator D is given by (cf. [F1], p. 66 and [F2], p. 59):

$$D=\frac{\partial^2}{\partial\tau^2}-2n\frac{\partial}{\partial\tau}+e^{2\tau}\sum_{i=1}^{2(n-1)}\frac{\partial^2}{\partial x_i^2}+(|x|^2e^{2\tau}+e^{4\tau})\frac{\partial^2}{\partial y^2}$$
$$+2e^{2\tau}\sum_{k=2}^{n}\left(x_{2k-3}\frac{\partial^2}{\partial y\partial x_{2k-2}}-x_{2k-2}\frac{\partial^2}{\partial y\partial x_{2k-3}}\right),$$

where $x_{2k-3}+ix_{2k-2}=z_k$ $(1\leqq k\leqq n)$ and $|x|^2=|z|^2=\sum_{i=1}^{2(n-1)}x_i^2$.

For simplicity we shall introduce the following notations:

$$\Delta=\frac{\partial^2}{\partial y^2}+\Delta_x=\frac{\partial^2}{\partial y^2}+\sum_{i=1}^{2(n-1)}\frac{\partial^2}{\partial x_i^2},$$
$$\|\nabla F\|^2=\left(\frac{\partial F}{\partial y}\right)^2+\|\nabla_x F\|^2=\left(\frac{\partial F}{\partial y}\right)^2+\sum_{i=1}^{2(n-1)}\left(\frac{\partial F}{\partial x_i}\right)^2$$

and

$$B=\sum_{k=2}^{n}\left(x_{2k-3}\frac{\partial}{\partial x_{2k-2}}-x_{2k-2}\frac{\partial}{\partial x_{2k-3}}\right).$$

LEMMA 2. *The hypotheses* (2a), (2b) *and* (2c) *in Lemma* 1 *are equivalent, under the horispheric coordinate, to the following conditions*:

(7a) $$\frac{\partial^2 F}{\partial\tau^2}-2n\frac{\partial F}{\partial\tau}+e^{2\tau}\Delta_x F+(|x|^2e^{2\tau}+e^{4\tau})\frac{\partial^2 F}{\partial\tau^2}+2e^{2\tau}B\left(\frac{\partial F}{\partial y}\right)=0,$$

(7b) $$\left(\frac{\partial F}{\partial\tau}\right)^2+e^{2\tau}\|\nabla_x F\|^2+(|x|^2e^{2\tau}+e^{4\tau})\left(\frac{\partial F}{\partial y}\right)^2+2e^{2\tau}\frac{\partial F}{\partial y}B(F)=4n^2F^2,$$

(7c) $$\frac{\partial F}{\partial\tau}(0)=2n;\quad \frac{\partial F}{\partial y}(0)=\frac{\partial F}{\partial x_1}(0)=\cdots=\frac{\partial F}{\partial x_{2(n-1)}}(0)=0.$$

PROOF. By the identity:

$$D(F^2)=2FDF+2\left[\left(\frac{\partial F}{\partial\tau}\right)^2+e^{2\tau}\|\nabla_x F\|^2+(|x|^2e^{2\tau}+e^{4\tau})\left(\frac{\partial F}{\partial y}\right)^2+2e^{2\tau}\frac{\partial F}{\partial y}B(F)\right],$$

(2a) and (2b) are equivalent to (7a) and (7b). By (3) the derivatives of Cartesian coordinate $(\zeta_1, \zeta_2, \cdots, \zeta_n)$ with respect to horispheric coordinate $(\xi, y, x_1, \cdots, x_{2(n-1)})$ are given by

$$\frac{\partial\zeta_p}{\partial\tau}(0)=\delta_{1p},\quad \frac{\partial\zeta_p}{\partial y}(0)=i\delta_{1p},\quad \frac{\partial\zeta_p}{\partial x_{2q-3}}(0)=\delta_{qp},\quad \frac{\partial\zeta_p}{\partial x_{2q-2}}(0)=i\delta_{qp}$$

for $1\leqq p, q\leqq n$. Then it follows that

$$\frac{\partial F}{\partial\tau}(0)=\frac{\partial F}{\partial\zeta_1}(0)+\frac{\partial F}{\partial\bar\zeta_1}(0),\qquad \frac{\partial F}{\partial y}(0)=i\left(\frac{\partial F}{\partial\zeta_1}(0)-\frac{\partial F}{\partial\bar\zeta_1}(0)\right),$$

$$\frac{\partial F}{\partial x_{2p-3}}(0)=\frac{\partial F}{\partial \zeta_p}(0)+\frac{\partial F}{\partial \bar{\zeta}_p}(0)\ , \qquad \frac{\partial F}{\partial x_{2p-2}}(0)=i\Big(\frac{\partial F}{\partial \zeta_p}(0)-\frac{\partial F}{\partial \bar{\zeta}_p}(0)\Big)\ .$$

Therefore, (2c) is equivalent to (7c). Q.E.D.

