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Weak Asymptotical Stability of Yang-Mills' Gradient Flow

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Introduction.

Let $J(\cdot)$ be a functional on some functional space X, and $u_0 \in X$ be a critical point of $J(\cdot)$, i.e., the solution of the variational problem

$$\operatorname{grad} J(u_0)=0 ,$$

where $-\operatorname{grad} J(\cdot)$ is the Euler-Lagrangian operator of $J(\cdot)$.

Concerning the variational problems, there are two important problems, the existence of critical points and their stability.

The classical Morse theory covers the analysis of the variational problems on finite-dimensional spaces. In differential geometry, we find several variational problems on infinite-dimensional spaces. For such problems in discussing the properties of a critical point, several authors study those of the corresponding gradient flow. The *gradient flow* $u(t)$ of $J(\cdot)$ with the initial value v is, if exists, a C^1-flow satisfying

$$\begin{cases} \dfrac{du(t)}{dt}=-\operatorname{grad} J(u(t)) & t\in(0, \infty) , \\ u(0)=v . \end{cases}$$

A typical variational problem in differential geometry is the harmonic map problem, i.e., that of critical maps of the energy integral defined on maps $f: M\to N$ between two Riemannian manifolds:

$$J(f)=E(f)=\frac{1}{2}\int_M |df|^2 *1 .$$

In 1964 Eells and Sampson showed the existence of harmonic maps by use of the gradient flow corresponding to the energy integral (the Eells-Sampson equation) in [2]. Recently, Naito [13] has clarified the

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relations between the stability of a harmonic map and the asymptotic behavior of the Eells-Sampson equation.

Concerning more general results on such relations we refer to the papers of Simon [15], Natio [14] and so on.

Another important variational problem in geometry is that of the Yang-Mills functional, which is given by the square integral of the curvature R^{∇} associated to a metric connection ∇ on a Riemannian vector bundle E over a Riemannian manifold M:

$$J(\nabla)=\mathscr{YM}(\nabla)=\frac{1}{2}\int_{M}|R^{\nabla}|^2 *1 .$$

In this paper, we discuss the asymptotical stability of some critical point for the Yang-Mills functional. We say that the critical point u_0 is *asymptotically stable* if there exists a neighborhood $U(u_0)$ of u_0 in X such that for any $v\in U(u_0)$ the gradient flow of $J(\cdot)$ with the initial value v exists and converges to u_0 as $t\to\infty$ in some topology.

Before stating our problem, we shortly discuss the variational problem for the Yang-Mills functional. Let ∇_0 be a fixed flat connection. It is well-known that every connection ∇ is uniquely expressed as

$$\nabla=\nabla_0+A ,$$

where A is an element of $\Omega^1(\mathfrak{g}_E)$ (for the definition of $\Omega^1(\mathfrak{g}_E)$, see §1). Then the Euler-Lagrange equation for the Yang-Mills functional, called the *Yang-Mills equation*, is written as

$$\delta^{\nabla_0}d^{\nabla_0}A+\delta^{\nabla_0}[A, A]-[A, d^{\nabla_0}A]-[A, [A, A]]=0 ,$$

where d^{∇_0} is the covariant exterior derivation operator of ∇_0 and δ^{∇_0} is its formal adjoint operator. The operator $\delta^{\nabla_0}d^{\nabla_0}$ is not uniformly elliptic. To recover the ellipticity, we impose farther the gauge condition

$$\delta^{\nabla_0}A=0 .$$

A gauge A satisfying this condition is called the *Coulomb gauge*. Under this condition, the Yang-Mills equation has uniform ellipticity, and one can use the argument in the framework of the elliptic partial differential equations. We refer to [16]-[18] and the references cited therein for the information of such gauges.

A similar situation occurs in studying Yang-Mills' gradient flow, that is, the system of equations defining the flow is not uniformly parabolic. However, it seems difficult for the authors to show the existence of the gradient flow satisfying the condition $\delta^{\nabla_0}A=0$.

Therefore in this paper we impose a different condition (2.2) on A which is weaker than $\delta^{\nabla_0}A=0$ and reduce the system to that of the semi-linear heat equations. We shall find easily that if there exists a stationary flow satisfying condition (2.2), then the flow gives a Coulomb gauge. This approach is due to Yokotani's paper [21] in which he proved the local existence of Yang-Mills' gradient flow.

By use of the standard technique for the semi-linear heat equations, we shall show the weak asymptotical stability of some critical points of the Yang-Mills functional. The meaning of the word "weak" will be clarified in §1.

§1. Main result.

First we introduce terminology used in our paper (basically we follow the notation in [10]). Let (M, g) be a smooth n-dimensional Riemannian manifold, where $n\geqq 2$. Suppose that $(E, \langle\ ,\ \rangle)$ is a Riemannian vector bundle over (M, g) of rank m. We denote the space of all smooth metric connections on E by $\mathscr{C}$. For $\nabla\in\mathscr{C}$ we can define a naturally induced connection on $\mathrm{Hom}(E, E)\cong E^*\otimes E$ in a canonical way. Namely, for $\nabla\in\mathscr{C}$ and a section $L\in\mathrm{Hom}(E, E)$, we define $\nabla(L)$ by

$$\nabla(L)(\varphi)=\nabla(L\varphi)-L(\nabla\varphi) \qquad \text{for any } \varphi\in\Gamma(E)\ .$$

The $\mathrm{Hom}(E, E)$-valued 2-form R^{∇} defined as follows is called the *curvature* of a connection ∇:

$$R^{\nabla}_{V,W}=\nabla_V\nabla_W-\nabla_W\nabla_V-\nabla_{[V,W]}$$

for any smooth vector field V, W on M. G_E and $\mathfrak{g}_E$ denote the bundles defined by

$$G_E=\{L\in\mathrm{Hom}(E, E)\,;\ \langle L\varphi, L\psi\rangle=\langle\varphi, \psi\rangle \text{ for all } \varphi, \psi\in E\}\ ,$$
$$\mathfrak{g}_E=\{L\in\mathrm{Hom}(E, E)\,;\ \langle L\varphi, \psi\rangle=-\langle\varphi, L\psi\rangle \text{ for all } \varphi, \psi\in E\}\ .$$

