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# Strongly Stable Factors, Crossed Products and Property $\Gamma$

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## §1. Introduction.

The important problems in the theory of factors of type $II_1$ are how to construct factors of type $II_1$ and classify them. As methods of their constructions, we know the operations of taking crossed product and tensor product. As a method of their rough classification, we know the method by using centralizing sequences. When we classify them by using centralizing sequences, first of all, we have two classes: one is factors with property $\Gamma$ and the other is full factors. The class with property $\Gamma$ contains an important subclass of factors, which are said to be strongly stable. For instance, the hyperfinite factor of type $II_1$ is strongly stable. On the other hand, the group von Neumann algebra associated with the free group on 2 generators is full.

In general, it is a basic problem whether a structure property of a von Neumann algebra is compatible with the operations of taking crossed product and tensor product or not. For instance, it is known [11] and is an important result that the crossed product of the hyperfinite factor of type $II_1$ by a free action of an amenable group is also hyperfinite. In the first part of the present paper (§3), we treat the class of strongly stable factors from the above point of view. We shall show (Theorem 3.1 and Corollary 3.2) that the crossed product of a strongly stable factor by a free action of an amenable group is again strongly stable. We shall then consider, as a converse of this result, the problem under what conditions the assumption the crossed product $M\times_\alpha G$ of an action $\alpha$ of a group $G$ being strongly stable implies that the original algebra $M$ is strongly stable. The property of $G$ for the case is non inner amenable (Proposition 3.3). Moreover we shall refer to the operations of tensor product and show that if a tensor product $M\otimes R$ between factors of type $II_1$ is strongly stable and $R$ has Connes and Jones' property $T$, then $M$

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is strongly stable (Corollary 3.7).

In the second part of this paper (§4 and §5), we treat the class of factors with property $\Gamma$ which is larger than that of strongly stable factors. In [11], A. Connes has shown that for a factor $R$ of type $\mathrm{II}_1$ on the standard Hilbert space $\mathfrak{H}$, $R$ has property $\Gamma$ if and only if the $C^*$-algebra $C^*(R, R')$ generated by $R$ and its commutant $R'$ does not contain compact operators except zero operator. For a subgroup $G$ of automorphism group $\mathrm{Aut}(R)$ on $R$, we write the $C^*$-algebra generated by $R$ and the unitaries induced by $G$ as $C^*(R, G)$. We denote by $\mathrm{Int}(R)$ the group of all inner automorphisms on $R$. Since we notice that the $C^*$-algebra $C^*(R, \mathrm{Int}(R))$ coincides with the $C^*$-algebra $C^*(R, R')$, it is natural to introduce the following problem:

PROBLEM. How large is a subgroup $G$ of $\mathrm{Aut}(R)$ satisfying

$$(*) \qquad C^*(R, G) \cap C(\mathfrak{H}) = \{0\}$$

for a factor $R$ of type $\mathrm{II}_1$ with property $\Gamma$? Here $C(\mathfrak{H})$ means the algebra of all compact operators on $\mathfrak{H}$.

In [17], as a solution of this problem, the author has shown the next relation:

$$(1) \qquad C^*(R, \mathrm{Cnt}(R)) \cap C(\mathfrak{H}) = \{0\}$$

where $\mathrm{Cnt}(R)$ denotes the group of all centrally trivial automorphisms on $R$. In §4, by mainly using Connes' result [11, Theorem 2.1] and the above relation (1), we shall obtain many characterizations of property $\Gamma$. In §5, we shall seek better solutions about the previous problem than the condition (1) and show (Theorem 5.2), for an automorphism $\theta$ on a factor $R$ with property $\Gamma$ such that for any nonzero $n$, $\theta^n$ is not centrally trivial,

$$(2) \qquad C^*(R, \mathrm{Cnt}(R) \vee \theta) \cap C(\mathfrak{H}) = \{0\}$$

where $\mathrm{Cnt}(R) \vee \theta$ means the subgroup of $\mathrm{Aut}(R)$ generated by $\mathrm{Cnt}(R)$ and $\theta$. Finally, we shall connect the arguments in §3 about actions on strongly stable factors with the previous problem. So far as strongly stable factors concerned, we shall obtain several better solutions than the condition (2) (Theorem 5.4, Corollary 5.5 and Corollary 5.6). In particular, we shall prove that if an amenable group $G$ acts on a strongly stable factor $M$, then we have

$$C^*(M, G) \cap C(\mathfrak{H}) = \{0\} .$$

## § 2. Preliminaries.

Throughout this paper, we assume that a factor of type $\mathrm{II}_1$ has the separable predual.

Let $M$ be a factor of type $\mathrm{II}_1$ with the faithful normal normalized trace $\tau$. We write the $L^2$-norm by $\tau$ as $\|x\|_2=\tau(x^*x)^{1/2}$, $x\in M$. We denote by $R_0$ the hyperfinite factor of type $\mathrm{II}_1$.

DEFINITION 2.1. We call $M$ a factor with property $\Gamma$ if there exists a sequence $\{u_n\}$ consisting of unitaries in $M$ satisfying

$$\lim_{n\to\infty}\|u_nx-xu_n\|_2=0,\ x\in M \quad\text{and}\quad \tau(u_n)=0,\ n\in \boldsymbol{N}\,.$$

DEFINITION 2.2. A factor $M$ is said to be strongly stable if it is isomorphic to its tensor product with $R_0$, that is to say,

$$M\cong M\otimes R_0\,.$$

Let $l^\infty(M)$ be the $C^*$-algebra consisting of the set of all bounded sequences in $M$.

DEFINITION 2.3. A bounded sequence $\{x_n\}$ in $M$ is said to be centralizing if

$$(*)\qquad \lim_{n\to\infty}\|x_na-ax_n\|_2=0\,,\qquad a\in M\,.$$

Two such sequences $\{x_n\}$ and $\{y_n\}$ are said to be equivalent if the sequence $\{x_n-y_n\}$ converges to zero in $\|\cdot\|_2$-norm. For a fixed free ultrafilter $\omega$ on the set of all natural numbers $\boldsymbol{N}$, $\{x_n\}$ is said to be $\omega$-centralizing if the above limit $(*)$ converges along the filter $\omega$. We define natural equivalence for $\omega$-centralizing sequences similarly as in the case of centralizing sequences.

We denote by $C(M)$ (respectively $C_\omega(M)$) the set of all centralizing (respectively $\omega$-centralizing) sequences of $M$. It is easily seen that $C(M)$ and $C_\omega(M)$ are $C^*$-algebras under natural operations and $l^\infty$-norm. In the algebra $C_\omega(M)$, we consider the set $I_\omega(M)$ of all bounded sequences converging to zero along the filter $\omega$ in $\|\cdot\|_2$-norm. It follows that $I_\omega(M)$ is a norm closed ideal of $C_\omega(M)$ and $l^\infty(M)$. Put the quotient $C^*$-algebras

$$M_\omega=C_\omega(M)/I_\omega(M)\quad\text{and}\quad M^\omega=l^\infty(M)/I_\omega(M)\,.$$

The following theorems are known and basic in our discussions (cf. [9], [10] and [11]).

THEOREM 2.A (A. Connes and D. McDuff). *In the above construction, the algebra $M_\omega$ is a finite von Neumann algebra with the trace $\tau_\omega$ defined by*

$$\tau_\omega(X) = \lim_{n\to\omega} \tau(x_n)$$

*for $X=\pi_\omega(\{x_n\})$ where $\pi_\omega$ is the quotient map of $C_\omega(M)$ to $M_\omega$.*

THEOREM 2.B (A. Connes and D. McDuff). *For a factor $M$ of type $\mathrm{II}_1$, we have*

(1) *$M$ has property $\Gamma$ if and only if $M_\omega$ is not trivial.*

(2) *$M$ is strongly stable if and only if $M_\omega$ is not abelian. In this case, $M_\omega$ is necessarily of type $\mathrm{II}_1$.*

For an automorphism $\theta \in \mathrm{Aut}(M)$ of $M$, the map

$$l^\infty(M) \ni \{x_n\} \to \{\theta(x_n)\} \in l^\infty(M)$$

leaves $C_\omega(M)$ and $I_\omega(M)$ invariant. Hence, this map induces an automorphism on $M_\omega$ and $M^\omega$. Such an automorphism on $M_\omega$ is said to be liftable and written as $\theta_\omega$.