Now we put

$$G(\tau,\ y,\ z)=e^{-2n\tau}F(\tau,\ y,\ z)\ .$$

Then by Lemma 2 and the formulas:

$$\frac{\partial F}{\partial \tau}=e^{2n\tau}\Big[2nG+\frac{\partial G}{\partial \tau}\Big]\ , \qquad \frac{\partial^2 F}{\partial \tau^2}=e^{2n\tau}\Big[4n^2G+4n\frac{\partial G}{\partial \tau}+\frac{\partial^2 G}{\partial \tau^2}\Big]\ ,$$

$$\frac{\partial F}{\partial y}=e^{2n\tau}\frac{\partial G}{\partial y}\ , \qquad \frac{\partial^2 F}{\partial y^2}=e^{2n\tau}\frac{\partial^2 G}{\partial y^2}\ , \qquad \frac{\partial F}{\partial x_i}=e^{2n\tau}\frac{\partial G}{\partial x_i}\ ,$$

$$\frac{\partial^2 F}{\partial x_i^2}=e^{2n\tau}\frac{\partial^2 G}{\partial x_i^2} \quad \text{and} \quad \frac{\partial^2 F}{\partial x_i\partial y}=e^{2n\tau}\frac{\partial^2 G}{\partial x_i\partial y}\ ,$$

we can rewrite Lemma 1 as the following

LEMMA 3. *Let $G(\tau,\ y,\ x_1,\ \cdots,\ x_{2(n-1)})$ be a real valued, C^2 function on $\boldsymbol{R}\times\boldsymbol{R}\times\boldsymbol{R}^{2(n-1)}$ which satisfies $G(0)=1$ and the following conditions:*

(8a) $$\frac{\partial^2 G}{\partial \tau^2}+2n\frac{\partial G}{\partial \tau}+e^{2\tau}\Delta_x G+(|x|^2e^{2\tau}+e^{4\tau})\frac{\partial^2 G}{\partial y^2}+2e^{2\tau}B\Big(\frac{\partial G}{\partial y}\Big)=0\ ,$$

(8b) $$\Big(\frac{\partial G}{\partial \tau}\Big)^2+e^{2\tau}\,\|\nabla_x G\|^2+(|x|^2e^{2\tau}+e^{4\tau})\Big(\frac{\partial G}{\partial y}\Big)^2+4nG\frac{\partial G}{\partial y}+2ne^{2\tau}\frac{\partial G}{\partial y}B(G)=0\ ,$$

(8c) $$\frac{\partial G}{\partial \tau}(0)=\frac{\partial G}{\partial y}(0)=\frac{\partial G}{\partial x_1}(0)=\cdots=\frac{\partial G}{\partial x_{2(n-1)}}(0)=0\ .$$

Then G is identically one.

§5. Integrals of polynomials.

The next lemma and the following corollaries will play an essential role in our calculation. Their proof will be given after the last corollary.

LEMMA 4. *The matrix coefficients c_i $(1\leq i\leq n)$ defined in §3 satisfy the equations:*

(i) $$\int_K 1dk=1\ ,$$

(ii) $\int_K |c_i|^2 dk = \frac{1}{n}$,

(iii) $$\int_K |c_i|^2 |c_j|^2 dk = \begin{cases} \frac{2}{n(n+1)} & \text{if } i=j \\ \frac{1}{n(n+1)} & \text{if } i \neq j , \end{cases}$$

(iv) $$\int_K |c_i|^2 |c_j|^2 |c_k|^2 dk = \begin{cases} \frac{6}{n(n+1)(n+2)} & \text{if } i=j=k \\ \frac{2}{n(n+1)(n+2)} & \text{if } i \neq j=k \\ \frac{1}{n(n+1)(n+2)} & \text{if } i>j>k . \end{cases}$$

Other integrals of polynomials of the variables c_i, $\bar{c}_i$ $(1 \leqq i \leqq n)$ *of degree* $\leqq 6$ *are all equal to* 0.

COROLLARY 5. *When* t *tends to* 0, *the coordinates* ξ, z_i $(2 \leqq i \leqq n)$ *satisfy the equations*:

(i) $\int_K 1 dk = 1$,

(ii) $\int_K |\xi|^2 dk = \frac{1}{n} t^2 + O(t^4)$ and $\int_K |z_i|^2 dk = \frac{1}{n} t^2 + O(t^4)$,

(iii) $$\int_K \xi |\xi|^2 dk = \int_K \bar{\xi} |\xi|^2 dk = \frac{2}{n(n+1)} t^4 + O(t^6)$$

and

$$\int_K \xi |z_i|^2 dk = \int_K \bar{\xi} |z_i|^2 dk = \frac{1}{n(n+1)} t^4 + O(t^6) ,$$

(iv) $\int_K |\xi|^4 dk = \frac{2}{n(n+1)} t^4 + O(t^6)$, $\int_K |\xi|^2 |z_i|^2 dk = \frac{1}{n(n+1)} t^4 + O(t^6)$

and

$$\int_K |z_i|^2 |z_j|^2 dk = \begin{cases} \frac{2}{n(n+1)} t^4 + O(t^6) & \text{if } i=j \\ \frac{1}{n(n+1)} t^4 + O(t^6) & \text{if } i \neq j . \end{cases}$$

Other integrals of polynomials of the variables ξ, $\bar{\xi}$, z_i, $\bar{z}_i$ $(2 \leqq i \leqq n)$ *of degree* $\leqq 4$ *are equal to* 0 *or* $O(t^6)$.