$\mathscr{G}$ and $\mathscr{Y}$ are spaces of all smooth sections of G_E and $\mathfrak{g}_E$ respectively. $g\in\mathscr{G}$ acts on $\nabla\in\mathscr{C}$ in the following way:

$$g(\nabla)=g\nabla g^{-1}\ .$$

DEFINITION 1.1. The *Yang-Mills functional* $\mathscr{YM}\colon\mathscr{C}\to[0, \infty]$ is given by

$$\mathscr{YM}(\nabla)=\frac{1}{2}\|R^{\nabla}\|_2^2=\frac{1}{2}\int_M\langle R^{\nabla}, R^{\nabla}\rangle_x\ .$$

REMARK 1. It is obvious that the Yang-Mills functional is gauge-invariant, i.e.,

$$\mathcal{YM}(\nabla)=\mathcal{YM}(g(\nabla)) \qquad \text{for} \quad \nabla \in \mathcal{C}\,, \; g \in \mathcal{G}\,.$$

Let $\Omega_0^1(\mathfrak{g}_E)$ be the subset of $\Omega^1(\mathfrak{g}_E)$ consisting of all elements with compact support. By direct calculation we find that if $\mathcal{YM}(\nabla)<\infty$, then for $\nabla^\varepsilon=\nabla+\varepsilon A$, $A \in \Omega_0^1(\mathfrak{g}_E)$,

$$\frac{d}{d\varepsilon}\mathcal{YM}(\nabla^\varepsilon)\Big|_{\varepsilon=0}=\int_M \langle R^\nabla, d^\nabla A\rangle_x=\int_M \langle \delta^\nabla R^\nabla, A\rangle_x\,.$$

Keeping this in mind, we define grad $\mathcal{YM}(\nabla)$ by

$$\operatorname{grad}\mathcal{YM}(\nabla)=\delta^\nabla R^\nabla$$

even for ∇ with $\mathcal{YM}(\nabla)=\infty$.

DEFINITION 1.2. A connection $\nabla \in \mathcal{C}$ is called the *Yang-Mills connection*, if

$$\delta^\nabla R^\nabla=0$$

is satisfied.

Now we fix a base connection ∇_0. Let $\tilde{\Omega}_0^1(\mathfrak{g}_E)$ be the completion of $\Omega_0^1(\mathfrak{g}_E)$ by the topology of $W_0^{1,n}(M)$ and we define $\tilde{\mathcal{C}}$ by

$$\tilde{\mathcal{C}}=\{\nabla\,;\; \nabla=\nabla_0+A,\; A \in \tilde{\Omega}_0^1(\mathfrak{g}_E)\}\,.$$

The completion of $\mathcal{G}$ by the topology of $L^\infty(M)$ is denoted by $\tilde{\mathcal{G}}$. We can define the above operations for the element ∇ in $\tilde{\mathcal{C}}$ and the action of the element g in $\tilde{\mathcal{G}}$ on $\tilde{\mathcal{C}}$ in the generalized sense.

In the following we restrict ourselves to the case where M is the Euclidean space $\boldsymbol{R}^n$ or a bounded domain $\Omega\subset\boldsymbol{R}^n$ with smooth boundary, where $n\geqq 2$. Suppose E be the trivial Riemannian vector bundle over (M, g_0) of rank m, where g_0 is the standard metric on $\boldsymbol{R}^n$. We denote by ∇_0 a canonical flat connection determined by the trivialization of the bundle E. Clearly ∇_0 is a Yang-Mills connection since it is flat. Then the connection ∇_0 is weakly asymptotically stable in the following sense:

THEOREM 1. *There exists a neighborhood $U(\nabla_0)$ of ∇_0 in $\tilde{\mathcal{C}}$ such that for any $\nabla \in U(\nabla_0)$ there exist a C^1-curve $g(t)$ in $\tilde{\mathcal{G}}$ with $g(0)=\mathrm{id.}$, and a C^1-flow $\nabla(t)$ which satisfy $g^{-1}(t)\nabla(t)g(t) \in \tilde{\mathcal{C}}$ and*

$$(1.1)\qquad \begin{cases} \dfrac{d\nabla(t)}{dt} = -\mathrm{grad}\,\mathscr{YM}(\nabla(t)) & t \in (0, \infty)\ , \\ \nabla(0) = \nabla\ , \end{cases}$$

$$(1.2)\qquad \lim_{t\to\infty} g^{-1}(t)\nabla(t)g(t) = \nabla_0 \quad \textit{in} \quad L^p(M) \qquad \textit{for} \quad n < p \leqq \infty\ .$$

REMARK 2. The rate of convergence (1.2) is evaluated by

$$\|g^{-1}(t)\nabla(t)g(t) - \nabla_0\|_{L^p(M)} \leqq C(n, p)t^{-(1-n/p)/2} \qquad \text{for} \quad n < p \leqq \infty\ .$$

This estimate is valid even for $p = n$.

REMARK 3. The stability of ∇_0 is "weak" in the following sense:
(1) Uniqueness of the gradient flow is uncertain.
(2) Large-time behavior of $g(t)$ is unknown.
(3) It is uncertain that $\nabla(t)$ is a flow in $\tilde{\mathscr{C}}$. We interpret the right-hand side of the first equation of (1.1) as

$$-\mathrm{grad}\,\mathscr{YM}(\nabla(t)) = -\mathrm{grad}\,\mathscr{YM}(g^{-1}(t)\nabla(t)g(t))$$

by virtue of Remark 1.

§2. Reduction of the proof of Theorem 1.

In this section, we reduce the assertions in Theorem 1 to the stability problem of the system of the semi-linear heat equations.

We shall seek $\nabla(t)$ in the form

$$\nabla(t) = g(t)\tilde{\nabla}(t)g^{-1}(t)\ .$$

By simple calculations, the equation (1.1) takes the form

$$(2.1)\qquad \frac{d\tilde{\nabla}(t)}{dt} - d^{\tilde{\nabla}(t)}Y(t) = -\delta^{\tilde{\nabla}(t)}R^{\tilde{\nabla}(t)}\ ,$$

where

$$Y(t) = g^{-1}(t)\frac{dg(t)}{dt}\ .$$

The right-hand side of (2.1) is degenerate. To avoid this difficulty, we utilize Yokotani's idea [21], i.e., we assume that $Y(t)$ satisfies the gauge condition

$$(2.2)\qquad Y(t) = g^{-1}(t)\frac{dg(t)}{dt} = -\delta^{\nabla_0}A(t)\ ,$$

where

$$A(t)=\tilde{\nabla}(t)-\nabla_0=g^{-1}(t)\nabla(t)g(t)-\nabla_0$$

is a Hom(E, E)-valued 1-form vanishing on ∂M if $\partial M\neq\emptyset$. As we stated in Introduction, we can say that condition (2.2) is weaker than the Coulomb gauge condition.