DEFINITION 2.4. An automorphism $\theta$ on $M$ is said to be centrally trivial if $\theta_\omega = \mathrm{id}$ on $M_\omega$. We denote by $\mathrm{Cnt}(M)$ the set of all centrally trivial automorphisms on $M$.

A homomorphism $\alpha$ of a discrete group $G$ into the automorphism group $\mathrm{Aut}(M)$ on $M$ is called an action of $G$ on $M$. An action $\alpha$ is said to be free if for every $g \in G$, $g \neq e$, $\alpha_g$ is an outer automorphism. We call an automorphism $\beta \in \mathrm{Aut}(M)$ strongly outer if the restriction of $\beta$ to the relative commutant of any countable $\beta$-invariant subset of $M$ is properly outer. A discrete group action $\alpha$ of $G$ on $M$ is said to be strongly free if for any $g \in G$, $g \neq e$, $\alpha_g$ is strongly outer. For a strongly stable factor $M$ and a free ultrafilter $\omega$, an action $\beta$ of a discrete group $G$ to $M_\omega$ is said to be liftable if for each $g \in G$, $\beta_g$ is liftable as an automorphism on $M$.

Let $N$ be a von Neumann algebra and $M$ be a von Neumann subalgebra of $N$. We denote by $\mathfrak{U}(N)$ the set of all unitaries of $N$. We set $\mathfrak{N}(M) = \{u \in \mathfrak{U}(N) \mid uMu^* = M\}$ and call it normalizer of $M$ in $N$.

## §3. Crossed products and strongly stable factors.

We shall prove the following:

THEOREM 3.1. *Let $N$ be a factor of type $\mathrm{II}_1$ and $M$ be a subfactor of $N$. Let $G$ be a countable subgroup of the normalizer $\mathfrak{N}(M)$. Suppose* *the factor $N$ is generated by the subfactor $M$ and the subgroup $G$. If $M$ is strongly stable and $G$ is amenable as a discrete group, then $N$ is strongly stable.*

PROOF. We write an element $g \in G$ as a unitary operator $u(g)$ in $N$. By the definition of the normalizer $\mathfrak{N}(M)$, a unitary $u(g)$ in $G$ induces an automorphism $\mathrm{Ad}\, u(g)$ on $M$, which is denoted by $\alpha_g$. Thus we obtain an action $\alpha$ of the amenable group $G$ on $M$.

Let $\omega$ be a fixed free ultrafilter on $\boldsymbol{N}$. Put

$$K=\{g \in G \mid \alpha_g \in \mathrm{Cnt}(M)\} .$$

Then $K$ is a normal subgroup of $G$, and as $G$ is amenable, the quotient group $H=G/K$ is also amenable. Since all elements in $K$ induce the trivial automorphism on $M_\omega$, we may consider the induced action $(\alpha)_\omega$ of $H$ on $M_\omega$. Take $s \in H$, $s \neq e$, and put $s=[g]$ for some $g \in G \setminus K$. Since $\alpha_g$ is not centrally trivial, by [20, Lemma 5.7], $(\alpha_g)_\omega$ is strongly outer. Thus the action $(\alpha)_\omega$ of $H$ on $M_\omega$ is strongly free.

Now assume that $M$ is strongly stable. Since the action $(\alpha)_\omega$ is liftable and strongly free, by [20, Lemma 8.3], the fixed point algebra $(M_\omega)^{(\alpha)_\omega}$ of $M_\omega$ under the action $(\alpha)_\omega$ of $H$ is of type $\mathrm{II}_1$. Hence we may take a $2\times 2$ matrix units $\{F_{ij}\}$, $i, j=1, 2$, in $M_\omega$ satisfying

$$(\alpha_s)_\omega(F_{ij})=F_{ij} , \qquad i, j=1, 2 , \quad s \in H .$$

Namely we have

$$(1) \qquad (\alpha_g)_\omega(F_{ij})=F_{ij} , \qquad i, j=1, 2 , \quad g \in G .$$

Now by [10, Proposition 1.1.3], there exist $\omega$-centralizing sequences consisting of $2\times 2$ matrix units $\{f_{ij}(n)\}$ for every $n \in \boldsymbol{N}$, such that $F_{ij}=\pi_\omega(\{f_{ij}(n)\})$, $i, j=1, 2$. Then the above relation (1) means

$$\pi_\omega(\{\alpha_g(f_{ij}(n))\})=\pi_\omega(\{f_{ij}(n)\}) , \qquad i, j=1, 2 , \quad g \in G ,$$

which implies

$$(2) \qquad \lim_{n\to\omega} \|\alpha_g(f_{ij}(n))-f_{ij}(n)\|_2=0 , \qquad i, j=1, 2 , \quad g \in G .$$

As $\alpha_g=\mathrm{Ad}\, u(g)$, $g \in G$, the equation (2) shows

$$(3) \qquad \lim_{n\to\omega} \|u(g)f_{ij}(n)-f_{ij}(n)u(g)\|_2=0 , \qquad i, j=1, 2 , \quad g \in G .$$

On the other hand, since the sequences $\{f_{ij}(n)\}$ are $\omega$-centralizing in $M$, we have

$$(4) \qquad \lim_{n\to\omega} \|xf_{ij}(n)-f_{ij}(n)x\|_2=0\,, \qquad i,\ j=1,\ 2\,, \quad x\in M\,.$$

With (3) and (4), and since the set $\{f_{ij}(n)\,|\,i,\ j=1,\ 2,\ n\in \boldsymbol{N}\}$ is bounded, one can verify that $\{f_{ij}(n)\}$, $i,\ j=1,\ 2$, are $\omega$-centralizing sequences in the factor $N$. As a consequence, we obtain a $2\times 2$ matrix units $\pi_\omega(\{f_{ij}(n)\})$, $i,\ j=1,\ 2$, in $N_\omega$. Therefore the von Neumann algebra $N_\omega$ is not abelian, whence by Theorem 2.B, we complete the proof.

COROLLARY 3.2. *Let $M$ be a factor of type* $\mathrm{II}_1$ *and $\alpha$ be a free action of a countable discrete amenable group $G$ on $M$. If $M$ is strongly stable, then the crossed product $M\times_\alpha G$ is again strongly stable.*

REMARK. In [21, Proposition 1.11], M. Pimsner and S. Popa showed that for a pair of factors of type $\mathrm{II}_1$ $N\subset M$, if the Jones' index $[M:N]$ is finite, then $M$ is strongly stable if and only if $N$ is strongly stable. As a special case of their result, we see that for a factor $M$ of type $\mathrm{II}_1$ and a free action $\alpha$ of a finite group $G$ on $M$, the following three conditions are equivalent: (a) $M$ is strongly stable, (b) the crossed product $M\times_\alpha G$ is strongly stable, and (c) the fixed point algebra $M^\alpha$ under $\alpha$ is strongly stable.

Next, we investigate the conditions under which if the crossed product $M\times_\alpha G$ by a discrete group action $\alpha$ is strongly stable, then $M$ is strongly stable.