COROLLARY 6. *When t tends to 0, the coordinates ξ, z_i $(2 \leqq i \leqq n)$ satisfy the equations*:

(i) $$\int_K \bar{c}_1^2 \xi dk = \frac{2}{n(n+1)} t^2 - \frac{4}{3n(n+1)} t^4 + O(t^6) ,$$

(ii) $$\int_K \bar{c}_1^2 \xi^2 dk = \frac{2}{n(n+1)} t^2 - \frac{4}{3n(n+1)} t^4 + O(t^6) ,$$

$$\int_K \bar{c}_1^2 |\xi|^2 dk = \frac{6}{n(n+1)(n+2)} t^4 + O(t^6)$$

and

$$\int_K \bar{c}_1^2 |z_i|^2 dk = \frac{2}{n(n+1)(n+2)} t^4 + O(t^6) ,$$

(iii) $$\int_K \bar{c}_1^2 \xi |\xi|^2 dk = \frac{12}{n(n+1)(n+2)} t^4 + O(t^6)$$

and

$$\int_K \bar{c}_1^2 \xi |z_i|^2 dk = \frac{2}{n(n+1)(n+2)} t^4 + O(t^6) ,$$

(iv) $$\int_K \bar{c}_1^2 \xi^2 |\xi|^2 dk = \frac{6}{n(n+1)(n+2)} t^4 + O(t^6)$$

and

$$\int_K \bar{c}_1^2 \xi^2 |z_i|^2 dk = \frac{2}{n(n+1)(n+2)} t^4 + O(t^6) .$$

Other integrals, with respect to the weighted measure $\bar{c}_1^2 dk$, of polynomials of the variables ξ, $\bar{\xi}$, z_i, $\bar{z}_i$ $(2 \leqq i \leqq n)$ of degree $\leqq 4$ are equal to 0 or $O(t^6)$.

PROOF. For $1 \leqq p \leqq n$, each c_p is a matrix coefficient (the p-th row in the first column) of the natural representation T_n of K on $\boldsymbol{C}^n$: $T_n(k)\zeta = k \cdot \zeta$ for $k \in K$ and $\zeta \in \boldsymbol{C}^n$. Then, using the fact that $\sum_{p=1}^n |c_p|^2 = 1$, we can prove Lemma 4. To prove the corollaries we shall recall that (see § 3):

$$\xi = c_1 \operatorname{th} t \sum_{l=0}^{\infty} (c_1 \operatorname{th} t)^l$$

and

$$z_p = c_p \operatorname{th} t \sum_{l=0}^{\infty} (c_1 \operatorname{th} t)^l \qquad (2 \leqq p \leqq n) .$$

Then, by Lemma 4 we can calculate the integrals of polynomials of the variables ξ, $\bar{\xi}$, z_i, $\bar{z}_i$ $(2\leqq i\leqq n)$ and obtain the desired expansions when t tends to 0. Q.E.D.

§6. K-finite eigenfunctions of the Laplace-Beltrami operator.

Let $\hat{K}$ denote the set of all equivalence classes of irreducible (finite dimensional) unitary representations of K. Let $(T, V)\in\hat{K}$ and F be a function on $B(\boldsymbol{C}^n)$. Then we define the projection F_T of F by

$$F_T(g)=\int_K \bar{\chi}_T(k)F(kg)dk ,$$

where χ_T is the character of T.

Let $A_T^l(G)$ $(l\in\boldsymbol{N})$ be the space of functions $F\colon G/K\cong B(\boldsymbol{C}^n)\to\boldsymbol{C}$ which satisfy the following conditions:

$$DF=4n^2l(l-1)F \quad \text{and} \quad F_T=F ;$$

then it follows (cf. [H2]) that

$$\dim A_T^l(G)=\dim V\cdot\dim V_T^M ,$$

where V_T^M is the space of M-fixed vectors in V. For $F\in A_T^l(G)$ we denote the restriction of F to A by F^A and put

$$A_T^l(A)=\{F^A\colon A\to\boldsymbol{C};\ F\in A_T^l(G)\} .$$

Then we see the following

PROPOSITION 7. *Suppose that* $\dim V_T^M=1$, *then*

(i) $A_T^l(A)=\boldsymbol{C}(P^l)_T^A$

and for each $a_t\in A$ *and* $F\in A_T^l(G)$

(ii) $F(a_t)=\int_K \bar{c}_M(k)F(ka_t)dk ,$

where P *is the Poisson kernel*: $P(n(y, z)a_\tau\cdot 0, e_1)=e^{2n\tau}$ *and* $c_M(k)=(T(k)e_M, e_M)$, e_M *the normalized* M-*fixed vector in* V_T^M.