Under the condition (2.2) we find that $A(t)$ must satisfy the system of semi-linear heat equations

$$\text{(2.3)} \qquad \begin{cases} \dfrac{dA(t)}{dt}=\Delta A(t)+F_1(A, \partial A)+F_2(A)\ , \\ A(t)|_{\partial M}=0 \qquad \text{if} \quad \partial M\neq\emptyset\ , \end{cases}$$

where the nonlinear terms F_1 and F_2 are respectively polynomials of order

$$\text{(2.4)} \qquad \begin{cases} F_1(A, B)\sim \text{const. } AB\ , \\ F_2(A)\sim \text{const. } A^3\ . \end{cases}$$

We should refer to [21] for the detail derivation of (2.3) and (2.4).

In §3 we shall prove

THEOREM 2. (i) *Let b be in $L^n(M)$. Then there exists a positive constant λ such that if $\|b\|_n<\lambda$ then there exists a unique solution $a(t)\in W_0^{1,n}(M)\cap W^{2,n}(M)$ for $t>0$ to*

$$\text{(2.5)} \qquad \begin{cases} a_t=\Delta a+F_1(a, \partial a)+F_2(a) \qquad on \quad M\ , \\ a(0)=b\ , \\ a|_{\partial M}=0 \qquad if \quad \partial M\neq\emptyset\ , \end{cases}$$

such that

$$\text{(2.6)} \qquad \begin{cases} t^{(1-n/p)/2}a\in BC([0, \infty);\ L^p(M)) \qquad for \quad n\leqq p<\infty\ , \\ t^{(1-n/(2q))}\partial a\in BC([0, \infty);\ L^q(M)) \qquad for \quad n\leqq q<\infty\ , \end{cases}$$

with values zero at $t=0$ except for the case $p=n$ in which $a(0)=b$. Moreover $a(t)$ belongs to $C^0([0, \infty); L^n(M))\cap C^1((0, \infty); L^n(M))$.

(ii) *We assume the hypothesis in* (i) *and $b\in W_0^{1,n}(M)$. Then the solution $a(t)$ constructed as above satisfies*

$$\text{(2.7)} \qquad \begin{cases} t^{1/2}\partial a\in BC([0, \infty);\ L^\infty(M)) \qquad for \quad M=\boldsymbol{R}^n\ , \\ t^{1/4+\beta/2}\partial a\in BC([0, \infty);\ L^\infty(M)) \qquad for \quad M=\Omega \quad (1/2<\beta<1)\ . \end{cases}$$

Moreover $a(t)$ belongs to $C^0([0, \infty); W_0^{1,n}(M))\cap C^1((0, \infty); W_0^{1,n}(M))$.

Applying Theorem 2 to (2.3), we find that under the hypotheses of Theorem 2 there exists an $A(t)$ satisfying (2.3) uniquely. It is clear that $\tilde{\nabla}(t)=\nabla_0+A(t)\in\tilde{\mathscr{C}}$. Besides Theorem 2 (ii) asserts the existence of a positive constant C_0 such that

$$\text{(2.8)}\qquad \begin{cases} \|u\delta^{\nabla_0}A(t)\|_{L^\infty(M)}\leqq C_0t^{-1/2}\|u\|_{L^\infty(M)} & \text{for } M=\boldsymbol{R}^n\,, \\ \|u\delta^{\nabla_0}A(t)\|_{L^\infty(M)}\leqq C_0(\beta)t^{-1/4-\beta/2}\|u\|_{L^\infty(M)} & \text{for } M=\Omega\quad (1/2<\beta<1)\,. \end{cases}$$

Next we seek $g(t)\in\tilde{\mathscr{G}}$ satisfying (2.2) and $g(0)=\mathrm{id}$. If we put

$$u(t)=g(t)-\mathrm{id.}\,,$$

then $u(t)$ must satisfy the system of integral equations

$$u(t)=-\int_0^t u(s)\delta^{\nabla_0}A(s)ds-\int_0^t\delta^{\nabla_0}A(s)ds\,.$$

Keeping in mind the estimates (2.8), we shall establish the following existence theorem in §4.

THEOREM 3. *Let $B(t)$ be a one-parameter $k\times k$-matrix-valued function on M which satisfies*

$$\text{(2.9)}\qquad \|uB(t)\|_{L^\infty(M)}\leqq Ct^{-\gamma}\|u\|_{L^\infty(M)}\qquad \textit{for } k\times k\textit{-matrix } u$$

for some $\gamma\in(0,1)$. Then there exists a unique solution to the system

$$\text{(2.10)}\qquad u(t)=-\int_0^t u(s)B(s)ds-\int_0^t B(s)ds$$

in $u(t)\in C^0([0,\infty);L^\infty(M))\cap C^1((0,\infty);L^\infty(M))$.

This theorem yields the existence of $g(t)$. It is obvious that $g(t)\in\tilde{\mathscr{G}}$.

The L^p-estimate ($n\leqq p<\infty$) in Remark 2 of §1 follows from (2.6). The L^∞-estimate follows from (2.6) and Gagliado-Nirenberg's inequality [11]:

$$\begin{aligned}\|g^{-1}(t)\nabla(t)g(t)-\nabla_0\|_{L^\infty(M)}&=\|A(t)\|_{L^\infty(M)}\\ &\leqq C(n)\|\partial A(t)\|^{1/2}_{L^{2n}(M)}\|A(t)\|^{1/2}_{L^{2n}(M)}\qquad \text{for } M=\boldsymbol{R}^n\,,\end{aligned}$$

$$\begin{aligned}\|g^{-1}(t)\nabla(t)g(t)-\nabla_0\|_{L^\infty(M)}&=\|A(t)\|_{L^\infty(M)}\\ &\leqq C(n)\|A(t)\|^{1/2}_{W^{1,2n}(M)}\|A(t)\|^{1/2}_{L^{2n}(M)}\qquad \text{for } M=\Omega\,.\end{aligned}$$

In the case of $M=\Omega$, the $W^{1,2n}(M)$-norm of $A(t)$ is majorized by the $L^{2n}(M)$-norm of $\partial A(t)$ because Ω is bounded and $A(t)$ vanishes on the boundary, [6, (7.44)]. Thus the convergence (1.2) follows.

