TOKYO J. MATH.
VOL. 12, NO. 1, 1989

Fourier Series with Nonnegative Coefficients on Compact Semisimple Lie Groups

Takeshi KAWAZOE and Hiroshi MIYAZAKI

Keio University

§1. Introduction.

Let G be a compact abelian group and $G^{\wedge}$ the dual of the group G. For f in $L^1(G)$, $f^{\wedge}$ denotes the Fourier transform of f. Then it is well known that functions in $L^1(G)$ with positive Fourier coefficients that are pth $(1<p\leqq 2)$ power integrable near the identity in G have Fourier coefficients in l^q, where $q=p/(p-1)$. When $p=2$, this result was proved by N. Wiener for $G=T$, the circle group, (cf. [B]) and by M. Rains for compact abelian groups (see [R]). For $1<p<2$ it was shown by J. M. Ash, M. Rains and S. Vági (see [ARV]). Recently, H. Miyazaki proved that the same result also holds for central functions on $SU(2)$ (see [M]). In this paper, applying the technique used in [ARV], we shall prove that the similar result holds for central and zonal functions on compact semisimple Lie groups.

When G is a compact abelian group, the characters χ_α $(\alpha\in G^{\wedge})$ satisfy $\chi_\alpha\chi_\beta=\chi_{\alpha+\beta}$ $(\alpha, \beta\in G^{\wedge})$, and thus, $(fg)^{\wedge}=f^{\wedge}*g^{\wedge}$; this property plays an important role in the proof of [ARV]. However, when G is an arbitrary compact group, the characters and the spherical functions on G don't satisfy such a simple formula; actually, the Clebsch-Gordan formula for characters and the addition formula for spherical functions offer the replacement. Then applying the same argument in [ARV], we can obtain an analogy on compact non abelian groups.

§2. Notation.

Let U be a compact semisimple Lie group and $T\subset U$ a maximal torus of U. Let $\mathfrak{u}$ and $\mathfrak{t}$ denote the Lie algebras of U and T respectively, $\mathfrak{g}_C$ and $\mathfrak{t}_C$ the complexifications. The Haar measures du and dt are normalized by $\int_U du=\int_T dt=1$. Let $U^{\wedge}$ denote the set of all equivalence classes of

Received January 11, 1989

irreducible (finite dimensional) unitary representations of U: they are parametrized by the dominant integral forms λ on $\mathfrak{t}_C$. If $\lambda \in U^\wedge$, let π_λ denote a member of the class λ acting on the d_λ-dimensional Hilbert space V_λ and χ_λ the character of π_λ. Then the Fourier series of $f \in L^1(U)$ is given by (cf. [H], p. 507 and [W], p. 205)

$$f(u) \sim \sum_{\lambda \in U^\wedge} d_\lambda \operatorname{Tr}(A_\lambda \pi_\lambda(u)) , \tag{2.1}$$

where A_λ is the Fourier coefficient of f defined by

$$A_\lambda = \int_U f(u) \pi_\lambda(u^{-1}) du . \tag{2.2}$$

If f is a central function, that is, $f(vuv^{-1}) = f(u)$ for all u, $v \in U$, (2.1) and (2.2) take the form

$$f(u) \sim \sum_{\lambda \in U^\wedge} f^\wedge(\lambda) \chi_\lambda(u)$$

(2.3) and

$$f^\wedge(\lambda) = \int_U f(u) \chi_\lambda(u^{-1}) du .$$

Especially, the characters χ_λ form a complete orthonormal system in $L^2_\natural(U)$, the space of central functions in $L^2(U)$. For $2 \leqq q < \infty$ let

$$\|f^\wedge\|_{\natural,q} = \Big(\sum_{\lambda \in U^\wedge} d_\lambda^{2-q} |f^\wedge(\lambda)|^q\Big)^{1/q} . \tag{2.4}$$