PROOF. Let $(e_1, e_2, \cdots, e_d)$ be an orthonormal basis of V, where we take $e_1=e_M$; let $c_{ij}(k)=(T(k)e_j, e_i)$ $(k\in K$ and $1\leqq i, j\leqq d)$ be the matrix coefficients of T; we see that $\chi_T(kk'^{-1})=\sum_{1\leq i,j\leq d} c_{ij}(k)\bar{c}_{ij}(k')$ for $k, k'\in K$. Then (ii) follows from the facts that $F=F_T$ and the function: $k\mapsto F(ka_t)$ $(k\in K)$ is right M-invariant; in fact

$$F(a_t)=F_T(a_t)=\int_K \bar{\chi}_T(k)F(ka_t)dk$$

$$=\int_K\int_M \bar{\chi}_T(k)F(kma_t)dmdk$$
$$=\int_K\left[\int_M \bar{\chi}_T(km^{-1})dm\right]F(ka_t)dk$$
$$=\int_K \bar{c}_M(k)F(ka_t)dk \qquad (c_M=c_{11}) .$$

To show (i) we note that the function: $k\mapsto P^l(ka_t, e_1)$ $(k\in K)$ is M-biinvariant, and thus,

$$(P^l)_T(ka_t)=\int_K \bar{\chi}_T(k')P^l(k'ka_t, e_1)dk'$$
$$=\int_K (\sum_{i,j} \bar{c}_{ij}(k')c_{ij}(k))P^l(k'a_t, e_1)dk'$$
$$=\int_K\int_M\int_M (\sum_{i,j} \bar{c}_{ij}(k')c_{ij}(k))P^l(m'k'ma_t, e_1)dk'dm'dm$$
$$=\int_K \sum_{i,j}\left[\int_M\int_M c_{ij}(m'^{-1}k'm^{-1})dm'dm\right]c_{ij}(k)P^l(k'a_t, e_1)dk'$$
$$=\int_K c_M(k)c_M(k')P^l(k'a_t, e_1)dk'$$
$$=c_M(k)\int_K c_M(k')P^l(k'a_t, e_1)dk'$$
$$=c_M(k)(P^l)^A_T(a_t) .$$

In particular, $A^l_T(A)\neq\{0\}$, because it contains $(P^l)^A_T\neq 0$. Moreover, we can choose k_i $(1\leqq i\leqq d)$, where $k_1=e_1$, for which the functions $g\mapsto(P^l)_T(k_ig)$ $(g\in G)$ form a basis of $A^l_T(G)$. Then we can choose another basis Q_i $(1\leqq i\leqq d)$ of $A^l_T(G)$, where $Q_1=(P^l)_T$, such that $\int_K c_M(k)Q_i(ka_t)dk\equiv 0$ for $2\leqq i\leqq d$. Then (i) follows from (ii). Q.E.D.

COROLLARY 8. *Let F be a real valued, C^2 function on $B(\boldsymbol{C}^n)$ which satisfies $F(0)=1$ and the equations* (2a) *and* (2b). *Then for all $l\in\boldsymbol{N}$*

(i) $\displaystyle\int_K F^l(ka_t\cdot 0)dk=\int_K P^l(ka_t\cdot 0, e_1)dk ,$

(ii) $\displaystyle\int_K \bar{c}_1^2F^l(ka_t\cdot 0)dk=\lambda_l\int_K \bar{c}_1^2P^l(ka_t\cdot 0, e_1)dk ,$

where $c_1(k)=\bar{u}v_{11}$ if $k=\begin{bmatrix} u & \\ & V\end{bmatrix}$, $V=(v_{ij})_{1\leq i,j\leq n}\in U(n)$ and λ_l's are constants depending on F and l.

PROOF. For each $T\in\hat{K}$ the function $(F^l)_T$ is an eigenfunction of the

Laplace-Beltrami operator D with eigenvalue $4n^2l(l-1)$. Therefore, $(F^l)_T$ belongs to $A_T^l(G)$ and $(F^l)_T^A$ to $A_T^l(A)$.

(i) Let T_0 be the trivial representation of K on a one-dimensional vector space V. Obviously, $V_{T_0}^M=V$ has one dimension; the character χ_{T_0} is identically one on K. Therefore, $(F^l)_{T_0}^A$ belongs to $C(P^l)_{T_0}^A$ by Proposition 7 and (i) follows by comparison with the values at the origin.

(ii) Let T_n be the natural representation of K on an n-dimensional vector space $\boldsymbol{C}^n$ (see the proof of Corollary 6). Then the M-fixed vectors of the representation $T=T_n\otimes T_n$ of K consist of $\boldsymbol{C}e_1\otimes e_1$, where $e_1={}^t(1, 0, \cdots, 0)\in\boldsymbol{C}^n$ and the corresponding matrix coefficient is given by

$$(T_n\otimes T_n(k)e_1\otimes e_1, e_1\otimes e_1)=(T_n(k)e_1, e_1)^2=c_1^2 .$$

Therefore, we can obtain the desired result by applying Proposition 7 to the irreducible component of $T_n\otimes T_n$ which contains $\boldsymbol{C}e_1\otimes e_1$. Q.E.D.