Consequently Theorems 2 and 3 give Theorem 1.

§3. Proof of Theorem 2.

In the remainder of this paper, for the sake of notational simplicity we denote the norms of $L^p(M)$ and $W^{1,p}(M)$ by $\|\cdot\|_p$ and $\|\cdot\|_{1,p}$ respectively. The symbol C denotes a generic constant with various values even in the same sentence.

In this section we shall prove Theorem 2 by a method similar to Kato [9].

Let $\Delta=\sum_{i=1}^n \partial_i^2$ be the Laplace operator in $L^p(M)$ $(1<p<\infty)$ with the domain $\mathscr{D}(\Delta)=W_0^{1,p}(M)\cap W^{2,p}(M)$. The following lemma is well-known:

LEMMA 3.1. *Δ generates a strongly continuous semigroup $\{e^{t\Delta}\}_{t\geqq 0}$ simultaneously on all $L^p(M)$ $(1<p<\infty)$, and satisfies*

$$\begin{cases} \|e^{t\Delta}a\|_\beta \leqq C(\alpha,\beta,n)t^{-(1/\alpha-1/\beta)n/2}\|a\|_\alpha & (1<\alpha\leqq\beta<\infty)\,, \\ \|\partial e^{t\Delta}a\|_\beta \leqq C(\alpha,\beta,n)t^{-(1+n/\alpha-n/\beta)/2}\|a\|_\alpha & (1<\alpha\leqq\beta<\infty)\,. \end{cases}$$

PROOF. We only give a simple proof of the above estimates. The estimates for $M=\boldsymbol{R}^n$ is a consequence of an application of Young's convolution inequality to a Gaussian kernel and its gradient (see [9]).

For $M=\Omega$, the first estimate follows from the one for $M=\boldsymbol{R}^n$ and the maximum principle (see [20]). Because Ω is bounded, the Laplace operator satisfies

$$\|\Delta e^{t\Delta}\|\leqq Ct^{-1}$$

(see [7]). The second estimate of the lemma is given by the first one and the above one using the interpolation argument (see [19]). □

If $a(t)$ is a solution to (2.5) satisfying (2.6), then it is easy to see that $a(t)$ is also a solution to the integral equation

$$\text{(3.1)} \qquad \begin{cases} a=a_0+S_1a+S_2a\,, \\ a_0=e^{t\Delta}b\,, \\ S_1a=\displaystyle\int_0^t e^{(t-s)\Delta}F_1(a(s),\partial a(s))ds\,, \\ S_2a=\displaystyle\int_0^t e^{(t-s)\Delta}F_2(a(s))ds \end{cases}$$

satisfying (2.6).

Conversely let $a(t)$ be a solution to (3.1) satisfying (2.6). Let A be

$\lambda-\Delta$, where $\lambda=1$ for $M=\boldsymbol{R}^n$, $\lambda=0$ for $M=\Omega$. A is a sectional operator satisfying $\Re\sigma(A)>\delta>0$, and $-A$ generates a strongly continuous semigroup $\{e^{-tA}\}_{t\geq 0}$ in $L^p(M)$. Then (3.1) is equivalent to

$$e^{-\lambda t}a(t)=e^{-tA}b+\int_0^t e^{-(t-s)A}e^{-\lambda t}\{F_1(a(s),\ \partial a(s))+F_2(a(s))\}ds\ .$$

By virtue of [7, Theorem 1.4.3], we can define the fractional power A^α $(0<\alpha\leqq 1)$ of A satisfying

$$\|(e^{-tA}-1)a\|_p\leqq C(\alpha,\ p)t^\alpha\|A^\alpha a\|_p\ .$$

Since nonlinear terms $F_1(a(t),\ \partial a(t))$ and $F_2(a(t))$ are polynomial order of their arguments, we can show their local Hölder continuity for $t>0$ by use of the above estimate and (2.6) in a similar manner to [5, Proposition 2.4]. An application of [8, Theorem 1.27] gives the fact that $e^{-\lambda t}a(t)$ belongs to $\mathscr{D}(A)$ for $t>0$, which is equivalent to $a(t)\in\mathscr{D}(\Delta)$ for $t>0$, and satisfies (2.5). Moreover $a(t)\in C^0([0,\ \infty);\ L^n(M))\cap C^1((0,\ \infty);\ L^n(M))$.

Therefore the equation (2.5) is converted into (3.1). Hence we shall construct the solution to (3.1). First we show

PROPOSITION 3.1. *Let r be a fixed index satisfying $(3/2)n<r<3n$. Under the hypotheses on Theorem 2 (i), there exists a unique solution $a=a(t)$ to (3.1) such that*

$$t^{(1-n/r)/2}a\in BC([0,\ \infty);\ L^r(M))\ ,\qquad t^{1/2}\partial a\in BC([0,\ \infty);\ L^n(M))$$

with

$$t^{(1-n/r)/2}a(t)|_{t=0}=t^{1/2}\partial a(t)|_{t=0}=0\ .$$

We shall prove the existence of the solution via successive approximation

$$\begin{cases}a_0=e^{t\Delta}b\ ,\\ a_{m+1}=a_0+S_1a_m+S_2a_m\ , \qquad m=0,\ 1,\ 2,\ \cdots\ .\end{cases}$$

Since the proof of Proposition 3.1 is lengthy, we divide it into 4 lemmas.

LEMMA 3.2. *Let K_m, K_m' and $\overline{K_m}$ be defined by*

$$K_m\equiv\sup_{t>0}t^{(1-n/r)/2}\|a_m\|_r\ ,\quad K_m'\equiv\sup_{t>0}t^{1/2}\|\partial a_m\|_n\ ,\quad \overline{K_m}=\max\{K_m,\ K_m'\}\ .$$

Then there exist positive constants C_1 and C_2 such that

$$\begin{cases}\overline{K_0}=C_1\|b\|_n\ ,\\ \overline{K_{m+1}}=\overline{K_0}+C_2\overline{K_m}^2(1+\overline{K_m})\ .\end{cases}$$

Proof. By virtue of Lemma 3.1, we have

$$\|a_0\|_r \leqq Ct^{-(1-n/r)/2}\|b\|_n , \qquad \|\partial a_0\|_n \leqq Ct^{-1/2}\|b\|_n . \tag{3.2}$$

Therefore the existence of C_1 is obvious.

