Inner Extensions of Automorphisms of Irrational Rotation Algebras to AF-Algebras

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Abstract. Let $A_{\theta}$ be an irrational rotation algebra. In the present paper we will show that automorphisms of $A_{\theta}$ with some properties can be extended to inner automorphisms of an AF-algebra. In other words, there are a monomorphism $\rho$ of $A_{\theta}$ into an AF-algebra $B$ and a unitary element $w \in B$ such that $\rho(\alpha(x))=w\rho(x)w^*$ for any $x \in A_{\theta}$.

§1. Introduction.

Let $\theta$ be an irrational number in $[0, 1]$ and let $\sigma$ be the rotation by the angle $2\pi\theta$ on the circle $T=\mathbb{R}/\mathbb{Z}$. Let $C(T)$ be the abelian $C^*$-algebra of all complex valued continuous functions on $T$. Then we can regard $\sigma$ as an automorphism of $C(T)$. Hence we can consider the crossed product $C(T) \times_{\sigma} \mathbb{Z}$ of $C(T)$ by $\sigma$ and we denote it by $A_{\theta}$, which is called the irrational rotation algebra by $\theta$. It is well known that $A_{\theta}$ has two generators $u$ and $v$ with $vu=e^{2\pi i\theta}uv$. Let $\text{Aut}(A_{\theta})$ be the group of all automorphisms of $A_{\theta}$ and $C^*(v)$ be the abelian $C^*$-subalgebra of $A_{\theta}$ generated by $v$. Furthermore throughout this paper we mean a unital $*$-monomorphism by a monomorphism.

DEFINITION. Let $\alpha \in \text{Aut}(A_{\theta})$. We say that $\alpha$ can be extended to an inner automorphism of an AF-algebra if there are a monomorphism $\rho$ of $A_{\theta}$ into an AF-algebra $B$ and a unitary element $w \in B$ such that $\rho(\alpha(x))=w\rho(x)w^*$ for any $x \in A_{\theta}$.

Now generally let $A$ be a unital $C^*$-algebra and for each $n \in \mathbb{N}$ let $M_n$ be the $n \times n$ matrix algebra. We identify $A \otimes M_n$ with the $n \times n$ matrix algebra $M_n(A)$ over $A$. Let $\alpha$ be an automorphism of $A$. For $i=0, 1$ we denote the $K_i$-group of $A$ by $K_i(A)$ and for any projection $p \in A \otimes M_n$ (resp. any unitary element $x \in A \otimes M_n$) $[p]$ (resp. $[x]$) denote the corresponding class in $K_0(A)$ (resp. $K_1(A)$). Let $\partial$ be the connecting map

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of $K_1(A \times_{\alpha} Z)$ into $K_0(A)$.

**Lemma 1.** With the above notations if $p \in A \otimes M_n$ satisfies $\alpha(p) = xp^*x^*$ for some unitary element $x \in A \otimes M_n$, then an element $w = (1 - p) + px^*yp \in (A \times_{\alpha} Z) \otimes M_n$ is a unitary element with $\partial([w]) = [p]$ where $y$ is a unitary element in $A \times_{\alpha} Z$ satisfying that $\alpha = \text{Ad}(y)$ and $A$ and $y$ generate $A \times_{\alpha} Z$.

**Proof.** We will use the notations in Pimsner and Voiculescu [6]. Let $K$ be the $C^*$-algebra of all compact operators on a countably infinite dimensional Hilbert space and $T$ be the Toeplitz algebra for $(A, \alpha)$. Let $J$ be a closed two sided ideal generated by a projection $Q = 1 \otimes I - (y \otimes S)(y \otimes S)^* = 1 \otimes P$. Then we obtain the connecting map $d$ of $K_1(T/J)$ into $K_0(J)$. By Pimsner and Voiculescu [6], $J$ is isomorphic to $A \otimes K$ and $T/J$ is isomorphic to $A \times_{\alpha} Z$. We denote the isomorphism of $A \otimes K$ onto $J$ by $\psi$ and the isomorphism of $A \times_{\alpha} Z$ onto $T/J$ by $\phi$. Then it is sufficient to show that $d([\phi(w)]) = [\psi(p)]$. By the definitions of $\phi$ and $\psi$, we have

$$\phi(w) = (1 - p) \otimes I + px^*yp \otimes S^*$$

and

$$\psi(p) = p \otimes P.$$ 

Let $z = \begin{bmatrix} (1 - p) \otimes I + px^*yp \otimes S^* & 0 \\ p \otimes P & (1 - p) \otimes I + py^*xp \otimes S \end{bmatrix}$ in $T \otimes M_{2n}$. Then $\pi(z) = \phi(w) \oplus \phi(w)^*$ where $\pi$ is the quotient map of $T$ onto $T/J$. Hence

$$d([\phi(w)]) = z \begin{bmatrix} 1 \otimes I & 0 \\ 0 & 0 \end{bmatrix} z^* - \begin{bmatrix} 1 \otimes I & 0 \\ 0 & 0 \end{bmatrix}.$$ 