Let $\mathfrak{u} = \mathfrak{k} + \mathfrak{p}^*$ be a Cartan decomposition of $\mathfrak{u}$ defined by an involution θ and K the analytic subgroup of U with Lie algebra $\mathfrak{k}$ (cf. [H], p. 187). The Haar measure dk is normalized by $\int_K dk = 1$. Let $U_K^\wedge$ denote the set of all equivalence classes of irreducible unitary representations of U of class one with respect to K. Then the spherical function ψ_λ on U corresponding to $\lambda \in U_K^\wedge$ is given by (cf. [H], p. 417)

$$\psi_\lambda(u) = \int_K \chi_\lambda(u^{-1}k) dk . \tag{2.5}$$

Then the Fourier series of f in $L^1(U//K)$, the space of K-biinvariant L^1 functions on U, is given by (see (2.1))

$$f(u) \sim \sum_{\lambda \in U_K^\wedge} d_\lambda f^\wedge(\lambda) \psi_\lambda(u) , \tag{2.6}$$

where $f^\wedge(\lambda)$ is defined by (2.3): it also can be defined by

$$f^{\wedge}(\lambda)=\int_U f(u)\psi_\lambda(u^{-1})du\ . \tag{2.7}$$

Especially, the spherical functions $d_\lambda^{1/2}\psi_\lambda$ form a complete orthonormal system in $L^2(U//K)$ (cf. [H], p. 507). For $2\leqq q<\infty$ let

$$\|f^{\wedge}\|_{\flat,q}=(\sum_{\lambda\in \hat{U}_K} d_\lambda|f^{\wedge}(\lambda)|^q)^{1/q}\ . \tag{2.8}$$

Since $|\chi_\lambda|\leqq d_\lambda$ ($\lambda\in U^{\wedge}$) and $|\psi_\lambda|\leqq 1$ ($\lambda\in \hat{U}_K$), it follows from (2.3) and (2.7) that

$$|f^{\wedge}(\lambda)|\leqq d_\lambda\|f\|_1 \quad \text{for} \quad f\in L^1_\sharp(U)$$

(2.9) and

$$|f^{\wedge}(\lambda)|\leqq \|f\|_1 \quad \text{for} \quad f\in L^1(U//K)\ .$$

Therefore, as in the case of the euclidean Fourier transform, the Riesz-Thorin interpolation theorem (cf. [RS], p. 27) between (2.9) and the Plancherel formula tells us that the Fourier transforms given by (2.3) and (2.7) respectively satisfy the Hausdorff-Young theorem:

Let $1<p\leqq 2$ and $1/p+1/q=1$. Then there exist constants C_p and $C'_p>0$ such that

$$\|f^{\wedge}\|_{\sharp,q}\leqq C_p\|f\|_p \quad \text{for} \quad f\in L^p_\sharp(U) \tag{2.10a}$$

$$\|f^{\wedge}\|_{\flat,q}\leqq C'_p\|f\|_p \quad \text{for} \quad f\in L^p(U//K)\ . \tag{2.10b}$$

§ 3. Fourier series of products.

We denote the Fourier series of the products $\chi_\lambda\chi_\mu$ and $\psi_\lambda\psi_\mu$ as

$$\chi_\lambda\chi_\mu=\sum A_{\lambda\mu}(\nu)\chi_\nu$$

(3.1) and

$$\psi_\lambda\psi_\mu=\sum B_{\lambda\mu}(\nu)\psi_\nu\ .$$

We note that $\chi_\lambda\chi_\mu$ is the character of the tensor product $\pi_\lambda\times\pi_\mu$, and thus, the decomposition into irreducible components α_i deduces that $\chi_\lambda\chi_\mu=\chi_{\alpha_1}+\cdots+\chi_{\alpha_n}$. Therefore, we easily see the following

LEMMA 3.1. *$A_{\lambda\mu}(\nu)\geqq 0$ for all $\nu\in U^{\wedge}$.*

Next we shall prove the positivity of $B_{\lambda\mu}(\nu)$. First we note that

LEMMA 3.2. *For $\lambda\in \hat{U}_K$ there exist C^∞ functions $\psi_{\lambda i}$ ($1\leqq i\leqq d_\lambda$) on U for which*

$$\psi_\lambda(x^{-1}y)=\sum_{1\leqq i\leqq d_\lambda}\psi_{\lambda i}(x)^-\psi_{\lambda i}(y) \qquad (x,\ y\in U)\ .$$

PROOF. Let $\{e_i;\ 1\leqq i\leqq d_\lambda\}$ be an orthonormal system of V_λ, where we take e_1 as a K-fixed vector of V_λ. Then $\psi_\lambda(u)=(\pi_\lambda(u)e_1,\ e_1)$ $(u\in U)$, and the desired relation is obvious if we let $\psi_{\lambda i}(u)=(\pi_\lambda(u)e_1,\ e_i)$ $(1\leqq i\leqq d_\lambda)$.

Q.E.D.