Lemma 3.1 and (2.4) yield

$$\begin{aligned}\|S_1 a_m(t)\|_r &\leqq C\int_0^t (t-s)^{-1/2}\|a_m(s)\|_r \|\partial a_m(s)\|_n ds\\ &\leqq CK_m K_m' \int_0^t (t-s)^{-1/2} s^{-(2-n/r)/2} ds\\ &\leqq CK_m K_m' t^{-(1-n/r)/2} ,\end{aligned}$$

which gives

$$\sup_{t>0} t^{(1-n/r)/2}\|S_1 a_m\|_r \leqq CK_m K_m' , \tag{3.3}$$

where C is a positive constant independent of m. In a similar way, we can show

$$\sup_{t>0} t^{(1-n/r)/2}\|S_2 a_m\|_r \leqq CK_m^3 . \tag{3.4}$$

Differentiating (3.1) and applying Lemma 3.1 again, we have

$$\sup_{t>0} t^{1/2}\|\partial S_1 a_m\|_n \leqq CK_m K_m' , \qquad \sup_{t>0} t^{1/2}\|\partial S_2 a_m\|_n \leqq CK_m^3 . \tag{3.5}$$

Here we use $(3/2)n<r<3n$. Combining (3.2)–(3.5), we know the existence of C_2. □

We may assume $C_2>1/4$. Then we have

Lemma 3.3. *If $\|b_n\|<(8C_1C_2)^{-1}$ holds, then there exists a positive constant K such that $\overline{K_m}<K$ holds for all m.*

Proof. Put

$$K=\frac{1-\sqrt{1-8C_2\overline{K_0}}}{4C_2} ; \tag{3.6}$$

By the assumption, K is one of the positive roots for

$$K = \overline{K_0}+2C_2K^2$$

and satisfies $K<1$. Consequently the assertion is proved by induction on m. □

We can replace the supremum taken over $t>0$ in the definition of K_m and K_m' by the supremum over $t\geqq 0$. Indeed the following lemma holds.

LEMMA 3.4. *$t^{(1-n/r)/2}a_m(t)$ and $t^{1/2}\partial a_m(t)$ are continuous at $t=0$ with values zero in the topology of $BC([0, \infty); L^r(M))$ and $BC([0, \infty); L^n(M))$ respectively.*

PROOF. Define

$$\overline{K_m}(t)\equiv\max\{\sup_{0<\tau\leqq t}\tau^{(1-n/r)/2}\|a_m\|_r,\ \sup_{0<\tau\leqq t}\tau^{1/2}\|\partial a_m\|_n\} .$$

In the same way as Lemma 3.2 we can show

$$\overline{K_{m+1}}(t)\leqq\overline{K_0}(t)+2C\overline{K_m}^2(t) .$$

Therefore to prove the assertion, it suffices to show

$$\overline{K_0}(t)\to 0 \qquad \text{as} \quad t\to 0 .$$

This fact is a direct consequence of the facts that the operator $t^{(1-n/r)/2}e^{t\Delta}$ is uniformly bounded from $L^n(M)$ to $L^r(M)$ and tends to zero strongly as $t\to 0$, and similarly for $t^{1/2}\partial e^{t\Delta}$ from $L^n(M)$ to itself. We can show these facts by the density of $C_0^\infty(M)$ in $L^n(M)$. □

The above three lemmas show that $\{t^{(1-n/r)/2}a_m\}$ and $\{t^{1/2}\partial a_m\}$ are bounded subsets in $BC([0, \infty); L^r(M))$ and $BC([0, \infty); L^n(M))$ respectively. To show Proposition 3.1 it remains to prove the uniform convergence of the sequences and the uniqueness of $a(t)$. These assertions follow from

LEMMA 3.5. *Let Φ be defined by*

$$\Phi(u)\equiv a_0+S_1u+S_2u ,$$

and X be

$$X\equiv\{u;\ |||u|||\leqq K<1\} ,$$

where

$$|||u|||\equiv\max\{\sup_{t\geqq 0}t^{(1-n/r)/2}\|u\|_r,\ \sup_{t\geqq 0}t^{1/2}\|\partial u\|_n\} .$$

Then Φ is contractive on X, i.e., for any $u, w\in X$,

$$|||\Phi(u)-\Phi(w)|||\leqq k|||u-w||| \qquad \textit{for some } k\in(0, 1)$$

holds.

PROOF. By use of (2.4), we have

$$\begin{aligned}\|(S_1u-S_1w)(t)\|_r &\leqq C\int_0^t (t-s)^{-1/2}\{\|u-w\|_r\|\partial u\|_n+\|w\|_r\|\partial u-\partial w\|_n\}ds\\ &\leqq CK|||u-w|||\int_0^t (t-s)^{-1/2}s^{-(1-n/(2r))}ds\\ &\leqq CK|||u-w|||t^{-(1-n/r)/2}\,.\end{aligned}$$

In a similar manner, we obtain

$$\begin{aligned}&\|S_2u-S_2w\|_r\leqq CK|||u-w|||t^{-(1-n/r)/2}\,,\\ &\|\partial S_iu-\partial S_iw\|_n\leqq CK|||u-w|||t^{-1/2}\qquad (i=1,\,2)\,.\end{aligned}$$

Here we use $K<1$. These yield

$$|||\Phi(u)-\Phi(w)|||\leqq 4C_3K|||u-w|||$$

for some $C_3>0$. Since C_3 is independent of C_2 in Lemma 3.2, we take C_2 greater than C_3 in advance. Then from (3.6) $4C_3K<1$ is valid. □

Now we complete the proof of Proposition 3.1.

Proposition 3.1 shows Theorem 2 (i) except (2.6). To show this we repeat an argument similar to that of the proof of Lemma 3.2.

PROPOSITION 3.2. *The solution* $a(t)$ *constructed in Proposition* 3.1 *satisfies* (2.6) *and*

$$\begin{aligned}&\|a(t)\|_p\leqq C(K)t^{-(1-n/p)/2}\qquad (n\leqq p<\infty)\,,\\ &\|\partial a(t)\|_q\leqq C(K)t^{-(1-n/(2q))}\qquad (n\leqq q<\infty)\,,\end{aligned}\tag{3.7}$$

where $C(K)$ *is a positive constant with the property*

$$C(K)\to 0\qquad as\quad K\to 0\,,$$

and K *is given by* (3.6).