PROPOSITION 3.3. *Let $N$ be a factor of type* $\mathrm{II}_1$ *and $\alpha$ be an action of a countable discrete non inner amenable group $G$ on $N$ (see* [13]). *If the crossed product $N\times_\alpha G$ is strongly stable, then $N$ is strongly stable.*

PROOF. Let $\{u(g)\,|\,g\in G\}$ be the unitaries in $N\times_\alpha G$ implementing the automorphisms $\{\alpha_g\,|\,g\in G\}$. From the assumption, there exist centralizing sequences consisting of $2\times 2$ matrix units $\{f_{ij}(n)\}$, $i,\ j=1,\ 2$, in $N\times_\alpha G$. Then we have

$$(*) \qquad \lim_{n\to\infty} \|u(g)f_{ij}(n)-f_{ij}(n)u(g)\|_2=0\,, \qquad i,\ j=1,\ 2\,, \quad g\in G\,.$$

By [6], the above relation $(*)$ implies

$$\lim_{n\to\infty} \|f_{ij}(n)-E(f_{ij}(n))\|_2=0\,, \qquad i,\ j=1,\ 2\,,$$

where the map $E$ is the canonical conditional expectation from $N\times_\alpha G$ onto $N$ satisfying $E(u(g))=0$, $g\in G$, $g\neq e$. This means that the sequence $\{E(f_{ij}(n))\}$ is a centralizing sequence in $N$. Since the sequence $\{f_{ij}(n)\}$ is equivalent to $\{E(f_{ij}(n))\}$, $\{E(f_{ij}(n))\}$, $i,\ j=1,\ 2$, induce a $2\times 2$ matrix units in $N_\omega$ for a free ultrafilter $\omega$ on $\boldsymbol{N}$. Hence the factor $N$ is strongly stable.

Since, by [1], an ICC group having Kazhdan's property $T$ (see [16]) is non inner amenable, the following corollary is a special case of Proposition 3.3.

COROLLARY 3.4. *Let $N$ be a factor of type* $\mathrm{II}_1$ *and $\alpha$ be an action of an ICC group $G$ on $N$. If the crossed product $N\times_\alpha G$ is strongly stable and the group $G$ has Kazhdan's property $T$, then $N$ is strongly stable.*

REMARK. As $N$ is a finite factor and $G$ is an ICC group, for any action $\alpha$ of $G$ on $N$ the crossed product $N\times_\alpha G$ is a factor of type $\mathrm{II}_1$. In fact, since $G$ is ICC, we have, as in [4, Lemma 1], that

$$u(G)'\cap(N\times_\alpha G)=N^\alpha$$

where $\{u(g)\mid g\in G\}$ are the unitaries in $N\times_\alpha G$ implementing the automorphisms $\{\alpha_g\mid g\in G\}$. Hence we have

$$\begin{aligned}(N\times_\alpha G)'\cap(N\times_\alpha G)&=N'\cap u(G)'\cap(N\times_\alpha G)\\&=N'\cap N^\alpha\\&\subset N'\cap N=\boldsymbol{C}.\end{aligned}$$

As an immediate consequence of the above corollary, we see, in case of trivial action of $G$ on $N$, that if the tensor product $N\otimes R(G)$ is strongly stable with the group $G$ having property $T$, then the factor $N$ becomes strongly stable, where $R(G)$ denotes the left group von Neumann algebra of the group $G$.

We shall however prove the following more general results. The following theorem was suggested by Prof. Y. Katayama to the author when the author reported the proof of Corollary 3.7 at the 21th Junior-Symposium at Izumisano in Japan.

THEOREM 3.5. *Let $N$ be a factor of type* $\mathrm{II}_1$ *and $R$ be a subfactor of $N$. If $R$ has property $T$ in the sense of Connes and Jones* [12], *then any centralizing sequence in $N$ is equivalent to a centralizing sequence in the relative commutant $R'\cap N$.*

PROOF. Assume $R$ has property $T$. By [12], there exist $y_1,\cdots,y_n\in R$, $\varepsilon>0$ and $K>0$ such that, for any $\delta<\varepsilon$, correspondence $H$ from $R$ to $R$ and vector $\zeta\in H$, $\|\zeta\|_2=1$ with $\|y_i\zeta-\zeta y_i\|_2<\delta$, $i=1,2,\cdots,n$, there exists a vector $\xi\in H$ satisfying $x\xi=\xi x$ for all $x\in R$ and $\|\zeta-\xi\|<K\delta$.

We denote by $L^2(N)$ the standard Hilbert space of $N$ constructed by the faithful normal normalized trace on $N$. As a correspondence from $R$ to $R$, we take the Hilbert space $L^2(N)$ and denote it by $H$.

Let $E$ be the conditional expectation from $N$ to $R'\cap N$ with respect to the canonical trace $\tau$ on $N$. We shall show that for any centralizing sequence $\{x_n\}$ in $N$,

$$\lim_{n\to\infty}\|x_n-E(x_n)\|_2=0 .$$

Suppose there exists a centralizing sequence $\{x_n\}$ in $N$ such that

$$\lim_{n\to\infty}\|x_n-E(x_n)\|_2\neq 0 .$$

Then we may take a positive number $r$ and a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that

$$\|x_{n(k)}-E(x_{n(k)})\|_2>r , \qquad \text{for all} \quad k\in \boldsymbol{N} .$$

Put $\delta=\min\{\varepsilon, 1/2K\}$. Since $\{x_{n(k)}\}$ is a centralizing sequence in $N$, there exists a positive integer $m$ such that

$$\|y_i x_{n(m)}-x_{n(m)}y_i\|_2<r\delta , \qquad \text{for} \quad i=1, 2, \cdots, n .$$

Put

$$x=\frac{x_{n(m)}-E(x_{n(m)})}{\|x_{n(m)}-E(x_{n(m)})\|_2} .$$

Denote by $\eta$ the canonical embedding of $N$ into $H$. Put $\zeta=\eta(x)$. Then we have $\zeta\in H$, $\|\zeta\|_2=1$ and $\|y_i\zeta-\zeta y_i\|_2<\delta$, $i=1, 2, \cdots, n$. Since $R$ has property $T$, there exists a vector $\xi\in H$ such that $x\xi=\xi x$ for all $x\in R$ and $\|\zeta-\xi\|_2<K\delta$. Since $\delta$ is less than $1/2K$, this implies

$$\|\zeta-\xi\|_2<\frac{1}{2} .$$

Since we see that $\xi$ is contained in $R'\cap L^2(N)$, as in [24], $\xi$ may be regarded as a closed operator affiliated with $(R'\cup N')'=R'\cap N$. Hence $\xi$ belongs to the subspace $L^2(R'\cap N)$ of $L^2(N)$ spanned by $R'\cap N$. It is clear that $\zeta$ is orthogonal to $R'\cap N$ and hence $L^2(R'\cap N)$. This means that $\zeta$ is orthogonal to $\xi$. It follows that

$$\begin{aligned}\|\zeta-\xi\|_2&=(\|\zeta\|_2^2+\|\xi\|_2^2)^{1/2}\\&=(1+\|\xi\|_2^2)^{1/2}\\&>1 .\end{aligned}$$

This contradicts that $\|\zeta-\xi\|_2<1/2$. Hence we showed that any centralizing sequence $\{x_n\}$ in $N$ is equivalent to $\{E(x_n)\}$.

COROLLARY 3.6. *Let $M$ and $R$ be factors of type* $\mathrm{II}_1$. *If $R$ has property $T$ in the sense of Connes and Jones, then, for any free ultrafilter $\omega$ on $\boldsymbol{N}$, we have*

$$(M\otimes R)_\omega = M_\omega\ .$$

REMARK. The above equality $(M\otimes R)_\omega=M_\omega$ means that the natural embedding of $M$ in $M\otimes R$ induces an isomorphism of $M_\omega$ onto $(M\otimes R)_\omega$.

PROOF. Put $N=M\otimes R$. Since $R$ has property $T$, by Theorem 3.5, any centralizing sequence in $N$ is equivalent to a centralizing sequence in $M$. Similarly as this argument, we can prove that any $\omega$-centralizing sequence in $N$ is equivalent to an $\omega$-centralizing sequence in $M$. This means

$$(M\otimes R)_\omega\subset M_\omega\ .$$

On the other hand, it is easy to see that $M_\omega$ is contained in $(M\otimes R)_\omega$. Therefore we have $(M\otimes R)_\omega=M_\omega$.