§ 7. Proof of Lemma 3.

The function F in Theorem is a solution of the equation (2a), which is elliptic (cf. [R], p. 52). Therefore, F is real analytic on $B(\boldsymbol{C}^n)$, and thus, G in Lemma 3 is also analytic on $\boldsymbol{R}\times\boldsymbol{R}\times\boldsymbol{R}^{2(n-1)}$; there exists a neighbourhood V of 0 on which G has a Taylor expansion:

$$G(\tau, y, z)=G_0+G_1(\tau, y, z)+\cdots+G_p(\tau, y, z)+\cdots ,$$

where for each $p\geqq 0$, G_p is a homogeneous polynomial of τ, y, x_i $(1\leqq i\leqq 2(n-1))$ of degree p. Since $G(0)=1$ and G satisfies (8c), we see that $G_0\equiv 1$ and $G_1\equiv 0$.

In the first step of the following arguments we shall prove that $\partial^2G_2/\partial y^2=0$ and deduce that $G_2\equiv 0$. Then, in the second step we shall show that $G_p\equiv 0$ for all $p\geqq 1$. Obviously, this means that $G\equiv 1$ and completes the proof of Lemma 3.

Step 1. On the neighbourhood V of 0, G has an expansion corresponding to the coordinate (ξ, z): $G(\xi, z)=1+\sum_{p=1}^{\infty}H_p(\xi, z)$, where for each $p\geqq 1$, H_p is a homogeneous polynomial of ξ, $\bar{\xi}$, z_i, $\bar{z}_i$ $(2\leqq i\leqq n)$ of degree p. Applying (8c) and (6), we see that

$$\frac{\partial G}{\partial \xi}(0)=\frac{\partial G}{\partial z_2}(0)=\cdots=\frac{\partial G}{\partial z_n}(0)=0 ;$$

consequently, $H_1\equiv 0$.

Here we put

$$H_2(\xi, z)=A\xi^2+B|\xi|^2+C\bar{\xi}^2+\sum_{i=2}^{n} D_i|z_i|^2+\cdots ,$$

then,

$$\frac{\partial^2 H_2}{\partial y^2}(0)=2\Big[\frac{\partial^2 H}{\partial\xi\partial\bar{\xi}}(0)\frac{\partial\xi}{\partial y}(0)\frac{\partial\bar{\xi}}{\partial y}(0)+\mathrm{Re}\Big(\frac{\partial^2 H}{\partial\xi^2}(0)\Big(\frac{\partial\xi}{\partial y}(0)\Big)^2\Big)\Big]$$
$$=2(B-2\,\mathrm{Re}\,A) .$$

Since $\partial^2G_2/\partial y^2=(\partial^2H_2/\partial y^2)(0)$ by (6), to prove that $\partial^2G_2/\partial y^2=0$ it is enough to show that $B-2\,\mathrm{Re}\,A=0$. We put

$$X=P(ka_t\cdot 0, e_1)-1=(1+(\xi+\bar{\xi})-|z|^2)^n-1$$

and

$$Y=G(ka_t\cdot 0)-1=H_2+\cdots .$$

Then the equations in Corollary 8 can be rewritten as

$$(9)\qquad \int_K(1+X)^l(1+Y)^l dk=\int_K(1+X)^l dk ,$$

$$(10)\qquad \int_K\bar{c}_1^2(1+X)^l(1+Y)^l dk=\lambda_l\int_K\bar{c}_1^2(1+X)^l dk .$$

When t tends to 0, we see that $X=O(t)$ and $Y=O(t^2)$; we can let t go to 0 after the integral over K, because K is compact; in particular, collecting the components of order t^6 in the equations for $l=1$, 2 and 3, we obtain that

$$(11)\qquad \begin{cases}\int_K(Y+XY)dk=O(t^6) ,\\ \int_K(2Y+4XY+2X^2Y+Y^2)dk=O(t^6) ,\\ \int_K(3Y+9XY+9X^2Y+3Y^2)dk=O(t^6)\end{cases}$$

and

$$(12)\qquad \begin{cases}\int_K\bar{c}_1^2(Y+XY)dk=(\lambda_1-1)\int_K\bar{c}_1^2Xdk+O(t^6) ,\\ \int_K\bar{c}_1^2(2Y+4XY+2X^2Y+Y^2)dk=(\lambda_2-1)\int_K\bar{c}_1^2(2X+X^2)dk+O(t^6) ,\\ \int_K\bar{c}_1^2(3Y+9XY+9X^2Y+3Y^2)dk=(\lambda_3-1)\int_K\bar{c}_1^2(3X+3X^2+X^3)dk+O(t^6) .\end{cases}$$

Then it follows from (11) that

(13) $$\int_K X^2 Y dk = O(t^6)\ ,$$

(14) $$\int_K Y dk = O(t^6)$$

and applying Corollary 5 to these integrals, we see that

$$\int_K X^2 Y dk = \frac{n}{n+1} 2\Big(A+2B+C+\sum_{i=2}^{n} D_i\Big)t^4 + O(t^6)$$

and

$$\int_K Y dk = \frac{1}{n}\Big(B+\sum_{i=2}^{n} D_i\Big)t^2 + O(t^4)\ .$$

Therefore, we obtain that

(15) $$B+\sum_{i=2}^{n} D_i = A+B+C = 0\ .$$

Since G is real valued, we have

(16) $$A = \bar{C}\ .$$

In particular;

(17) when $n=1$, $B=0$ and $\mathrm{Re}\, A=0$.