PROOF. It is enough to show the above assertion for $a_m(t)$ instead of $a(t)$. Lemma 3.1 gives the assertion for $a_0(t)$. Since we have already known

$$\|a_m(t)\|_r\leqq C(K)t^{-(1-n/r)/2}\,,\qquad \|\partial a_m(t)\|_n\leqq C(K)t^{-1/2}$$

by Lemmas 3.2 and 3.3, where r is given in Proposition 3.1, we can show

$$\begin{aligned}\|S_1a_m(t)\|_p &\leqq C\int_0^t (t-s)^{-(1/r+1/n-1/p)n/2}\|a_m(s)\|_r\|\partial a_m(s)\|_n ds\\ &\leqq C(K)\int_0^t (t-s)^{-(1/r+1/n-1/p)n/2}s^{-1+n/(2r)}ds\end{aligned}$$

$$\leqq C(K)t^{-(1-n/p)/2} \qquad \text{for} \quad n \leqq p < \infty .$$

For a similar reason,

$$\|S_2 a_m(t)\|_p \leqq C(K)t^{-(1-n/p)/2} ,$$
$$\|\partial S_1 a_m(t)\|_q + \|\partial S_2 a_m(t)\|_q \leqq C(K)t^{-(1-n/(2q))}$$

hold.

The continuity up to $t=0$ is proved in the same way as in Lemma 3.4. □

Next we shall prove Theorem 2 (ii). First we consider the case when $M=\boldsymbol{R}^n$. By virtue of Gagliado-Nirenberg's inequality

$$\|a\|_\infty \leqq C\|\partial a\|_{2n}^{1/2}\|a\|_{2n}^{1/2} ,$$

we have to show

PROPOSITION 3.3. *Under the hypotheses on Theorem* 2 (ii), *there exists a positive constant* C_4 *such that*

$$\|\partial a\|_{2n} \leqq C_4 t^{-1/4} , \qquad \|\partial^2 a\|_{2n} \leqq C_4 t^{-3/4} .$$

PROOF. We shall show the above estimate for a_m instead of a. Since the operator ∂ can commute with $e^{t\Delta}$ when $M=\boldsymbol{R}^n$, $\{a_m\}$ satisfies

$$\partial a_{m+1}(t) = e^{t\Delta}\partial b + \int_0^t e^{(t-s)\Delta}\partial(F_1(a_m(s), \partial a_m(s)) + F_2(a_m(s)))ds . \tag{3.8}$$

Lemma 3.1 yields

$$\|e^{t\Delta}\partial b\|_{2n} \leqq Ct^{-1/4}\|\partial b\|_n .$$

Now we choose indices p, q, r such that

$$\frac{2}{3}n < p < n , \quad q \geqq n , \quad r \geqq n , \quad \frac{1}{p} = \frac{1}{q} + \frac{1}{2n} = \frac{2}{r} + \frac{1}{2n} .$$

It follows from (2.4) and Gagliado-Nirenberg's inequality that the estimates

$$\|\partial F_1(a_m, \partial a_m)\|_p \leqq C(\|\partial a_m\|_{2p}^2 + \|a_m\|_q \|\partial^2 a_m\|_{2n})$$
$$\leqq C\|a_m\|_q \|\partial^2 a_m\|_{2n} ,$$

$$\|\partial F_2(a_m)\|_p \leqq C\|\partial a_m\|_{2n}\|a_m\|_r^2$$

hold. Therefore by use of Lemma 3.1 and (3.7) for a_m (see the proof of

Proposition 3.2) we have

$$\begin{aligned}
&\left\|\int_0^t e^{(t-s)\Delta}\partial F_1(a_m(s),\ \partial a_m(s))ds\right\|_{2n}\\
&\qquad\leqq C\int_0^t (t-s)^{-(n/p-1/2)/2}\|a_m(s)\|_q\|\partial^2 a_m(s)\|_{2n}ds\\
&\qquad\leqq C(K)\sup_{\tau>0}\tau^{3/4}\|\partial^2 a_m(\tau)\|_{2n}\int_0^t (t-s)^{-(n/p-1/2)/2}s^{-(3-n/p)/2}ds\\
&\qquad\leqq C(K)t^{-1/4}\sup_{\tau>0}\tau^{3/4}\|\partial^2 a_m(\tau)\|_{2n}\ ,
\end{aligned}$$

$$\begin{aligned}
&\left\|\int_0^t e^{(t-s)\Delta}\partial F_2(a_m(s))ds\right\|_{2n}\\
&\qquad\leqq C\int_0^t (t-s)^{-(n/p-1/2)/2}\|\partial a_m(s)\|_{2n}\|a_m(s)\|_r^2 ds\\
&\qquad\leqq C(K)\sup_{\tau>0}\tau^{1/4}\|\partial a_m(\tau)\|_{2n}\int_0^t (t-s)^{-(n/p-1/2)/2}s^{-(3-n/p)/2}ds\\
&\qquad\leqq C(K)t^{-1/4}\sup_{\tau>0}\tau^{1/4}\|\partial a_m(\tau)\|_{2n}\ .
\end{aligned}$$

For a similar reason, we also have

$$\begin{aligned}
&\|\partial e^{t\Delta}\partial b\|_{2n}\leqq Ct^{-3/4}\|\partial b\|_{2n}\ ,\\
&\left\|\int_0^t \partial e^{(t-s)\Delta}\partial F_1(a_m(s),\ \partial a_m(s))ds\right\|_{2n}\leqq C(K)t^{-3/4}\sup_{\tau>0}\tau^{3/4}\|\partial^2 a_m(\tau)\|_{2n}\ ,\\
&\left\|\int_0^t \partial e^{(t-s)\Delta}\partial F_2(a_m(s))ds\right\|_{2n}\leqq C(K)t^{-3/4}\sup_{\tau>0}\tau^{1/4}\|\partial a_m(\tau)\|_{2n}\ .
\end{aligned}$$