LEMMA 3.3. *$B_{\lambda\mu}(\nu)\geqq 0$ for all $\nu\in \hat{U_K}$.*

PROOF. The proof is similar in the case of non compact symmetric spaces given by [FK]. Since $(f*g)^\wedge(\lambda)=f^\wedge(\lambda)g^\wedge(\lambda)$ for $f,\ g\in L^1(U//K)$, it follows that

$$(f*g,\ g)=\sum_{\lambda\in \hat{U_K}}d_\lambda f^\wedge(\lambda)|g^\wedge(\lambda)|^2\ ,$$

for all $f\in L^1(U//K)$ and $g\in C^\infty(U//K)$. Especially, $f^\wedge(\lambda)\geqq 0$ for all λ if and only if $(f*g,\ g)\geqq 0$ for all $g\in C^\infty(U//K)$. Therefore, it is enough to prove that $((\psi_\lambda\psi_\mu)*g,\ g)\geqq 0$ for all $g\in C^\infty(U//K)$. Then we see from Lemma 3.2 that

$$\begin{aligned}((\psi_\lambda\psi_\mu)*g,\ g)&=\iint_{U\times U}\psi_\lambda(yx)\psi_\mu(yx)g(x^{-1})g(y)^-dxdy\\&=\sum_{1\leqq i,j\leqq d_\lambda}\left|\int_U g(x^{-1})\psi_{\lambda i}(x)\psi_{\mu j}(x)dx\right|^2\geqq 0\ .\end{aligned}$$

Q.E.D.

LEMMA 3.4. *$A_{\lambda 0}(\lambda)=B_{\lambda 0}(\lambda)=1$.*

PROOF. Since $\chi_0\equiv\psi_0\equiv 1$, this is clear from the definition (3.1).

Q.E.D.

§4. Main result.

Let Ξ be a neighborhood of the origin of U and let

$$\Xi_\sharp=\bigcup_{u\in U}u\Xi u^{-1}\quad\text{and}\quad \Xi_\flat=K\Xi K\ . \tag{4.1}$$

For a function f on U and a neighborhood Ξ of U we denote by f_Ξ the function on U which coincides with f on Ξ and vanishes outside of Ξ. Then we can obtain the following

THEOREM 4.1. *Let $1<p\leqq 2$ and $q=p/(p-1)$. Let $\Xi_\sharp$ and $\Xi_\flat$ be as above.*

(1) *If $f\in L^1_\sharp(U)$ has nonnegative Fourier coefficients and $f_{\Xi_\sharp}\in L^p(U)$, then $\|f^\wedge\|_{\sharp,q}<\infty$.*

(2) *If* $f \in L^1(U//K)$ *has nonnegative Fourier coefficients and* $f_{\Xi_\flat} \in L^p(U)$, *then* $\|f^\wedge\|_{\flat,q} < \infty$.

The proof of the theorem will be done as in [R] and [ARV] after we find a function satisfying the following lemma. For a function h on U we let

$$h_\sharp(u) = \int_U h(vuv^{-1})dv \quad \text{and} \quad h_\flat(u) = \iint_{K\times K} h(k_1uk_2)dk_1dk_2 . \tag{4.2}$$

LEMMA 4.2. *There exists a function* h *in* $C_c^\infty(U)$ *such that*
(1) $\operatorname{supp} h_\sharp \subset \Xi_\sharp$ (*resp.* $\operatorname{supp} h_\flat \subset \Xi_\flat$),
(2) $\|h\|_\infty < \infty$,
(3) $h^\wedge(\lambda) \geqq 0$ *for all* $\lambda \in U^\wedge$ (*resp.* $\lambda \in U_K^\wedge$),
(4) $h^\wedge(0) = 1$,
where the Fourier coefficient $h^\wedge(\lambda)$ *is defined by* (2.3) (*resp.* (2.7)).

PROOF. We choose a neighborhood W of the origin of U with $(WW^{-1})_\sharp \subset \Xi_\sharp$. Then we can find a C^∞ function g on U such that $\operatorname{supp} g \subset W$, $\|g\|_\infty < \infty$ and $\int_U g(u)du = 1$. Then the desired function is given by $h = g*(g^\sim)$, where $g^\sim(u) = g(u^{-1})^-$ $(u \in U)$; actually, (1), (2) and (4) are clear and (3) follows from

$$\begin{aligned} h^\wedge(\lambda) &= \iint_{U\times U} g(xy^{-1})g^\sim(y) \sum_{1\leqq i\leqq d_\lambda} (\pi_\lambda(x)e_i, e_i)^- dxdy \\ &= \iint_{U\times U} g(x)g(y^{-1})^- \sum_{1\leqq i,j\leqq d_\lambda} (\pi_\lambda(x)e_i, e_j)^-(\pi_\lambda(y)e_i, e_j)^- dxdy \\ &= \sum_{1\leqq i,j\leqq d_\lambda} \left| \int_U g(x)(\pi_\lambda(x^{-1})e_i, e_j)dx \right|^2 \geqq 0 , \end{aligned}$$

where $\{e_i; 1\leqq i\leqq d_\lambda\}$ is an orthonormal system of V_λ.