COROLLARY 3.7. *Let $M$ and $R$ be factors of type* $\mathrm{II}_1$. *If the tensor product von Neumann algebra $M\otimes R$ is strongly stable and $R$ has property $T$, then $M$ is strongly stable.*

The author is indebted to Professor M. Choda for the formulation of Corollary 3.7.

REMARK. We showed, in Corollary 3.6, for factors $M$ and $R$ of type $\mathrm{II}_1$, if $R$ has property $T$, then $(M\otimes R)_\omega=M_\omega$ for a free ultrafilter $\omega$ on $\boldsymbol{N}$. Since $R$ has property $T$, $R$ is full. Hence $R_\omega=\boldsymbol{C}$. This implies

$$(M\otimes R)_\omega \cong M_\omega\otimes R_\omega\ .$$

In general, it is an open problem whether for any factors $M$ and $N$ of type $\mathrm{II}_1$, $(M\otimes N)_\omega$ is isomorphic to $M_\omega\otimes N_\omega$ or not. This problem was given in [18].

Next, we discuss the cases that von Neumann algebras are not necessarily factors.

PROPOSITION 3.8. *Let $A$ be a finite von Neumann algebra with a faithful normal normalized trace $\tau$. Let $\alpha$ be a $\tau$-preserving action of an ICC group $G$ on $A$. If the crossed product $A\times_\alpha G$ is a strongly stable* *factor of type* $\mathrm{II}_1$, *then the group* $G$ *does not have property* $T$ *or the fixed point algebra* $A^{\alpha}$ *under* $\alpha$ *is not commutative.*

PROOF. Let $\{u(g) \mid g \in G\}$ be the unitaries in $A \times_{\alpha} G$ implementing the automorphisms $\{\alpha_g \mid g \in G\}$. Write $\beta_g = \mathrm{Ad}\, u(g)$, $g \in G$. Let $\varepsilon$ be the $\tau$-preserving conditional expectation of $A \times_{\alpha} G$ onto the fixed point algebra $(A \times_{\alpha} G)^{\beta}$ of $A \times_{\alpha} G$ under $\beta$. Suppose $A \times_{\alpha} G$ is strongly stable. We assume that $G$ has property $T$ and $A^{\alpha}$ is commutative. Fix a free ultrafilter $\omega$ on $N$. Let $\{x_n\}$ be an $\omega$-centralizing sequence in $A \times_{\alpha} G$. Taking a suitable subsequence of $\{x_n\}$, the sequence $\{x_n\}$ is regarded as a centralizing sequence of $A \times_{\alpha} G$. Then we have

$$\lim_{n \to \infty} \|\beta_g(x_n) - x_n\|_2 = \lim_{n \to \infty} \|u(g)x_n - x_n u(g)\|_2 = 0\,, \qquad g \in G\,.$$

Since $G$ has property $T$, by [3, Theorem 1], we have

$$(*) \qquad \lim_{n \to \infty} \|x_n - \varepsilon(x_n)\|_2 = 0\,.$$

By the following equation

$$(A \times_{\alpha} G)^{\beta} = u(G)' \cap (A \times_{\alpha} G) = A^{\alpha}\,,$$

$\varepsilon(x_n)$ are in $A^{\alpha}$. Since $\{x_n\}$ is an $\omega$-centralizing sequence in $A \times_{\alpha} G$, the relation $(*)$ implies $\varepsilon(x_n)$ is an $\omega$-centralizing sequence in $A^{\alpha}$ and it is equivalent to $\{x_n\}$. Hence for two $\omega$-centralizing sequences $\{x_n\}$ and $\{y_n\}$ in $A \times_{\alpha} G$, we have

$$\begin{aligned}
\pi_{\omega}(\{x_n\})\pi_{\omega}(\{y_n\}) &= \pi_{\omega}(\{\varepsilon(x_n)\})\pi_{\omega}(\{\varepsilon(y_n)\}) \\
&= \pi_{\omega}(\{\varepsilon(x_n)\varepsilon(y_n)\}) \\
&= \pi_{\omega}(\{\varepsilon(y_n)\varepsilon(x_n)\}) \\
&= \pi_{\omega}(\{y_n\})\pi_{\omega}(\{x_n\})\,.
\end{aligned}$$

It follows that the algebra $(A \times_{\alpha} G)_{\omega}$ is commutative. Therefore the crossed product $A \times_{\alpha} G$ is not strongly stable. This is a contradiction.

Let $\{X, \mu\}$ be a non atomic probability measure space. Let $\alpha$ be an action on the abelian von Neumann algebra $L^{\infty}(X)$ induced by a measure preserving free ergodic action on $X$ of a countable discrete group $G$. Then the crossed product $L^{\infty}(X) \times_{\alpha} G$ is usually called the group measure space construction algebra.

PROPOSITION 3.9. *Keep the above notations. If the algebra* $L^{\infty}(X) \times_{\alpha} G$

*is strongly stable, then the group $G$ is inner amenable.*

PROOF. Suppose the group $G$ is not inner amenable. Then, similarly as the proof of Proposition 3.3, an $\omega$-centralizing sequence in the crossed product $L^\infty(X)\times_\alpha G$ is equivalent to an $\omega$-centralizing sequence in the algebra $L^\infty(X)$. Since $L^\infty(X)$ is commutative, all $\omega$-centralizing sequences in it induce mutually commuting elements in $(L^\infty(X)\times_\alpha G)_\omega$. Hence the algebra $(L^\infty(X)\times_\alpha G)_\omega$ is commutative. It follows that $L^\infty(X)\times_\alpha G$ is not strongly stable, a contradiction.

## §4. Characterizations of property $\Gamma$.

In this section, we shall give many characterizations of property $\Gamma$ in factors of type $\mathrm{II}_1$. First of all, we must prepare for some notations.

Let $R$ be a factor of type $\mathrm{II}_1$ with the canonical trace $\tau$. Let $R$ act standardly on the Hilbert space $\mathfrak{H}=L^2(R, \tau)$. For an automorphism $\theta\in\mathrm{Aut}(R)$, we define the unitary operator $u(\theta)$ on $\mathfrak{H}$ by $u(\theta)\eta(x)=\eta(\theta(x))$, $x\in R$, where $\eta$ is the canonical embedding of $R$ into $\mathfrak{H}$. Following [5], for a subset $G\subset\mathrm{Aut}(R)$, we write the $C^*$-algebra generated by the unitaries $u(\theta)$, $\theta\in G$ and $R$ as $C^*(R, G)$. We denote by $C^*(R, R')$ the $C^*$-algebra generated by $R$ and its commutant $R'$ on $\mathfrak{H}$ and by $C(\mathfrak{H})$ the algebra of all compact operators on $\mathfrak{H}$. We notice $C^*(R, R')=C^*(R, \mathrm{Int}(R))$ for any factor $R$ of type $\mathrm{II}_1$.

In [11], A. Connes has characterized a factor of type $\mathrm{II}_1$ with property $\Gamma$ as a factor satisfying the following condition:

$$(*)\qquad C^*(R, R')\cap C(\mathfrak{H})=\{0\}\ .$$

As corresponding to this result, in [17], the author has shown that the above condition $(*)$ can be replaced by the next one:

$$C^*(R, \mathrm{Cnt}(R))\cap C(\mathfrak{H})=\{0\}\ .$$

Moreover, A. Connes has shown another three conditions as characterizations of property $\Gamma$ in factors of type $\mathrm{II}_1$, by using inner automorphisms [11, Theorem 2.1]. The first four conditions of the following theorem analogize with those of Connes' result [11, Theorem 2.1].

THEOREM 4.1. *Let $R$ be a factor of type* $\mathrm{II}_1$ *acting on the standard Hilbert space $\mathfrak{H}$ and $J$ be the canonical involution on $\mathfrak{H}$. Let $p$ be the rank one projection onto $C\cdot\eta(1)$. Then the following six assertions are equivalent*:

(1) *$R$ has property $\Gamma$.*

(2) *For any finitely generated subgroup G of* Cnt($R$), *there exists a G-invariant singular state on R.*

(3) *For* $\theta_1, \cdots, \theta_n \in \mathrm{Cnt}(R)$, *there exists a sequence* $\{\xi_k\}$ *of unit vectors of* $\mathfrak{H}$ *satisfying*

$$\lim_{k\to\infty} \|\theta_j(\xi_k)-\xi_k\|_2=0 , \qquad j=1, 2, \cdots, n ,$$

$$\limsup_{k\to\infty} |(\eta(1) \mid \xi_k)| \neq 1 .$$

(4) $C^*(R, \mathrm{Cnt}(R)) \cap C(\mathfrak{H})=\{0\}$.