Summing up these estimates we get

$$\begin{cases}
\sup_{t>0} t^{1/4}\|\partial a_0(t)\|_{2n}+\sup_{t>0} t^{3/4}\|\partial^2 a_0(t)\|_{2n}\leqq C\|\partial b\|_n\ ,\\
\sup_{t>0} t^{1/4}\|\partial a_{m+1}(t)\|_{2n}+\sup_{t>0} t^{3/4}\|\partial^2 a_{m+1}(t)\|_{2n}\\
\qquad\leqq C\|\partial b\|_n+C(K)\{\sup_{t>0} t^{1/4}\|\partial a_m(t)\|_{2n}+\sup_{t>0} t^{3/4}\|\partial^2 a_m(t)\|_{2n}\}\ ,\quad m=0,\ 1,\ 2,\cdots.
\end{cases}$$

Put

$$C_4=\frac{C\|\partial b\|_n}{1-C(K)}\ .$$

If $\|b\|_n$ is sufficiently small, then so is K, and C_4 is a positive constant. The assertion of the proposition follows by induction on m. □

Next we consider the case when $M=\Omega$. In this case we shall use the fractional power of $-\Delta$ (see [1, 3, 7]). We denote by $(-\Delta)^\alpha$ and $\mathscr{D}((-\Delta)^\alpha)$ the fractional power of $-\Delta$ of order α $(0<\alpha<1)$ and its domain respectively. The $\mathscr{D}((-\Delta)^\alpha)$-norm is defined by

$$\|a\|_{\mathscr{D}((-\Delta)^\alpha)}=\|(-\Delta)^\alpha a\|_n ,$$

which is equivalent to the graph norm $\|a\|_n+\|(-\Delta)^\alpha a\|_n$ for $\alpha>0$ and the bounded domain Ω. It is well-known that

$$\|(-\Delta)^\alpha e^{t\Delta}a\|_\beta \leqq C(\beta, \gamma, n)t^{-(2\alpha+n/\gamma-n/\beta)/2}\|a\|_\gamma \qquad (1<\gamma\leqq\beta<\infty) , \tag{3.9}$$

$$\|a\|_{k,p}\leqq C\|(-\Delta)^{k/2}a\|_p \tag{3.10}$$

hold, where $\|\cdot\|_{k,p}$ is the norm of the Bessel potential space $\mathscr{L}^{k,p}(M)$ (see [4]). We shall prove

PROPOSITION 3.4. *Under the hypotheses in Theorem* 2 (ii), *there exists a positive constant* C_5 *such that*

$$\|(-\Delta)^{1/2}a(t)\|_{2n}\leqq C_5 t^{-1/4} , \qquad \|(-\Delta)^{1/2+\alpha}a(t)\|_{2n}\leqq C_5 t^{-1/4-\alpha} \quad (0<\alpha<1/2) .$$

PROOF. It suffices to show the estimates for $a_m(t)$. Since $W_0^{1,n}(M)$ and $\mathscr{D}((-\Delta)^{1/2})$ coincide as vector spaces and carry equivalent norms ([1]), our hypotheses imply $(-\Delta)^{1/2}b\in L^n(M)$. The operators $(-\Delta)^{1/2}$ and $(-\Delta)^{1/2+\alpha}$ commute with $e^{t\Delta}$, and $\{a_m\}$ satisfies

$$\begin{aligned}(-\Delta)^{1/2}a_{m+1}(t)&=e^{t\Delta}(-\Delta)^{1/2}b\\&\quad+\int_0^t(-\Delta)^{1/2}e^{(t-s)\Delta}(F_1(a_m(s), \partial a_m(s))+F_2(a_m(s)))ds ,\end{aligned} \tag{3.11}$$

$$\begin{aligned}(-\Delta)^{1/2+\alpha}a_{m+1}(t)&=(-\Delta)^\alpha e^{t\Delta}(-\Delta)^{1/2}b\\&\quad+\int_0^t(-\Delta)^{1/2+\alpha}e^{(t-s)\Delta}(F_1(a_m(s), \partial a_m(s))+F_2(a_m(s)))ds .\end{aligned} \tag{3.12}$$

From Lemma 3.1 and (3.9) we have

$$\begin{aligned}&\|e^{t\Delta}(-\Delta)^{1/2}b\|_{2n}\leqq C_6 t^{-1/4}\|(-\Delta)^{1/2}b\|_n ,\\&\|(-\Delta)^\alpha e^{t\Delta}(-\Delta)^{1/2}b\|_{2n}\leqq C_6 t^{-1/4-\alpha}\|(-\Delta)^{1/2}b\|_n .\end{aligned}$$

It follows from (2.4), Hölder's and Gagliado-Nirenberg's inequalities, (3.10) and (3.7) for $a_m(t)$ that

$$\begin{aligned}\|F_1(a_m(s), \partial a_m(s))\|_n&\leqq C\|a_m(s)\|_{2n}\|\partial a_m(s)\|_{2n}\\&\leqq C(K)s^{-1/4}\|(-\Delta)^{1/2}a_m(s)\|_{2n} ,\end{aligned}$$

$$\begin{aligned}\|F_2(a_m(s))\|_{2n} &\leqq C\|a_m(s)\|_\infty^2\|a_m(s)\|_{2n}\\ &\leqq C\|a_m(s)\|_{2n}^2\|(-\Delta)^{1/2}a_m(s)\|_{2n}\\ &\leqq C(K)s^{-1/2}\|(-\Delta)^{1/2}a_m(s)\|_{2n}\end{aligned}$$

hold. Using the above estimates and (3.9) appropriately, we have

$$\begin{aligned}&\left\|\int_0^t(-\Delta)^{1/2}e^{(t-s)\Delta}F_1(a_m(s),\,\partial a_m(s))ds\right\|_{2n}\\ &\quad\leqq C(K)\sup_{\tau>0}\tau^{1/4}\|(-\Delta)^{1/2}a_m(\tau)\|_{2n}\int_0^t(t-s)^{-3/4}s^{-1/2}ds\\ &\quad\leqq C(K)t^{-1/4}\sup_{\tau>0}\tau^{1/4}\|(-\Delta)^{1/2}a_m(\tau)\|_{2n}\,,\end{aligned}$$

$$\begin{aligned}&\left\|\int_0^t(-\Delta)^{1/2}e^{(t-s)\Delta}F_2(a_m(s))ds\right\|_{2n}\\ &\quad\leqq C(K)\sup_{\tau>0}\tau^{1/4}\|(-\Delta)^{1/2}a_m(\tau)\|_{2n}\int_0^t(t-s)^{-1/2}s^{-3/4}ds\\ &\quad\leqq C(K)t^{-1/4}\sup_{\tau>0}\tau^{1/4}\|(-\Delta)^{1/2}a_m(\tau)\|_{2n}\,.\end{aligned}$$

Therefore we have

$$\sup_{t>0}t^{1/4}\|(-\Delta)^{1/2}a_{m+1}(t)\|_{2n}\leqq C_6\|(-\Delta)^{1/2}b\|_n+C(K)\sup_{t>0}t^{1/4}\|(-\Delta)^{1/2}a_m(t)\|_{2n}\,,$$

which yields the existence of $C_7>0$ such that

$$\sup_{t>0}t^{1/4}\|(-\Delta)^{1/2}a_m(t)\|_{2n}\leqq C_7$$

holds, if $\|b\|_n$ is sufficiently small.