We note that $\psi_\lambda(u) = (\pi_\lambda(u)e_k, e_k)$ $(u \in U)$, where e_k is a K-fixed vector of U. Therefore, if we choose a neighborhood W of the origin of U with $(WW^{-1})_\flat \subset \Xi_\flat$, the case of a zonal function $h_\flat$ follows from the same argument as above. Q.E.D.

THE PROOF OF THEOREM 4.1. (1) Let h be the function obtained in Lemma 4.2 such that $\operatorname{supp} h_\sharp \subset \Xi_\sharp$. Since $h_\sharp^\wedge(\lambda) = (h, \chi_\lambda) = h^\wedge(\lambda)$, it follows from (3.1) that

$$(fh_\sharp)^\wedge(\nu) = \sum_{\lambda,\mu \in U^\wedge} f^\wedge(\lambda)h^\wedge(\mu)A_{\lambda\mu}(\nu) .$$

Here we recall that $f^\wedge(\lambda) \geqq 0$ by the assumption on f, $h^\wedge(\mu) \geqq 0$ and

$h^\wedge(0)=1$ by Lemma 4.2 (3), (4), and $A_{\lambda\mu}(\nu)\geqq 0$ and $A_{\nu 0}(\nu)=1$ by Lemmas 3.1 and 3.4. Especially, we see that

$$(fh_\sharp)^\wedge(\nu)\geqq f^\wedge(\nu)$$

for all $\nu\in U^\wedge$. Therefore, noting (2.10a), we can deduce that

$$\begin{aligned}\|f^\wedge\|_{\sharp,q}&\leqq\|(fh_\sharp)^\wedge\|_{\sharp,q}\\ &\leqq C_p\|fh_\sharp\|_p\\ &\leqq C_p\|h\|_\infty\|f_{B_\sharp}\|_p<\infty\ .\end{aligned}$$

(2) Let h be the function obtained in Lemma 4.2 such that $\operatorname{supp} h_\flat\subset\Xi_\flat$. Since $h_\flat^\wedge(\lambda)=(h,\psi_\lambda)=h^\wedge(\lambda)$, it follows from (3.1) that

$$(fh_\flat)^\wedge(\nu)=\sum_{\lambda,\mu\in\hat{U}_K}d_\lambda d_\mu d_\nu^{-1}f^\wedge(\lambda)h^\wedge(\mu)B_{\lambda\mu}(\nu)\ .$$

Then, by repeating the argument in the previous case, the rest of the proof follows from Lemma 3.3, Lemma 3.4 and (2.10b).

This completes the proof of the theorem. Q.E.D.

References

[ARV] J. M. Ash, M. Rains and S. Vági, Fourier series with positive coefficients, Proc. Amer. Math. Soc., **101** (1987), 392–393.

[B] R. P. Boas, *Entire Functions*, Academic Press, New York, 1964.

[FK] M. Flensted-Jensen and T. H. Koornwinder, Jacobi functions: the addition formula and the positivity of the dual convolution structure, Ark. Mat., **17** (1979), 139–151.

[H] S. Helgason, *Groups and Geometric Analysis*, Academic Press, New York, 1984.

[M] H. Miyazaki, Central functions on $SU(2)$ with nonnegative Fourier coefficients, Keio Sci. Tech. Rep., **42** (1989), 1–5.

[R] M. Rains, On functions with nonnegative Fourier transforms, Indian J. Math., **27** (1985), 41–48.

[RS] M. Reed and B. Simon, *Methods of Modern Mathematical Physics II. Fourier Analysis, Self-Adjointness*, Academic Press, New York, 1975.

[W] G. Warner, *Harmonic Analysis on Semi-Simple Lie Groups I*, Springer-Verlag, New York, 1972.

Present Address:
Department of Mathematics, Faculty of Science and Technology, Keio University
Hiyoshi, Kohoku-ku, Yokohama 223, Japan