Since $z \begin{bmatrix} 1 \otimes I & 0 \\ 0 & 0 \end{bmatrix} z^* = \begin{bmatrix} 1 \otimes I & 0 \\ 0 & p \otimes P \end{bmatrix}$, we obtain that $d([\phi(w)]) = [p \otimes P]$. Q.E.D.

§ 2. The case of $\alpha(u) = fu$ and $\alpha(v) = v$.

In this section we will show that if $\alpha \in \text{Aut}(A_{\theta})$ with $\alpha(u) = fu$ and $\alpha(v) = v$ where $f$ is a unitary element in $C^*(v)$, there are an AF-algebra $B$, a monomorphism $\rho$ and a unitary element $w \in B$ such that $\rho(\alpha(x)) = \rho(x)^*w^*$ for any $x \in A_{\theta}$. Now we consider the crossed product $A_{\theta} \times_{\alpha} Z$ of $A_{\theta}$ by $\alpha$. Then there is a unitary element $z \in A_{\theta} \times_{\alpha} Z$ such that $\alpha(x) = zzz^*$ for any $x \in A_{\theta}$ and $A_{\theta}$ and $z$ generate $A_{\theta} \times_{\alpha} Z$. Hence we have the following relations;

$$zuz^* = fu.$$
Let $C^*(v, z)$ be the $C^*$-subalgebra of $A_{\theta} \times_{\alpha} Z$ generated by $v$ and $z$ and let $\beta$ be the automorphism of $C^*(v, z)$ defined by $\beta(v) = uvu^* = e^{-2\pi i \theta}$ and $\beta(z) = uzu^* = f^*z$.

**Lemma 2.** With the above assumptions $\text{Sp}(z) = T$.

**Proof.** Suppose that $\text{Sp}(z) \subsetneqq T$. Then we can find a selfadjoint element $a \in A_{\theta} \times_{\alpha} Z$ such that $z = e^{ia}$. Hence $[z] = 0$ in $K_1(A_{\theta} \times_{\alpha} Z)$. On the other hand by the Pimsner-Voiculescu six terms exact sequence we have the following sequence:

$$0 \longrightarrow \text{Im}(id - \alpha_{\ast}) \longrightarrow K_1(A_{\theta} \times_{\alpha} Z) \xrightarrow{\partial} K_0(A_{\theta}) \longrightarrow 0.$$ 

Then by Lemma 1, $\partial([z]) = [1]$. Thus $[z] \neq 0$ in $K_1(A_{\theta} \times_{\alpha} Z)$. This is a contradiction. Q.E.D.

By Lemma 2, $C^*(v, z)$ is isomorphic to $C(T^2)$ and we identify $C^*(v, z)$ with $C(T^2)$ and regard $\beta$ as a homeomorphism of $T^2$. Then clearly $A_{\theta} \times_{\alpha} Z$ is isomorphic to $C(T^2) \times_{\beta} Z$. Let $\tau$ be the unique faithful tracial state of $A_{\theta}$ and $\bar{\tau}$ be a faithful tracial state of $A_{\theta} \times_{\alpha} Z$ defined by $\bar{\tau}(g) = \tau(g(0))$ for each $g \in l^1(Z, A_{\theta})$. Thus $C(T^2) \times_{\beta} Z$ has a faithful tracial state. Recall that a separable unital $C^*$-algebra $A$ is quasidiagonal if there is a monomorphism $\pi$ of $A$ into $B(H)$ such that $\pi(A) \cap K(H) = 0$ where $K(H)$ denotes the $C^*$-algebra of all compact operators on a Hilbert space $H$ and a sequence $\{p_n\}_{n \in \mathbb{N}}$ of finite dimensional orthogonal projections in $B(H)$ such that

$$\cdots \leq p_n \leq p_{n+1} \leq \cdots,$$ 

and for every $a \in A$

$$\|p_n \pi(a) - \pi(a) p_n\| \to 0.$$

Moreover $A$ is finite if no proper projection is algebraically equivalent to 1 and $A$ is stably finite if $M_n(A)$ is finite for any $n \in \mathbb{N}$. By the above definition we can easily see that $C(T^2) \times_{\beta} Z$ is finite since it has a faithful tracial state.