By use of this estimate, (2.4), (3.7) for a_m we have

$$\begin{aligned}\|F_1(a_m(s),\,\partial a_m(s))\|_{2n} &\leqq C\|a_m(s)\|_\infty\|\partial a_m(s)\|_{2n}\\ &\leqq C(\|a_m(s)\|_\infty^2+1)\|\partial a_m(s)\|_{2n}\\ &\leqq C(\|(-\Delta)^{1/2}a_m(s)\|_{2n}^3+\|\partial a_m(s)\|_{2n})\\ &\leqq Cs^{-3/4}\,,\end{aligned}$$

$$\begin{aligned}\|F_2(a_m(s))\|_{2n} &\leqq C\|a_m(s)\|_{2n}^2\|(-\Delta)^{1/2}a_m(s)\|_{2n}\\ &\leqq C\|(-\Delta)^{1/2}a_m(s)\|_{2n}^3\\ &\leqq Cs^{-3/4}\,.\end{aligned}$$

From these estimates and (3.9),

$$\left\| \int_0^t (-\Delta)^{1/2+\alpha} e^{(t-s)\Delta}(F_1(a_m(s), \partial a_m(s)) + F_2(a_m(s))ds \right\|_{2n}$$
$$\leqq C\int_0^t (t-s)^{-(1/2+\alpha)} s^{-3/4} ds$$
$$\leqq Ct^{-1/4-\alpha}$$

follows. Thus we obtain

$$\|(-\Delta)^{1/2+\alpha} a_{m+1}(t)\|_{2n} \leqq C_8 t^{-1/4-\alpha} .$$

The assertion is valid for $C_5 = \max\{C_6, C_7, C_8\}$. □

Sobolev's imbedding theorem gives

$$\|\partial a\|_\infty \leqq C(\beta)\|\partial a\|_{\beta,2n} \leqq C(\beta)\|a\|_{1+\beta,2n} \qquad \text{if} \quad \beta > 1/2 .$$

Consequently Proposition 3.4 and (3.10) yield

$$\|\partial a\|_\infty \leqq C(\beta) t^{-1/4-\beta/2} \qquad \text{for} \quad 1/2 < \beta < 1 .$$

The above arguments show $t^{1/2}\partial a$ (for $M = \boldsymbol{R}^n$), $t^{1/4+\beta/2}\partial a$ (for $M = \Omega$) $\in L^\infty((0, \infty); L^\infty(M))$. The continuity of these functions on $[0, \infty)$ follows from the property of $e^{t\Delta}$.

Finally, repeating an argument similar to that of the paragraph just before Proposition 3.1, we get the facts that $b \in W_0^{1,n}(M)$ implies $a(t)$, $\partial a(t)$ (for $M = \boldsymbol{R}^n$), $(-\Delta)^{1/2} a(t)$ (for $M = \Omega$) $\in \mathscr{D}(\Delta)$ for $t > 0$ and that $a(t) \in C^0([0, \infty); W_0^{1,n}(M)) \cap C^1((0, \infty); W_0^{1,n}(M))$.

Thus we complete the proof of Theorem 2.

§4. Proof of Theorem 3.

We solve the system of equations (2.10) by successive approximation

$$u_0(t) = -\int_0^t B(s)ds ,$$

$$u_{m+1}(t) = -\int_0^t u_m(s)B(s)ds + u_0(t) , \qquad m = 0, 1, 2, \cdots .$$

It follows from (2.9) that

$$\|u_0(t)\|_\infty \leqq C\int_0^t s^{-\gamma} ds = \frac{Ct^{1-\gamma}}{1-\gamma}$$

holds, where C is a positive constant in (2.9). Hence by induction on m, we have

$$\|(u_{m+1}-u_m)(t)\|_\infty \leqq \left(\frac{C}{1-\gamma}\right)^{m+2} \frac{t^{(m+2)(1-\gamma)}}{(m+2)!}, \qquad m=0, 1, 2, \cdots .$$

Consequently $\{u_m\}$ converges to u, and u satisfies

$$u_t(t)=-u(t)B(t)-B(t) ,$$

$$\|u(t)\|_\infty \leqq \exp\left\{\frac{Ct^{1-\gamma}}{1-\gamma}\right\}-1 ,$$

which imply $u \in C^0([0, \infty); L^\infty(M)) \cap C^1((0, \infty); L^\infty(M))$.

Next we prove the uniqueness. If u and v are solutions to (2.10), then

$$(u-v)(t)=-\int_0^t (u-v)(s)B(s)ds$$

holds. Using (2.9) we have

$$\|(u-v)(t)\|_\infty \leqq C \sup_{0\leqq\tau\leqq t} \|(u-v)(\tau)\|_\infty \int_0^t s^{-\gamma} ds . \tag{4.1}$$

Since there exists a $T>0$ such that

$$C\int_0^T s^{-\gamma} ds=\frac{1}{2} ,$$

we have $u\equiv v$ on $[0, T]$ from (4.1).

We assume that $[0, T^*)$ is the maximal interval on which $u\equiv v$ holds, and that $T^*<\infty$. Using a similar argument, we have

$$\|(u-v)(t)\|_\infty \leqq C \sup_{T^*\leqq\tau\leqq t} \|(u-v)(\tau)\|_\infty \int_{T^*}^t s^{-\gamma} ds \qquad \text{for} \quad t>T^* .$$

If $t-T^*$ is sufficiently small, then

$$C\int_{T^*}^t s^{-\gamma} ds<1 .$$

This contradicts the maximality of T^*.

Thus the proof of Theorem 3 is complete.

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