On the other hand, it follows from (12) that

(18) $$\int_K \bar{c}_1^2 X^2 Y dk = \int_K \bar{c}_1^2 \Big[(\lambda_3 - 2\lambda_2 + \lambda_1)X + (\lambda_3 - \lambda_2)X^2 + \frac{\lambda_3 - 1}{3} X^3\Big] dk + O(t^6)\ ,$$

(19) $$\begin{cases} \int_K \bar{c}_1^2 Y dk = \int_K \bar{c}_1^2 (\lambda_2 - 1) X dk + O(t^4)\ , \\ \int_K \bar{c}_1^2 [2(\lambda_2 - \lambda_1)X + (\lambda_2 - 1)X^2] dk = O(t^4)\ , \\ \int_K \bar{c}_1^2 [(\lambda_3 - \lambda_1)X + (\lambda_3 - 1)X^2] dk = O(t^4)\ . \end{cases}$$

Here we note from Corollary 6, (15) and (16) that

(20) $$\begin{aligned} \int_K \bar{c}_1^2 X^2 Y dk &= \frac{1}{n(n+1)(n+2)}\Big(12n^2 A + 6n^2 B + 2n^2 \sum_{i=2}^{n} D_i\Big)t^4 + O(t^6) \\ &= \frac{4n}{(n+1)(n+2)}(2A - \bar{A})t^4 + O(t^6)\ , \end{aligned}$$

$$
(21)\qquad \begin{cases} \displaystyle\int_K \bar{c}_1^2 Y dk = \frac{2}{n(n+1)} A t^2 + O(t^4) , \\ \displaystyle\int_K \bar{c}_1^2 X dk = t^2 + \frac{2(2n^2-9n+1)}{3(n+1)(n+2)} t^4 + O(t^6) , \\ \displaystyle\int_K \bar{c}_1^2 X^2 dk = \frac{2n}{n+1} t^2 + \frac{4n(6n^2+5n-5)}{3(n+1)(n+2)} t^4 + O(t^6) , \end{cases}
$$

$$
(22)\qquad \int_K \bar{c}_1^2 X^3 dk = \frac{6n^2(5n+1)}{(n+1)(n+2)} t^4 + O(t^6) .
$$

Therefore, substituting (21) for (19), we can obtain that

$$
(23)\qquad \begin{cases} \dfrac{2A}{n(n+1)} = \lambda_1 - 1 , \\ \lambda_2 = \dfrac{n+(n+1)\lambda_1}{2n+1} , \\ \lambda_3 = \dfrac{2n+(n+1)\lambda_1}{3n+1} . \end{cases}
$$

In particular, we have

$$
(24)\qquad \begin{cases} \lambda_3 - 2\lambda_2 + \lambda_1 = \dfrac{2n^2}{(2n+1)(3n+1)} (\lambda_1 - 1) , \\ \lambda_3 - \lambda_2 = \dfrac{-n(n+1)}{(2n+1)(3n+1)} (\lambda_1 - 1) , \\ \lambda_3 - 1 = \dfrac{n+1}{3n+1} (\lambda_1 - 1) . \end{cases}
$$

Finally, using the formulas (18), (20)-(24), we can deduce the following equation:

$$
(6n^3+11n^2+10n-3)A = (6n^3+11n^2+6n+1)\bar{A} ,
$$

which implies that Re $A=0$ when $n \geqq 2$, and thus, $B=0$ by (15) and (16). Therefore, for each $n \geqq 1$ (see (17) for $n=1$) we conclude that

$$
(8d)\qquad \frac{\partial^2 G_2}{\partial y^2} = 2(B - 2 \operatorname{Re} A) = 0 .
$$

Now we shall prove that $G_2 \equiv 0$. Collecting the homogeneous components of degrees 1 and 2 in (8b), we obtain that

$$
(25)\qquad \frac{\partial G_2}{\partial \tau} \equiv 0 ,
$$

(26)
$$\frac{\partial G_3}{\partial \tau} = -\frac{1}{4n} \| \nabla G_2 \|^2 ;$$

in particular,

(27) G_2 and $\dfrac{\partial G_3}{\partial \tau}$ are not dependent on τ,

(28)
$$\Delta\left(\frac{\partial G_3}{\partial \tau}\right) = -\frac{1}{2n}\left[\left(\frac{\partial^2 G_2}{\partial y^2}\right)^2 + 2\sum_{i=1}^{2(n-1)}\left(\frac{\partial^2 G_2}{\partial x_i \partial y}\right)^2 + \sum_{i,j=1}^{2(n-1)}\left(\frac{\partial^2 G_2}{\partial x_i \partial x_j}\right)^2\right].$$