**Lemma 3.** Let $T$ be a homeomorphism of a compact metrizable space $X$ and $\alpha_T$ be an automorphism of $C(X)$ induced by $T$. Then the following conditions for $C(X) \times_{\alpha_T} Z$ are equivalent;

$$zvz^* = v,$$

$$vu = e^{2\pi i \theta} uv.$$
(1) quasidiagonal,
(2) finite,
(3) stably finite.

**Proof.** (1) implies (3); By Pimsner [5, Theorem 9] there exists an embedding of $C(X) \times_{\alpha} Z$ into an AF-algebra. Hence $C(X) \times_{\alpha} Z$ is stably finite since we can regard it as a $C^{*}$-subalgebra of the AF-algebra.

(3) implies (2); This is trivial.

(2) implies (1); Suppose that $C(X) \times_{\alpha} Z$ is not quasidiagonal. Then it follows from Pimsner [5, Proposition 8 and Theorem 9] that we can find a non unitary isometry in $C(X) \times_{\alpha} Z$. However this contradicts (2). Q.E.D.

**Proposition 4.** If $\alpha \in \text{Aut}(A_{\theta})$ with $\alpha(u) = fu$ and $\alpha(v) = v$ where $f$ is a unitary element in $C^{*}(v)$, there are an AF-algebra $B(\alpha)$, and a monomorphism $\rho_{\alpha}$ of $A_{\theta} \times_{\alpha} Z$ into $B(\alpha)$.

**Proof.** By Lemma 3, $C(T) \times_{\alpha} Z$ is quasidiagonal and $A_{\theta} \times_{\alpha} Z$ is isomorphic to $C(T) \times_{\alpha} Z$. Hence by Pimsner [5, Theorem 9] we can find an AF-algebra $B(\alpha)$ and a monomorphism $\rho_{\alpha}$ of $A_{\theta} \times_{\alpha} Z$ into $B(\alpha)$. Q.E.D.

§ 3. The case of $\alpha(u) = fu$ and $\alpha(v) = e^{2\pi i t}v$.

For each $t \in \mathbb{R}$ let $\beta_{t}^{(1)} \in \text{Aut}(A_{\theta})$ be defined by $\beta_{t}^{(1)}(u) = e^{2\pi i t} u$ and $\beta_{t}^{(1)}(v) = v$ and let $\beta_{t}^{(2)} \in \text{Aut}(A_{\theta})$ be defined by $\beta_{t}^{(2)}(u) = u$ and $\beta_{t}^{(2)}(v) = e^{2\pi i t} v$. And we define $\beta_{(s,t)} = \beta_{s}^{(1)} \circ \beta_{t}^{(2)}$. Let $SL(2, \mathbb{Z})$ be the group of all $2 \times 2$ matrices over $\mathbb{Z}$ with determinant 1 and let $G = \{ g \in SL(2, \mathbb{Z}); g = \begin{bmatrix} 1 & 0 \\ n & 1 \end{bmatrix} \}$. For each $g \in SL(2, \mathbb{Z})$ let $\beta_{g} \in \text{Aut}(A_{\theta})$ be defined by $\beta_{g}(u) = u^{a} v^{0}$ and $\beta_{g}(v) = u^{b} v^{c}$ where $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $a, b, c, d \in \mathbb{Z}$.

In this section we will show that if $\alpha = \beta_{s} \circ \beta_{(s,t)}$ with $g \in G$ and $s, t \in \mathbb{R}$, there are an AF-algebra $B$, a monomorphism $\rho$ of $A_{\theta}$ into $B$ and a unitary element $w \in B$ such that $\rho(\alpha(x)) = w \rho(x) w^{*}$ for any $x \in A_{\theta}$. For each $n \in \mathbb{N}$ let $U_{n} \in M_{n}$ be defined by

$$U_{n} = \begin{bmatrix} 0 & 0 & \cdots & 0 & (-1)^{n-1} \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & . \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & 0 & 0 \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}$$