(5) *There exists a state* $\varphi$ *on* $\mathscr{L}(\mathfrak{H})$ *satisfying*

$$\varphi(p)=0 \quad and \quad \varphi(uJuJ)=1, \ u \in \mathfrak{U}(R) .$$

(6) *For any finitely generated subgroup G of* Cnt($R$), *there exists a state* $\psi$ *on* $\mathscr{L}(\mathfrak{H})$ *satisfying*

$$\psi(p)=0 \quad and \quad \psi(u(g))=1, \ g \in G ,$$

*where* $\mathscr{L}(\mathfrak{H})$ *means the algebra of all bounded linear operators on* $\mathfrak{H}$.

PROOF. By [11, Theorem 2.1], the fact $\mathrm{Int}(R) \subset \mathrm{Cnt}(R)$ and [17], we have only to show the following implications (1)→(3), (1)→(2), (1)→(5), (5)→(1), (1)→(6) and (6)→(1).

Proof of (1)→(3): We shall show the implication by the similar way as the proof in [11, Theorem 2.1]. By [17], we have known that the condition (1) is equivalent to (4), so we prove that if (3) does not hold, then $C^*(R, \mathrm{Cnt}(R)) \cap C(\mathfrak{H}) \neq \{0\}$. Since (3) does not hold, we can take $\theta_1, \cdots, \theta_n \in \mathrm{Cnt}(R)$ such that for a sequence $\{\xi_k\}$ of $\mathfrak{H}$,

$$\lim_{k\to\infty} \|\theta_j(\xi_k)-\xi_k\|_2=0 , \qquad j=1, 2, \cdots, n , \quad \|\xi_k\|_2=1$$

implies

$$\limsup_{k\to\infty} | (\eta(1) \mid \xi_k) |=1 .$$

We define an operator $T \in C^*(R, \mathrm{Cnt}(R))$ by $T=\sum_{j=1}^n u(\theta_j)$, that is to say

$$T\xi=\sum_{j=1}^n \theta_j(\xi) , \qquad \xi \in \mathfrak{H} .$$

Adding $\{\theta_1^{-1}, \cdots, \theta_n^{-1}\}$ to $\{\theta_1, \cdots, \theta_n\}$, we can assume $T$ is a self-adjoint operator on $\mathfrak{H}$. Since we easily see $T\eta(1)=n\eta(1)$, we have $\|T\|=n$. We shall show the eigenvalue $n$ of $T$ is simple. If $n$ is an element of the spectrum on the orthogonal complement $\{C\cdot\eta(1)\}^{\perp}$ of $C\cdot\eta(1)$, then there exists a sequence $\{\xi_k\} \in \mathfrak{H}$ such that

$$\lim_{k\to\infty} \|(T-n)\xi_k\|_2=0\ , \quad \|\xi_k\|_2=1\ , \quad (\xi_k \mid \eta(1))=0\ , \quad k\in \boldsymbol{N}\ .$$

It follows that

$$\lim_{k\to\infty} \left\|\sum_{j=1}^{n} \theta(\xi_k)-n\xi_k\right\|_2=0\ .$$

Hence we have

$$\lim_{k\to\infty} \left\|\sum_{j=1}^{n} \theta(\xi_k)\right\|_2=n\ ,$$

and

$$\begin{aligned} 0&=\lim_{k\to\infty}\left(n^2-\left\|\sum_{j=1}^{n}\theta(\xi_k)\right\|_2^2\right)\\ &=\lim_{k\to\infty}\sum_{i,j=1}^{n}\{1-(\theta_i(\xi_k)\mid\theta_j(\xi_k))\}\\ &=\lim_{k\to\infty}\sum_{i>j}^{n}\{2-(\theta_i(\xi_k)\mid\theta_j(\xi_k))-(\theta_j(\xi_k)\mid\theta_i(\xi_k))\}\\ &=\lim_{k\to\infty}\sum_{i>j}^{n}\{2-2\,\mathrm{Re}(\theta_i(\xi_k)\mid\theta_j(\xi_k))\}\\ &=\lim_{k\to\infty}\sum_{i>j}^{n}\|\theta_i(\xi_k)-\theta_j(\xi_k)\|_2^2\ . \end{aligned}$$

Thus we have

$$\lim_{k\to\infty} \|\theta_i(\xi_k)-\theta_j(\xi_k)\|_2=0\ , \qquad i,\ j=1,\ 2,\ \cdots,\ n\ .$$

Since we have

$$n\|\theta_i(\xi_k)-\xi_k\|_2\leqq\left\|\sum_{j=1}^{n}\theta_j(\xi_k)-n\xi_k\right\|_2+\sum_{j=1}^{n}\|\theta_j(\xi_k)-\theta_i(\xi_k)\|_2\ ,$$

it follows that

$$\lim_{k\to\infty} \|\theta_i(\xi_k)-\xi_k\|_2=0\ , \qquad i=1,\ 2,\ \cdots,\ n\ .$$

But, this means, by the properties of $\{\theta_1, \cdots, \theta_n\}$,

$$\lim_{k\to\infty} |(\eta(1)\mid\xi_k)|=1\ .$$

This is a contradiction because $\{\xi_k\}$ are orthogonal to $\eta(1)$. Therefore $n$ is an isolated point in the spectrum of $T$. Hence the one dimensional projection $p$ onto $\boldsymbol{C}\cdot\eta(1)$ belongs to the $C^*$-algebra generated by $T$ so that $p$ is contained in $C^*(R, \mathrm{Cnt}(R))$. Consequently, we have

$$C^*(R, \mathrm{Cnt}(R)) \cap C(\mathfrak{H}) \neq \{0\} .$$

Proof of (1)→(2): We assume that $R$ has property $\Gamma$. Let $G$ be a finitely generated subgroup of $\mathrm{Cnt}(R)$ with generators $\theta_1, \cdots, \theta_n$. First of all, we shall prove that for a free ultrafilter $\omega$ on $N$, the fixed point algebra $(R^\omega)^{\theta_1,\cdots,\theta_n}$ of $R^\omega$ under $\theta_1, \cdots, \theta_n$ is infinite dimensional.

Since $R$ has property $\Gamma$, by [18, Corollary of Theorem 5], the center of the von Neumann algebra $R_\omega$ is non atomic. Hence $R_\omega$ is infinite dimensional. The fixed point algebra $(R^\omega)^{\theta_1,\cdots,\theta_n}$ is given by the following subset of $R^\omega$

$$\{\pi^\omega(\{x_n\}) \in R^\omega \mid \lim_{n\to\omega} \|\theta_j(x_n) - x_n\|_2 = 0,\ j=1, 2, \cdots, n\} ,$$

where $\pi^\omega$ means the quotient map from $l^\infty(R)$ onto $R^\omega$. Now $\theta_1, \cdots, \theta_n$ are centrally trivial so that the automorphisms induced by them on $R_\omega$ are identity. Hence $R_\omega$ is contained in the algebra $(R^\omega)^{\theta_1,\cdots,\theta_n}$. Therefore $(R^\omega)^{\theta_1,\cdots,\theta_n}$ is infinite dimensional.