On the other hand, collecting the homogeneous components of degrees 1 and 2 in (8a) and using (8d) and (27), we obtain that

(29)
$$\Delta G_2 \equiv 0 ,$$

(30)
$$\Delta G_3 = -2B\left(\frac{\partial G_2}{\partial y}\right) .$$

In particular, we have

(31)
$$\Delta\left(\frac{\partial G_3}{\partial \tau}\right) = 0 .$$

The equations (27), (28) and (31) imply that all second derivatives of G_2, which is a homogeneous polynomial of τ, y, x_i $(1 \leqq i \leqq 2(n-1))$ of degree 2, are equal to 0. Thus it follows that $G_2 \equiv 0$ on $B(C^n)$.

Step 2. We shall prove that $G_p \equiv 0$ for all $p \geqq 1$ by induction. Let us suppose that $G_1 \equiv G_2 \equiv \cdots \equiv G_p \equiv 0$ for $p \geqq 2$ (see Step 1); we shall show that $G_{p+1} \equiv 0$. First, collecting the homogeneous components of degree $m \geqq 2$ in (8b), we obtain that

(32)
$$\left[\left(\frac{\partial G}{\partial \tau}\right)^2\right]_m + \sum_{q=0}^{m} \frac{(2\tau)^q}{q!} \sum_{i=1}^{2(n-1)} \left[\left(\frac{\partial G}{\partial x_i}\right)^2\right]_{m-q} + |x|^2 \sum_{q=0}^{m-2} \frac{(2\tau)^q}{q!} \left[\left(\frac{\partial G}{\partial y}\right)^2\right]_{m-q-2}$$
$$+ \sum_{q=0}^{m} \frac{(4\tau)^q}{q!} \left[\left(\frac{\partial G}{\partial y}\right)^2\right]_{m-q} + 4n \sum_{q=0}^{m} G_q \frac{\partial G_{m-q+1}}{\partial \tau}$$
$$+ 2\sum_{q=0}^{m} \frac{(2\tau)^q}{q!} \sum_{k=0}^{m-q} \frac{\partial G_{k+1}}{\partial y} B(G_{m-q-k}) = 0 ,$$

where

$$\left[\left(\frac{\partial G}{\partial \tau}\right)^2\right]_m = \frac{\partial G_1}{\partial \tau} \frac{\partial G_{m+1}}{\partial \tau} + \frac{\partial G_2}{\partial \tau} \frac{\partial G_m}{\partial \tau} + \cdots + \frac{\partial G_{m+1}}{\partial \tau} \frac{\partial G_1}{\partial \tau} ,$$

and thus,

$$\left[\left(\frac{\partial G}{\partial \tau}\right)^2\right]_m = \begin{cases} 0 & \text{if} \quad m \leqq 2p-1 \\ \left(\dfrac{\partial G_{p+1}}{\partial \tau}\right)^2 & \text{if} \quad m = 2p \, . \end{cases}$$

Moreover,

$$\left[\left(\frac{\partial G}{\partial y}\right)^2\right]_{m-q} = \begin{cases} 0 & \text{if} \quad m \leqq 2p-1 \\ 0 & \text{if} \quad m = 2p \text{ and } q \geqq 1 \\ \left(\dfrac{\partial G_{p+1}}{\partial y}\right)^2 & \text{if} \quad m = 2p \text{ and } q = 0 \, , \end{cases}$$

$$\left[\left(\frac{\partial G}{x_j}\right)^2\right]_{m-q} = \begin{cases} 0 & \text{if} \quad m \leqq 2p-1 \\ 0 & \text{if} \quad m = 2p \text{ and } q \geqq 1 \\ \left(\dfrac{\partial G_{p+1}}{\partial x_i}\right)^2 & \text{if} \quad m = 2p \text{ and } q = 0 \, , \end{cases}$$

$$\sum_{q=0}^{m} G_q \frac{\partial G_{m-q+1}}{\partial \tau} = \frac{\partial G_{m+1}}{\partial \tau} \qquad \text{if} \quad m \leqq 2p$$

and

$$\frac{\partial G_{k+1}}{\partial y} B(G_{m-q-k}) = 0 \qquad \text{if} \quad m \leqq 2p, \ 0 \leqq q \leqq m \text{ and } 0 \leqq k \leqq m-q \, .$$

Therefore, (32) implies that

(33) $$\frac{\partial G_{m+1}}{\partial \tau} = 0 \qquad \text{if} \quad m \leqq 2p-1$$

and

(34) $$\frac{\partial G_{2p+1}}{\partial \tau} = -\frac{1}{4n}\left[\left(\frac{\partial G_{p+1}}{\partial y}\right)^2 + \sum_{i=1}^{2(n-1)}\left(\frac{\partial G_{p+1}}{\partial x_i}\right)^2\right].$$

In particular, we deduce that

(35) $G_{p+1}, G_{p+2}, \cdots, G_{2p}$ are not dependent on τ ,

(36) $\dfrac{\partial G_{2p+1}}{\partial \tau}$ is not dependent on τ .