and let $I_{n}$ be the unit element of $M_{n}$. 
Lemma 5. Let $\alpha \in \text{Aut}(A_{\theta})$. If there exist an $n \in N$, a monomorphism $\rho_{a^n}$ of $A_{\theta}$ into an AF-algebra $B(\alpha^n)$ and a unitary element $w_{a^n}$ such that $\rho_{a^n}(\alpha^n(x)) = w_{a^n}\rho_{a^n}(x)w_{a^n}^*$ for any $x \in A_{\theta}$, there are a monomorphism $\rho_{a}$ of $A_{\theta}$ into an AF-algebra $B(\alpha)$ and a unitary element $w_{a}$ such that $\rho_{a}(\alpha(x)) = w_{a}\rho_{a}(x)w_{a}^*$ for any $x \in A_{\theta}$.

Proof. Let $B(\alpha) = B(\alpha^n) \otimes M_n$ and $\rho_{a}$ be a monomorphism of $A_{\theta}$ into $B(\alpha)$ defined by $\rho_{a}(x) = \bigoplus_{j=0}^{n-1}\rho_{a^n}(\alpha^{j}(x))$ for each $x \in A_{\theta}$. Then for any $x \in A_{\theta}$

$$\rho_{a}(\alpha(x)) = \bigoplus_{j=0}^{n-1}\rho_{a^n}(\alpha^{j+1}(x))$$

$$= (I_{n-1} \oplus w_{a^n})^*\left(\bigoplus_{j=0}^{n-1}\rho_{a^n}(\alpha^{j}(x))\right)(I_{n-1} \oplus w_{a^n})$$

since $\rho_{a^n}(\alpha^n(x)) = w_{a^n}\rho_{a^n}(x)w_{a^n}^*$.

Q.E.D.

Corollary 6. Let $\alpha \in \text{Aut}(A_{\theta})$ with $\alpha(u) = fu$ and $\alpha(v) = e^{2\pi it}v$ where $f$ is a unitary element in $C^*(v)$ and $t \in \mathbb{R}$. If $t \in \mathbb{Q}$, there are an AF-algebra $B(\alpha)$, a monomorphism $\rho_{a}$ of $A_{\theta}$ into $B(\alpha)$ and a unitary element $w_{a} \in B(\alpha)$ such that $\rho_{a}(\alpha(x)) = w_{a}\rho_{a}(x)w_{a}^*$ for any $x \in A_{\theta}$.

Proof. Since $t \in \mathbb{Q}$, there is an $n \in N$ such that $\alpha^n(u) = gu$ and $\alpha^n(v) = v$ where $g$ is a unitary element in $C^*(v)$. By Proposition 4, $\alpha$ satisfies the assumptions of Lemma 5. Therefore we obtain the conclusion. Q.E.D.

For any automorphism $\alpha$ of a $C^*$-algebra we denote the Connes spectrum by $\Gamma(\alpha)$.

Corollary 7. Let $\alpha \in \text{Aut}(A_{\theta})$ with $\Gamma(\alpha) \subseteq T$. Then there are an AF-algebra $B(\alpha)$, a monomorphism $\rho_{a}$ of $A_{\theta}$ into $B(\alpha)$ and a unitary element $w_{a} \in B(\alpha)$ such that $\rho_{a}(\alpha(x)) = w_{a}\rho_{a}(x)w_{a}^*$ for any $x \in A_{\theta}$.

Proof. By Pimsner and Voiculescu [7] we have a monomorphism $\rho$ of $A_{\theta}$ into an AF-algebra $B_{\theta}$. And since $\Gamma(\alpha) \subseteq T$, there are an $n \in N$ and a unitary element $z \in A_{\theta}$ such that $\alpha^n = \text{Ad}(z)$ and $\alpha(z) = z$. Hence $\rho(\alpha^n(x)) = \rho(z)\rho(x)\rho(z)^*$ for any $x \in A_{\theta}$. Thus $\alpha$ satisfies the assumptions of Lemma 5. Therefore we obtain the conclusion. Q.E.D.
Let $\tilde{u}$ and $\tilde{v}$ be generators of $C(T^2)$. Then $\tilde{u}$ and $\tilde{v}$ are generators of $C(T^2)$, for any $g$ in $SL(2, \mathbb{Z})$ let $\tilde{\beta}_g = \text{Aut}(C(T^2))$ be defined by $\tilde{\beta}_g(\tilde{u}) = \tilde{u}^g$ and $\tilde{\beta}_g(\tilde{v}) = \tilde{v}^g$ where $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $a, b, c, d \in \mathbb{Z}$. We note that $\tilde{\beta}_g$ is induced by a toral automorphism of $T^2$. For any $s, t \in \mathbb{R}$ let $\tilde{\beta}_{(s,t)} \in \text{Aut}(C(T^2))$. For any $s, t \in \mathbb{R}$ let $\tilde{\beta}_{(s,t)} \in \text{Aut}(C(T^2))$ be defined by $\tilde{\beta}_{(s,t)}(\tilde{u}) = e^{2\pi is} \tilde{u}$ and $\tilde{\beta}_{(s,t)}(\tilde{v}) = e^{2\pi it} \tilde{v}$. Then we have the following lemma:

**Lemma 8.** With the above notations the crossed product $C(T^2) \times_{\alpha} \mathbb{Z}$ is quasidiagonal where $\alpha = \tilde{\beta}_s \circ \tilde{\beta}_{(s,t)}$.

**Proof.** Let $\mu$ be the Haar measure of $T^2$ with $\mu(T^2) = 1$ and let $tr$ be a faithful finite trace of $C(T^2)$ defined by $tr(x) = \int_{T^2} x \, d\mu$ for any $x \in C(T^2)$. Since $\mu$ is two sided invariant and $\tilde{\beta}_g$ is induced by a toral automorphism of $T^2$ leaving $\mu$ fixed, $tr(\tilde{\beta}_g(x)) = tr(x)$ for any $x \in C(T^2)$. Hence if $\widetilde{tr}$ is defined by $\widetilde{tr}(y) = tr(y(0))$ for $y \in \mathcal{L}(C(T^2))$, $\widetilde{tr}$ is a faithful finite trace of $C(T^2) \times_{\alpha} \mathbb{Z}$. Thus $C(T^2) \times_{\alpha} \mathbb{Z}$ is quasidiagonal by Lemma 3. Q.E.D.

**Proposition 9.** With the above notations let $\alpha = \beta_s \circ \beta_{(s,t)} \in \text{Aut}(A_\theta)$ where $s, t \in \mathbb{R}$ and $g \in G$. Then there are an AF-algebra $B(\alpha)$, a monomorphism $\rho_\alpha$ of $A_\theta$ into $B(\alpha)$ and a unitary element $w_\alpha \in B(\alpha)$ such that $\rho_\alpha(\alpha(x)) = w_\alpha \rho_\alpha(x) w_\alpha^*$ for any $x \in A_\theta$.

**Proof.** By Corollaries 6 and 7 we can assume that $t \in Q$ and $\Gamma(\alpha) = T$. Since $g \in G$, there is an $n \in \mathbb{Z}$ such that $g = \begin{pmatrix} 1 & 0 \\ n & 1 \end{pmatrix}$. Let $\gamma \in \text{Aut}(A_\theta)$ be defined by $\gamma(u) = e^{2\pi is} u^n v^n = e^{2\pi i(s - n\theta)} u^n$ and $\gamma(v) = v$. Then there are an AF-algebra $B(\gamma)$ and a monomorphism $\rho_\gamma$ of $A_\theta \times_{\gamma} \mathbb{Z}$ into $B(\gamma)$ by Proposition 4. Let $u, v$ and $w$ be generators of $A_\theta \times_{\gamma} \mathbb{Z}$ with $uv = e^{2\pi i s} uv$ and $\gamma(w) = Ad(w)$. Let $\tilde{\gamma} \in \text{Aut}(C(T^2))$ be defined by $\tilde{\gamma} = \tilde{\beta}_s \circ \tilde{\beta}_{(0,t)}$, i.e., $\tilde{\gamma}(u) = \tilde{u}^n$ and $\tilde{\gamma}(v) = e^{2\pi it} \tilde{v}$. Then by Lemma 8 and Pimsner [5, Theorem 9] there are an AF-algebra $B(\tilde{\gamma})$ and a monomorphism $\rho_{\tilde{\gamma}}$ of $C(T^2) \times_{\tilde{\gamma}} \mathbb{Z}$ into $B(\tilde{\gamma})$. Let $\tilde{u}$, $\tilde{v}$ and $\tilde{w}$ be generators of $C(T^2) \times_{\tilde{\gamma}} \mathbb{Z}$ with $\tilde{w} = v^n u^n$ and $\tilde{\gamma}(v) = Ad(\tilde{w})$, and let $u_\alpha$, $v_\alpha$ and $w_\alpha$ be generators of $A_\theta \times_{\alpha} \mathbb{Z}$ with $v_\alpha u_\alpha = e^{2\pi i s} u_\alpha v_\alpha$ and $\alpha = Ad(\tilde{w})$. We define a homomorphism $\rho_\alpha$ of $A_\theta \times_{\alpha} \mathbb{Z}$ into $B(\gamma) \otimes B(\tilde{\gamma})$ as follows:

$$
\rho_\alpha(u_\alpha) = \rho_\gamma(u) \otimes \rho_{\tilde{\gamma}}(\tilde{u})
$$

$$
\rho_\alpha(v_\alpha) = \rho_\gamma(v) \otimes \rho_{\tilde{\gamma}}(\tilde{v})
$$

$$
\rho_\alpha(w_\alpha) = \rho_\gamma(w) \otimes \rho_{\tilde{\gamma}}(\tilde{w})
$$

Then we can easily see that.
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$p_{\alpha}(v_{\alpha})\rho_{\alpha}(u_{\alpha})=e^{2\pi i\theta}\rho_{\alpha}(u_{\alpha})p_{\alpha}(v_{\alpha})$,

$\rho_{\alpha}(w_{\alpha})p_{\alpha}(u_{a})p_{\alpha}(w_{\alpha})^{*}=e^{2\pi i\epsilon}\rho_{\alpha}(u_{\alpha})\rho_{\alpha}(v_{\alpha})^{n}$,

and

$\rho_{\alpha}(w_{\alpha})\rho_{\alpha}(v_{\alpha})\rho_{\alpha}(w_{\alpha})^{*}=e^{2\pi it}\rho_{\alpha}(v_{\alpha})$.

Hence the above definition of $\rho_{\alpha}$ is well defined. Since $\Gamma(\alpha)=T$ and $A_{\theta}$ is simple, $A_{\theta}\rtimes_{\alpha} \mathbb{Z}$ is simple. Thus $\rho_{\alpha}$ is injective. Q.E.D.

§ 4. The main theorem.

PROPOSITION 10. Let $\alpha \in \text{Aut}(A_{\theta})$ with $\alpha(u)=fu^{*}$ and $\alpha(v)=e^{2\pi it}v^{*}$ where $f$ is a unitary element in $C^{*}(v)$ and $t \in \mathbb{R}$. Then there are an AF-algebra $B(\alpha)$, a monomorphism $\rho_{\alpha}$ and a unitary element $w_{\alpha} \in B(\alpha)$ such that $\rho_{\alpha}(\alpha(x))=w_{\alpha}\rho_{\alpha}(x)w_{\alpha}^{*}$ for any $x \in A_{\theta}$.

PROOF. We have that $\alpha^{2}(u)\in C^{*}(v)u$ and $\alpha^{2}(v)=v$. Hence by Proposition 4 and Lemma 5 we obtain the conclusion. Q.E.D.

THEOREM 11. Let $\alpha \in \text{Aut}(A_{\theta})$ be defined by $\alpha(u)=e^{2\pi it}uv^{n}$ and $\alpha(v)=e^{2\pi it}v^{*}$, where $s$, $t \in \mathbb{R}$ and $n \in \mathbb{Z}$. Then for any unitary element $z$ in $A_{\theta}$, $\text{Ad}(z)\circ \alpha$ can be extended to an inner automorphism of an AF-algebra.

PROOF. By Propositions 9 and 10 this is clear. Q.E.D.

Before we state a corollary, we need some notations. Let $A_{\theta}^{\infty}$ be the dense $*$-subalgebra of all smooth elements of $A_{\theta}$ with respect to the canonical action of $T^{2}$ and let $A_{\theta}^{\infty}$ be the $*$-subalgebra of finite linear combinations of monomials in $u$ and $v$.

COROLLARY 12. Let $\alpha \in \text{Aut}(A_{\theta})$ be leaving invariant a canonical subalgebra isomorphic to $C(T)$. If $\theta$ has the generic Diophantine property and $\alpha(A_{\theta}^{\infty})=A_{\theta}^{\infty}$ or if $\alpha(A_{\theta}^{\infty})=A_{\theta}^{\infty}$, $\alpha$ can be extended to an inner automorphism of an AF-algebra.

PROOF. By Elliott [3] and Brenken [1] $\alpha$ satisfies the assumptions of Theorem 11. Hence we obtain the conclusion. Q.E.D.

References


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