Next, we shall construct a $\theta_1, \cdots, \theta_n$-invariant singular state on $R$. Put $M=(R^\omega)^{\theta_1,\cdots,\theta_n}$. Since the dimension of $M$ is infinite, there exists an infinite dimensional abelian von Neumann subalgebra of $M$. Hence there exists an infinite sequence of mutually commuting projections in it so that we can take, for each $k \in N$, a projection $e_k$ in $M$ such that $0<\tau^\omega(e_k)<1/k$, where $\tau^\omega$ is the trace on $R^\omega$ induced by $\tau$ on $R$. As each projection in $M$ is represented by a sequence consisting of projections of $R$, we can find a sequence $\{f_i\}_{i\in N}$ of projections in $R$ such that

$$\|\theta_j(f_i)-f_i\|_1 \leqq \frac{\|f_i\|_1}{i} , \qquad 0<\tau(f_i)\leqq \frac{1}{i} ,$$
$$j=1, 2, \cdots, n , \quad i \in N$$

where $L^1$-norm $\|\cdot\|_1$ is defined by $\|x\|_1=\tau(|x|)$, $x \in R$. For a $k \in N$, we define a normal state $\varphi_k$ on $R$ by $\varphi_k(x)=\tau(f_k x)/\tau(f_k)$, $x \in R$. Since we see $\varphi_k(f_k)=1$ and $\tau(f_k)<1/k$, the set $\{\varphi_k\}_{k\in N}$ is not $\sigma(R^*, R)$-relative compact in $R^*$ (cf. [25, Theorem 5.4]). Hence we can find a non normal state $\varphi$ in $\sigma(R^*, R)$-limit points of $\{\varphi_k\}_{k\in N}$. The state $\varphi$ is $\theta_1, \cdots, \theta_n$-invariant, so that, by taking its singular part, we get a $G$-invariant singular state.

Proof of (1)→(5): We assume $R$ has property $\Gamma$. By definition, we can take a centralizing sequence $\{v_n\}$ of unitaries in $R$ satisfying $\tau(v_n)=0$, $n \in N$. Put $\varphi_n(x)=(x\eta(v_n) \mid \eta(v_n))$, $x \in \mathscr{L}(\mathfrak{H})$. Since the state space on $\mathscr{L}(\mathfrak{H})$ is $\sigma(\mathscr{L}(\mathfrak{H})^*, \mathscr{L}(\mathfrak{H}))$-compact, there exists a $\sigma(\mathscr{L}(\mathfrak{H})^*, \mathscr{L}(\mathfrak{H}))$-limit point $\varphi$ of $\{\varphi_n\}_{n\in N}$. The state is the desired one. In fact, we have $\varphi_n(p)=|\tau(v_n)|^2=0$ and for a unitary $u$ in $R$,

$$\begin{aligned}|\varphi_n(uJuJ)-1| &= |(uJuJ\eta(v_n)\,|\,\eta(v_n))-(\eta(v_n)\,|\,\eta(v_n))| \\ &\leqq \|uJuJ\eta(v_n)-\eta(v_n)\|_2\,\|\eta(v_n)\|_2 \\ &= \|uv_nu^*-v_n\|_2\,.\end{aligned}$$

Therefore we see that $\varphi$ satisfies the condition (5).

Proof of (5)$\to$(1): We assume the condition (5). For unitaries $u_1, \cdots, u_n$ in $R$, put $u_0=1$ and $T=\sum_{j=0}^{n} u_jJu_jJ$. Let $\mathfrak{K}$ be the orthogonal complement of the subspace $C\cdot\eta(1)$ in $\mathfrak{H}$, that is to say

$$\mathfrak{K}=(1-p)\mathfrak{H}=\boldsymbol{C}\cdot\eta(1)\,.$$

Since we see $T^*\eta(1)=(n+1)\eta(1)$, we may regard $T$ as a bounded linear operator on $\mathfrak{K}$. Let $\varphi$ be a state satisfying the condition (5). By the condition $\varphi(p)=0$, $\varphi$ may be regarded as a state on the $C^*$-algebra $\mathscr{L}(\mathfrak{K})$. Then we have

$$\varphi(T)=\sum_{j=0}^{n}\varphi(u_jJu_jJ)=n+1\,.$$

Hence the operator norm of $T$ on $\mathfrak{K}$ is equal to $n+1$. Therefore for each $k\in \boldsymbol{N}$ we can take a unit vector $\xi_k$ in $\mathfrak{K}$ such that

$$\|T\xi_k\|_2^2\geqq(n+1)^2-\frac{1}{k^2}\,.$$

It follows that

$$\begin{aligned}\frac{1}{k^2} &\geqq (n+1)^2-\left\|\sum_{j=0}^{n}u_jJu_jJ\xi_k\right\|_2^2 \\ &= \sum_{i,j=0}^{n}\{1-(u_i\xi_ku_i^*\,|\,u_j\xi_ku_j^*)\} \\ &= 2\sum_{i>j}\{1-\mathrm{Re}(u_i\xi_ku_i^*\,|\,u_j\xi_k u_j^*)\} \\ &\geqq 2\{1-\mathrm{Re}(u_i\xi_ku_i^*\,|\,\xi_k)\} \\ &= \|\xi_k-u_i\xi_ku_i^*\|_2^2\,.\end{aligned}$$

Consequently, we have

$$\lim_{k\to\infty}\|u_i\xi_k-\xi_ku_i\|_2=0\,,\quad i=1, 2, \cdots, n\,,\qquad \text{and}$$

$$(\xi_k\,|\,\eta(1))=0\,,\qquad k\in\boldsymbol{N}\,.$$

Thus, by [11, Theorem 2.1], $R$ has property $\Gamma$.

Proof of (1)$\to$(6): We assume $R$ has property $\Gamma$. Let $G$ be a finitely generated subgroup of $\mathrm{Cnt}(R)$ with generators $\theta_1, \cdots, \theta_n$. Since we have shown the implication (1)$\to$(3), we may take a sequence $\{\xi_k\}$ of unit vectors in $\mathfrak{H}$ satisfying the condition (3). For each $\xi_k$, by decomposing it along the direct summand $\mathfrak{H}=C\cdot\eta(1)\oplus\{C\cdot\eta(1)\}^{\perp}$, picking up the vector belonging to the subspace $\{C\cdot\eta(1)\}^{\perp}$ and normalizing it, we may assume $\xi_k$ is orthogonal to $C\cdot\eta(1)$. The rest of the proof may be done by the similar way as in the proof of the implication (1)→(5). In fact, we may find a state $\psi$ which is a $\sigma(R^*, R)$-limit point of vector states $\{(\cdot\xi_k\,|\,\xi_k)\}_{k\in N}$ and satisfies

$$\psi(u(\theta_j))=1\;,\quad j=1, 2, \cdots, n \qquad \text{and} \quad \psi(p)=0\;.$$

By the above condition $\psi(u(\theta_j))=1$ and Schwarz's inequality, it follows that

$$(*)\qquad \psi(u(\theta_j)x)=\psi(x)=\psi(xu(\theta_j))\;,\qquad x\in\mathscr{L}(\mathfrak{H})\;,\quad j=1, 2, \cdots, n\;.$$

Hence, we have inductively $\psi(u(g))=1$ for all $g\in G$.

Proof of (6)→(1): We shall show the condition (3) under the condition (6). For centrally trivial automorphisms $\theta_1, \cdots, \theta_n$ on $R$, put $T=\sum_{j=0}^{n} u(\theta_j)$ where $\theta_0$ is the identity automorphism. Similarly as in the proof of the implication (5)→(1), we see that the operator norm of $T$ on the subspace $(1-p)\mathfrak{H}$ of $\mathfrak{H}$ is equal to $n+1$. Continuing the same argument as the proof of (5)→(1), we complete the proof.

Remark. It is easy to see that a state $\psi$ on $\mathscr{L}(\mathfrak{H})$ satisfying the condition (6) of Theorem 4.1 is $G$-invariant, by using the identity (*) in the proof of the implication (1)→(6).