Next collecting the homogeneous components of degree $(m-2)$ in the equation: $e^{-2\tau}[DG + 4n(\partial G/\partial \tau)] = 0$, which is nothing but (8a), we obtain that

$$\sum_{q=0}^{m-2}\frac{(-2\tau)^q}{q!}\Big(\frac{\partial^2 G_{m-q}}{\partial\tau^2}+2n\frac{\partial G_{m-q-1}}{\partial\tau}\Big)+|x|^2\frac{\partial^2 G_{m-2}}{\partial y^2}$$
$$+\Delta_x(G_m)+\sum_{q=0}^{m-2}\frac{(2\tau)^q}{q!}\frac{\partial^2 G_{m-q}}{\partial y^2}+2B\Big(\frac{\partial G_{m-1}}{\partial y}\Big)=0\ . \tag{37}$$

Then, taking account of (35) and (36), we see that for $m\leqq 2p$ the first sum in (37) is identically zero; $\Delta_x(G_m)$ and $B(\partial G_{m-1}/\partial y)$ are not dependent on τ; thus, we obtain that

$$\Delta G_m+2B\Big(\frac{\partial G_{m-1}}{\partial y}\Big)=0 \qquad \text{for}\quad m\leqq 2p\ , \tag{38}$$

$$\frac{\partial^2 G_{m-1}}{\partial y^2}=0 \qquad \text{for}\quad m\leqq 2p\ . \tag{39}$$

In particular, for $m=p+1$ we have

$$\Delta G_{p+1}=0\ . \tag{40}$$

Applying the differential operator $\partial^2/\partial y^2$ to (38), we see from (39) that

$$\Delta\Big(\frac{\partial^2 G_m}{\partial y^2}\Big)=0 \qquad \text{for}\quad m\leqq 2p\ . \tag{41}$$

When $m=2p+1$, the equation (37) means that

$$\Delta(G_{2p+1})+2B\Big(\frac{\partial G_{2p}}{\partial y}\Big)+2\tau\frac{\partial^2 G_{2p}}{\partial y^2}=0\ . \tag{42}$$

Now we shall apply the differential operator $\Delta\circ(\partial/\partial\tau)$ to (42). Then we obtain from (35) and (41) that

$$\Delta\Big[\Delta\Big(\frac{\partial G_{2p+1}}{\partial\tau}\Big)\Big]=0\ ; \tag{43}$$

substituting (34) for (43) and noting (40), we finally deduce that

$$\begin{aligned}0&=\Delta\Big[\Delta\Big(\frac{\partial G_{2p+1}}{\partial\tau}\Big)\Big]\\&=-\frac{1}{4n}\Delta\Delta\Big[\Big(\frac{\partial G_{p+1}}{\partial y}\Big)^2+\sum_{i=1}^{2(n-1)}\Big(\frac{\partial G_{p+1}}{\partial x_i}\Big)^2\Big]\\&=-\frac{1}{2n}\Delta\Big[\Big(\frac{\partial^2 G_{p+1}}{\partial y^2}\Big)^2+2\sum_{i=1}^{2(n-1)}\Big(\frac{\partial^2 G_{p+1}}{\partial y\partial x_i}\Big)^2+\sum_{i,j=1}^{2(n-1)}\Big(\frac{\partial^2 G_{p+1}}{\partial x_i\partial x_j}\Big)^2\Big]\\&=-\frac{1}{n}\Big[\Big(\frac{\partial^3 G_{p+1}}{\partial y^3}\Big)^2+3\sum_{i=1}^{2(n-1)}\Big(\frac{\partial^3 G_{p+1}}{\partial y^2\partial x_i}\Big)^2+3\sum_{i,j=1}^{2(n-1)}\Big(\frac{\partial^3 G_{p+1}}{\partial y\partial x_i\partial x_j}\Big)^2\end{aligned}$$

$$+\sum_{i,j,k=1}^{2(n-1)}\left(\frac{\partial^3 G_{p+1}}{\partial x_i \partial x_j \partial x_k}\right)^2\Bigg] .$$

Therefore, G_{p+1} is a homogeneous polynomial of y, x_i $(1\leqq i\leqq 2(n-1))$ (see (35)) of degree $\geqq 3$, whose third derivatives:

$$\frac{\partial^3 G_{p+1}}{\partial y^3}, \quad \frac{\partial^3 G_{p+1}}{\partial y^2 \partial x_i}, \quad \frac{\partial^3 G_{p+1}}{\partial y \partial x_i \partial x_j} \quad \text{and} \quad \frac{\partial^3 G_{p+1}}{\partial x_i \partial x_j \partial x_k} \qquad \text{for} \quad 1\leqq i,\ j,\ k\leqq 2(n-1)$$

are all identically zero. Obviously, it follows that $G_{p+1}\equiv 0$ on $B(C^n)$.

This completes the proof of Lemma 3. Q.E.D.

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