TOKYO J. MATH.
VOL. 12, NO. 1, 1989

On Certain Homogeneous Diophantine Equations of Degree $n(n-1)$

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(Communicated by Y. Ito)

1. In [3] Hilbert treated the Diophantine equation $D=D(x_0, x_1, \cdots, x_n)=\pm 1$, where

$$D=x_0^{2n-2}\prod(t_i-t_k)^2 \qquad (i=1, 2, \cdots, n;\ k=i+1, i+2, \cdots, n)$$

is the discriminant of

$$x_0t^n+x_1t^{n-1}+\cdots+x_n=0 ,$$

with undetermined coefficients, and roots $t_1, t_2, \cdots, t_n$. He showed that, if $n>3$, the equation $D=\pm 1$ has no integer solutions. The proof is based on the theorem that the discriminant of an algebraic number field of degree $n>1$ is distinct from ± 1. Is his method applicable to other Diophantine equations?

In the present paper we discuss the homogeneous equation

$$a^s(n-1)^{n-1}x^{n(n-1)}+n^ny^{n(n-1)}=Az^{n(n-1)} , \tag{1.1}$$

where a, s, n, A are rational integers satisfying the following conditions:

(1) a is square-free, $|a|\neq 1$;

(2) $s\geqq 1$, $n\geqq 3$, $s<2(n-1)$, $A\neq 0$;

(3) $(n, asA)=((n-1)a, A)=1$.

The equation (1.1) may have non-trivial integer solutions; for example, if $A=a^s(n-1)^{n-1}+n^n$, then $x=y=z=1$ is a solution of (1.1). However, if A satisfies a certain condition, (1.1) has no integer solutions except $x=y=z=0$ (Theorem 1). The proof depends on a result of Komatsu [4] and Minkowski's inequality on the discriminant of an algebraic number field.

2. For simplicity, we shall use the following notation: For a prime

Received September 19, 1988

number p and a rational integer b, we denote by b_p the largest integer m such that b is divisible by p^m; similarly, for a prime ideal P and an algebraic integer α, we denote by α_P the largest integer m such that α is divisible by P^m.

We require the following lemma:

LEMMA 1. *Let $a(0), a(1), \cdots, a(n-1)$ $(n\geqq 1)$ be rational integers such that there exists a prime number p satisfying*

$$0<a(0)_p\leqq a(i)_p \quad (0\leqq i\leqq n-1), \qquad (a(0)_p, n)=1 .$$

Then the polynomial

$$f(u)=u^n+a(n-1)u^{n-1}+\cdots+a(1)u+a(0)$$

is irreducible over $\boldsymbol{Q}$.

PROOF. Let α be an arbitrary root of $f(u)=0$, and let P be a prime ideal in $\boldsymbol{Q}(\alpha)$ which divides p. Since

$$-\alpha^n=\sum_{i=0}^{n-1} a(i)\alpha^i ,$$

we see that α is divisible by P and so $\alpha_P>0$. Hence

$$n\alpha_P=a(0)_p p_P .$$

Since $(a(0)_p, n)=1$, p_P is divisible by n. Hence we obtain

$$n\leqq p_P\leqq[\boldsymbol{Q}(\alpha):\boldsymbol{Q}]\leqq n ,$$

which proves our lemma.

3. Now we state our theorem.

THEOREM 1. *Let a, s, n, A be rational integers which satisfy the following conditions:*

(1) *a is square-free, $|a|\neq 1$;*

(2) *$s\geqq 1$, $n\geqq 3$, $s<2(n-1)$, $A\neq 0$;*

(3) *$(n, asA)=((n-1)a, A)=1$.*

Let B^2 denote the largest square dividing A, and let A_0 denote the square-free integer defined by

$$A=A_0B^2 .$$

If there is no algebraic number field of degree n with discriminant $a^{n-1}A_0$ or $-a^{n-1}A_0$, then the only integer solution of the equation

$$a^s(n-1)^{n-1}x^{n(n-1)}+n^n y^{n(n-1)}=Az^{n(n-1)} \tag{3.1}$$

is given by $x=y=z=0$. *In particular, if* $|a|=2$, 3 *or* 5, *and if* A_0 *satisfies the inequality*

$$|A_0|<\frac{1}{|a|^{n-1}}\left(\frac{\pi}{4}\right)^n\left(\frac{n^n}{n!}\right)^2, \tag{3.2}$$

then the equation (3.1) *has no integer solutions except* $x=y=z=0$.

PROOF. We may assume that A is $n(n-1)$-th power free. Let (x, y, z) be an integer solution of (3.1) with no common prime factors. Then $(y, z)=1$. In fact, if p is a common prime factor of y and z, then by (3.1)

$$\begin{aligned} n(n-1) &\leqq (a^s(n-1)^{n-1})_p = sa_p+(n-1)(n-1)_p \leqq s+(n-1)(n-1)_p \\ &\leqq s+(n-1)(n-2) , \end{aligned}$$

since $m_p \leqq (m-1)$ for every $m \geqq 2$. This is a cotradiction, since $s<2(n-1)$. Similarly, $(x, z)=(x, y)=1$. Hence

$$(a, y)=((n-1)ax, Az)=1 . \tag{3.3}$$

By Lemma 1, we see that

$$f(u)=u^n+a^s x^{n-1}u-a^s y^n$$

is irreducible over $\boldsymbol{Q}$. Now let α be a root of $f(u)=0$; let $\delta=f'(\alpha)$, $D=\operatorname{norm}\delta$ (in $\boldsymbol{Q}(\alpha)$). Then

$$\begin{aligned} D &= (-1)^{n-1}(n-1)^{n-1}(a^s x^{n-1})^n+n^n(-a^s y^n)^{n-1} \\ &= (-1)^{n-1}a^{s(n-1)}Az^{n(n-1)} . \end{aligned}$$

Let d denote the discriminant of $\boldsymbol{Q}(\alpha)$. Then, by (3.3) and Komatsu [4] (Theorem 2, Theorem 3), we see that

$$|d|=|a^{n-1}A_0| ,$$

which proves the first assertion. Now let $n=r+2t$, where r denotes the number of real conjugate fields of $\boldsymbol{Q}(\alpha)$. From Minkowski's inequality on the discriminant of an algebraic number field (Hilbert [2], §18), we obtain

$$|d| \geqq \left(\frac{\pi}{4}\right)^{2t}\left(\frac{n^n}{n!}\right)^2 \geqq \left(\frac{\pi}{4}\right)^n\left(\frac{n^n}{n!}\right)^2 ,$$

which completes the proof.

REMARK. The right-hand side of (3.2) diverges to infinity (as $n \to \infty$) if $|a|=2$, 3 or 5. In fact, by Stirling's formula, we have

$$\frac{n^n}{n!} > \frac{e^n}{\sqrt{2\pi n}} e^{-1/12n} .$$

On the other hand,

$$\frac{\pi e^2}{4|a|} > 1$$

if $|a|=2$, 3 or 5.

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