Corollary 4.2. *Let $R$ be a factor of type* $\mathrm{II}_1$ *with property $\Gamma$ and $\varphi$ be a state on $\mathscr{L}(\mathfrak{H})$ satisfying the condition* (5) *of Theorem* 4.1. *Then $\varphi$ is an extension of the canonical trace $\tau$ on $R$. Namely, we have $\varphi(x)=\tau(x)$, $x\in R$.*

Proof. Since, for a unitary $u$ in $R$, we have $\varphi(uJuJ)=1$, as describing in the proof of the implication (1)→(6) of Theorem 4.1, we obtain

$$\varphi(uJuJx)=\varphi(x)=\varphi(xuJuJ)\;,\qquad x\in\mathscr{L}(\mathfrak{H})\;,\quad u\in\mathfrak{U}(R)\;.$$

For any $a$ in $R$, by substituting $Ju^*Ja$ for $x$ in the above identity, we have $\varphi(ua)=\varphi(au)$. By the uniqueness of the trace on $R$, we see the restriction of $\varphi$ to $R$ coincides with $\tau$.

Remark. In [7] and [8], M. Choda has given a characterization of approximately inner automorphisms on a factor of type $\mathrm{II}_1$ by describing a similar statement to the conditions (5) and (6) of Theorem 4.1. So we may consider the conditions (5) and (6) and Corollary 4.2 as property $\Gamma$

versions of her results. However our proof of the implication $(5)\rightarrow(1)$ is not similar to that of her theorem [8, Theorem 2]. By suitably modifying the proof of $(5)\rightarrow(1)$, we can prove her theorem more easily.

## § 5. $C^*$-algebras associated with factors having property $\Gamma$ and compact operators.

We recall the problem mentioned in § 1. Namely, "How large is a subgroup $G$ of $\mathrm{Aut}(R)$ satisfying

$$(*) \qquad C^*(R, G)\cap C(\mathfrak{H})=\{0\}$$

for a factor $R$ of type $\mathrm{II}_1$ with property $\Gamma$?"

As we stated in § 1, we already have known

$$(1) \qquad C^*(R, \mathrm{Cnt}(R))\cap C(\mathfrak{H})=\{0\}\ .$$

This section is devoted to seeking better solutions about the problem than the above condition (1). The following lemma plays important roles in order to find larger groups satisfying the condition $(*)$ than $\mathrm{Cnt}(R)$.

LEMMA 5.1. *Let $R$ be a factor of type* $\mathrm{II}_1$ *having property $\Gamma$ with the canonical trace $\tau$ and $G$ be a subgroup of* $\mathrm{Aut}(R)$. *If there exists a centralizing sequence $\{u_n\}$ in $R$ consisting of unitaries satisfying*

$$\lim_{n\to\infty}\|g(u_n)-u_n\|_2=0\ , \quad g\in G \qquad \textit{and} \quad \limsup_{n\to\infty}|\tau(u_n)|\neq 1\ ,$$

*then we have*

$$C^*(R, G)\cap C(\mathfrak{H})=\{0\}\ .$$

This lemma is proved by essentially using M. Choda's result [5].

PROOF. Suppose that there exists a centralizing sequence $\{u_n\}$ in $R$ consisting of unitaries satisfying

$$(*) \qquad \lim_{n\to\infty}\|g(u_n)-u_n\|_2=0\ , \qquad g\in G\ ,$$
$$\limsup_{n\to\infty}|\tau(u_n)|\neq 1\ .$$

Let $\bar{G}$ be the subgroup of $\mathrm{Aut}(R)$ generated by $G$ and $\mathrm{Int}(R)$. Since $\{u_n\}$ is a centralizing sequence, all $g$ in $\mathrm{Int}(R)$ satisfy the condition $(*)$. Hence the condition $(*)$ is fulfilled for all automorphisms in $\bar{G}$. On the other hand, since we have $\|u_n-\tau(u_n)\|_2^2=2(1-|\tau(u_n)|^2)$, it follows that

$$\limsup_{n\to\infty}\|u_n-\tau(u_n)\|_2\neq 0\ .$$

This means that $\bar{G}$ does not act strongly ergodically on $R$ (cf. [5]). Therefore, by [5, Theorem 4], we see that $C^*(R, \bar{G})$ does not contain $C(\mathfrak{H})$. Now $\bar{G}$ contains $\mathrm{Int}(R)$ so that $C^*(R, \bar{G})$ contains $C^*(R, R')$. Thus $C^*(R, \bar{G})$ is irreducible and hence we have

$$C^*(R, \bar{G}) \cap C(\mathfrak{H}) = \{0\} .$$

This implies $C^*(R, G) \cap C(\mathfrak{H}) = \{0\}$.

By using Lemma 5.1, we may find a better solution than before as follows.

THEOREM 5.2. *Let $\theta$ be an automorphism on a factor $R$ of type* $\mathrm{II}_1$ *with property $\Gamma$ such that for any nonzero integer $n$, $\theta^n$ is not centrally trivial. Then we have*

$$C^*(R, \mathrm{Cnt}(R) \vee \theta) \cap C(\mathfrak{H}) = \{0\}$$

*where* $\mathrm{Cnt}(R) \vee \theta$ *is the subgroup of* $\mathrm{Aut}(R)$ *generated by* $\mathrm{Cnt}(R)$ *and* $\theta$.

PROOF. Let $\theta$ be an automorphism such that for any nonzero integer $n$, $\theta^n$ is not centrally trivial. Then, by using [10, Proposition 2.1.2], we have for a free ultrafilter $\omega$ on $N$, the automorphism $\theta_\omega$ is aperiodic on $R_\omega$, namely, for any nonzero integer $n$, $(\theta_\omega)^n$ is properly outer on $R_\omega$. As in the proof of [22, Proposition 1.4], by applying Connes' non commutative Rohklin's theorem [10, Theorem 1.2.5] to the automorphism $\theta_\omega$ on $R_\omega$, we may find a unitary $V$ in $R_\omega$ such that

$$\theta_\omega(V) = V \quad \text{and} \quad \tau_\omega(V) = 0 .$$

By [10, Proposition 1.1.5], we may take an $\omega$-centralizing sequence $\{v_n\}$ consisting of unitaries in $R$ such that

$$\lim_{n\to\omega} \|\theta(v_n) - v_n\|_2 = 0 \quad \text{and} \quad \lim_{n\to\omega} \tau(v_n) = 0 .$$

By choosing a suitable subsequence of $\{v_n\}$, we get a centralizing sequence $\{u_n\}$ of unitaries in $R$ satisfying

$$\lim_{n\to\infty} \|\theta(u_n) - u_n\|_2 = 0 \quad \text{and} \quad \lim_{n\to\infty} \tau(u_n) = 0 .$$

Hence it follows that

$$\lim_{n\to\infty} \|g(u_n) - u_n\|_2 = 0 , \qquad \text{for all} \quad g \in \mathrm{Cnt}(R) \vee \theta .$$

This implies, by Lemma 5.1,

$$C^*(R, \mathrm{Cnt}(R) \vee \theta) \cap C(\mathfrak{H}) = \{0\} .$$

Consequently we have shown that for a factor $R$ with property $\Gamma$ and a suitable automorphism $\theta$ on $R$, three $C^*$-algebras $C^*(R, R')$, $C^*(R, \mathrm{Cnt}(R))$ and $C^*(R, \mathrm{Cnt}(R) \vee \theta)$ do not intersect with $C(\mathfrak{H})$ except zero operator as yet. The relation between these $C^*$-algebras is of course

$$C^*(R, R') \subset C^*(R, \mathrm{Cnt}(R)) \subset C^*(R, \mathrm{Cnt}(R) \vee \theta) .$$

If the above relations of inclusion are not proper, it is not significant much to consider $C^*$-algebras $C^*(R, G)$ for larger groups than $\mathrm{Int}(R)$. In order to make clear the differences between them, we provide the following proposition. It means that the relations of inclusion of the groups

$$\mathrm{Int}(R) \subset \mathrm{Cnt}(R) \subset \mathrm{Cnt}(R) \vee \theta$$

are exactly compatible with those of the $C^*$-algebras

$$C^*(R, R') \subset C^*(R, \mathrm{Cnt}(R)) \subset C^*(R, \mathrm{Cnt}(R) \vee \theta) .$$

PROPOSITION 5.3. *Let $\theta$ be an automorphism on a factor $R$ of type $\mathrm{II}_1$.*

(1) $\theta \in \mathrm{Int}(R)$ *if and only if* $u(\theta) \in C^*(R, R')$.

(2) $\theta \in \mathrm{Cnt}(R)$ *if and only if* $u(\theta) \in C^*(R, \mathrm{Cnt}(R))$.

PROOF. The assertion (1) has proved in [17] by the author. So, we have only to show the if part of the assertion (2). Thus we assume $u(\theta) \in C^*(R, \mathrm{Cnt}(R))$. Let $\eta$ be the natural embedding of $R$ into $\mathfrak{H} = L^2(R, \tau)$. Put $\tilde{\tau}(a) = (a\eta(1) \mid \eta(1))$, $a \in C^*(R, \mathrm{Cnt}(R))$. Then $\tilde{\tau}$ is a state on $C^*(R, \mathrm{Cnt}(R))$ which is an extension of the canonical trace $\tau$ on $R$. We define the seminorm $\|\cdot\|_{\tilde{\tau}}$ on $C^*(R, \mathrm{Cnt}(R))$ by $\|a\|_{\tilde{\tau}} = \tilde{\tau}(a^*a)^{1/2}$, $a \in C^*(R, \mathrm{Cnt}(R))$. It is easy to see that $\|xu(g)\|_{\tilde{\tau}} = \|x\|_2$, $x \in R$, $g \in \mathrm{Cnt}(R)$. Let $\{x_n\}$ be a centralizing sequence in $R$. Then we have

$$\|u(g)x_n - x_n u(g)\|_{\tilde{\tau}} = \|g(x_n) - x_n\|_2 , \qquad g \in \mathrm{Cnt}(R) .$$

This implies

$$\lim_{n\to\infty} \|u(g)x_n - x_n u(g)\|_{\tilde{\tau}} = 0 , \qquad g \in \mathrm{Cnt}(R) .$$

On the other hand, as $\{x_n\}$ is centralizing in $R$, we have

$$\lim_{n\to\infty} \|yx_n - x_n y\|_{\tilde{\tau}} = 0 , \qquad y \in R .$$

Therefore, for an element $b$ in the dense $*$-subalgebra of $C^*(R, \mathrm{Cnt}(R))$ algebraically generated by $R$ and $u(g)$, $g \in \mathrm{Cnt}(R)$, it follows that

$$\lim_{n\to\infty} \|bx_n - x_n b\|_\tau = 0 .$$

By noticing the following inequality

$$\|xy\|_\tau \leqq \|x\| \, \|y\| , \qquad x, y \in C^*(R, \mathrm{Cnt}(R))$$

and the boundedness of $\{x_n\}$, we may see

$$\lim_{n\to\infty} \|ax_n - x_n a\|_\tau = 0 , \qquad a \in C^*(R, \mathrm{Cnt}(R)) .$$

Now, we assume that $u(\theta)$ belongs to $C^*(R, \mathrm{Cnt}(R))$. Thus we have

$$\lim_{n\to\infty} \|\theta(x_n) - x_n\|_2 = \lim_{n\to\infty} \|u(\theta)x_n - x_n u(\theta)\|_\tau = 0 .$$

This means that $\theta$ is centrally trivial.

By Proposition 5.3, we may know many factors of type $\mathrm{II}_1$ with property $\Gamma$ and automorphisms on them satisfying the following relations

$$C^*(R, R') \subsetneqq C^*(R, \mathrm{Cnt}(R)) \subsetneqq C^*(R, \mathrm{Cnt}(R) \vee \theta) \quad \text{and}$$
$$C^*(R, \mathrm{Cnt}(R) \vee \theta) \cap C(\mathfrak{H}) = \{0\} .$$

For instance, let $R(F_2)$ be the left group von Neumann algebra constructed by the free group $F_2$ on 2 generators. We consider the tensor product von Neumann algebra $R(F_2) \otimes R_0$ between $R(F_2)$ and the hyperfinite factor $R_0$ of type $\mathrm{II}_1$ and denote it by $M$. As we have seen in [17], the centrally trivial automorphism group on $M$ is exactly larger than inner automorphism group on it. Since there exists an automorphism $\theta_0$ on $R_0$ such that for any nonzero $n$, $(\theta_0)^n$ is not centrally trivial, each nonzero power of the automorphism $\theta = \mathrm{id} \otimes \theta_0$ on $R(F_2) \otimes R_0 = M$ is not centrally trivial. Therefore, by Theorem 5.2 and Proposition 5.3, we have

$$C^*(M, M') \subsetneqq C^*(M, \mathrm{Cnt}(M)) \subsetneqq C^*(M, \mathrm{Cnt}(M) \vee \theta) \quad \text{and}$$
$$C^*(M, \mathrm{Cnt}(M) \vee \theta) \cap C(\mathfrak{H}) = \{0\} .$$

Finally we shall discuss in restricting factors with property $\Gamma$ to strongly stable ones. Then we shall connect the arguments of § 1 with the previous problem. We have the following theorem.

THEOREM 5.4. *Let $M$ be a strongly stable factor of type* $\mathrm{II}_1$ *and $G$ be a subgroup of* $\mathrm{Aut}(M)$. *If the homomorphic image of $G$ into the quotient group* $\mathrm{Aut}(M)/\mathrm{Cnt}(M)$ *is countable and amenable as a discrete group, then we have*

$$C^*(M, \mathrm{Cnt}(M) \vee G) \cap C(\mathfrak{H}) = \{0\} .$$

PROOF. We fix a free ultrafilter $\omega$ on $N$. We denote by $G_\omega$ the homomorphic image of $G$ into $\mathrm{Aut}(M)/\mathrm{Cnt}(M)$. By assumption, $G_\omega$ is a countable amenable group of which action on $M_\omega$ is liftable and strongly free. Hence, by [20, Lemma 8.3], the fixed point algebra $(M_\omega)^{G_\omega}$ of $M_\omega$ under $G_\omega$ is of type $\mathrm{II}_1$. Thus there exists a $2\times 2$ matrix unit $\{F_{ij}\}_{i,j=1,2}$ in $(M_\omega)^{G_\omega}$. Put $U=F_{11}-F_{22}$. Then $U$ is a unitary in $M_\omega$ such that

$$g_\omega(U)=U\ , \quad g\in G \quad \text{and} \quad \tau_\omega(U)=0\ .$$

Similarly as in the proof of Theorem 5.2, we may find a centralizing sequence $\{u_n\}$ of unitaries in $M$ such that

$$\lim_{n\to\infty} \|g(u_n)-u_n\|_2=0\ , \quad g\in G \quad \text{and} \quad \lim_{n\to\infty} \tau_\omega(u_n)=0\ .$$

Hence we see for any element $g$ in the group $\mathrm{Cnt}(M)\vee G$ generated by $\mathrm{Cnt}(M)$ and $G$,

$$\lim_{n\to\infty} \|g(u_n)-u_n\|_2=0\ .$$

By Lemma 5.1, we have

$$C^*(M, \mathrm{Cnt}(M)\vee G)\cap C(\mathfrak{H})=\{0\}\ .$$

COROLLARY 5.5. *If $M$ is a strongly stable factor of type* $\mathrm{II}_1$, *then for any automorphism $\theta$ on $M$, we have*

$$C^*(M, \mathrm{Cnt}(M)\vee \theta)\cap C(\mathfrak{H})=\{0\}\ .$$

COROLLARY 5.6. *Let $M$ is a factor of type* $\mathrm{II}_1$ *and $\alpha$ be an action of a countable discrete amenable group $G$ on $M$. If $M$ is strongly stable, then we have*

$$C^*(M, G)\cap C(\mathfrak{H})=\{0\}\ .$$

*Here $G$ is identified with its homomorphic image by the action $\alpha$ into* $\mathrm{Aut}(M)$, *namely, $C^*(M, G)$ means $C^*(M, \alpha_G)$ exactly.*

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