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# 3-Dimensional Fano Varieties with Canonical Singularities

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## §0. Introduction.

In this article by a variety we mean an irreducible reduced projective variety over the field of complex numbers.

Let $X$ be a 3-dimensional Fano variety with canonical singularities and $H$ be a Cartier divisor satisfying $-K_X \sim r(X)H$ for the index $r(X)$ of $X$. The purpose of this article is to study the rational map $\Phi_{|H|}$ and singularities of a general member of $|H|$. In particular in the case of $r(X)=2$, which is the most essential, we find $\Phi_{|H|}$ to be as follows.

(1) When $d=H^3 \geqq 3$, a closed immersion into $\boldsymbol{P}^{d+1}$.

(2) When $d=2$, a double covering over $\boldsymbol{P}^3$.

And

(3) when $d=1$, a rational map that is defined except exactly one point and the closure of whose general fiber is a smooth elliptic curve.

And furthermore a general member $S$ of $|H|$ has rational double points at $S \cap \mathrm{Sing}(X)$.

(0.1) Definition. A variety $V$ is called a *Fano variety* whenever the following conditions are satisfied.

(1) $V$ is normal.

(2) $V$ is Gorenstein, i.e. $K_V$ is a Cartier divisor.

(3) The anticanonical divisor $-K_V$ is ample.

(0.2) Definition. For an $n$-dimensional Fano variety $V$, we define the *index* of $V$ to be $\max\{m \in \boldsymbol{Z} \mid \exists$ a Cartier divisor $H$ such that $-K_V \sim mH\}$.

When $V$ is an $n$-dimensional Fano variety, we use the following notation.

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$r(V)$: the index of $V$,
$H$: a Cartier divisor such that $-K_V \sim r(V)H$,
$d(V)=H^n$: the degree of $V$.

(0.3) **Definition.** An $n$-dimensional Fano variety of index $n-1$ is called a *del-Pezzo variety*.

The classification theory of 3-dimensional smooth Fano varieties, which originated with G. Fano, was completed by Iskovskih ([I1], [I2], [I3]), Shokurov ([S1], [S2]) and Mori-Mukai ([MM]). Iskovskih classified all the smooth Fano 3-folds of indices $\geqq 2$ and those of indices $=1$ and $b_2=1$ assuming that two conjectures are true. One of them conjectures that there exists a smooth member in the linear system $|H|$ and the other does that there exists a quasi-line on a smooth Fano 3-fold of index 1. Later all two of them are solved affirmatively by Shokurov. For smooth Fano 3-folds of indices $=1$ and $b_2>1$, Mori and Mukai completed the classification theory.

The motivation of this article is to extend their method to the case where objects have some singularities, say, canonical singularities. Since, as will be shown in §3, Fano 3-folds with canonical singularities whose indices $\geqq 3$ are immediately classified by applying Fujita's $\Delta$-genera theory ([F1]), we may restrict our concern to the cases of indices 1 and 2.

In Iskovskih and Shokurov's theory of classification of smooth del-Pezzo 3-folds, the existence of a smooth member of $|H|$ plays a crucial role. For the singular case Reid shows the following theorem by extending the arguments in [S1].

(0.4) **Theorem** ([R]). *Let $X$ be a 3-dimensional variety which has only canonical singularities. Assume the Weil divisor $-K_X$ is a numerically effective Cartier divisor and $(-K_X)^3>0$. Then a general member of $|-K_X|$ is a K3-surface which has at worst rational double points as its singularities and the dimension of the image of $X$ by a rational map associated with $|-K_X|$ is 2 or 3.*

**Remark.** $\dim \mathrm{Bs}|-K_X| \leqq 1$.

**Remark.** By [KMM, Theorem 3-1-1] and [KMM, Remark 3-1-2] we may assume $-K_X$ to be ample in the above theorem.

Going a little further starting from Reid's theorem, we obtain the following theorem, which is one of our main theorems.

(0.5) **Theorem.** *Assume all the hypotheses in* (0.4). *Then we have*

(1) *If the index $r$ is greater than* 1, *then* $\mathrm{Bs}|-K_X|=\varnothing$.

(2) *If* $\dim \mathrm{Bs}|-K_X|=1$, *then scheme-theoretically* $\mathrm{Bs}|-K_X|\cong \boldsymbol{P}^1$ *and* $\mathrm{Bs}|-K_X|\cap \mathrm{Sing}(S)=\varnothing$ *for a general member* $S\in|-K_X|$, *in particular,* $\mathrm{Bs}|-K_X|\cap \mathrm{Sing}(X)=\varnothing$.

(3) *If* $\dim \mathrm{Bs}|-K_X|=0$ *then* $\mathrm{Bs}|-K_X|$ *consists of exactly one point and a general member of* $S\in|-K_X|$ *has an ordinary double point at* $\mathrm{Bs}|-K_X|$. *In this case* $\mathrm{Bs}|-K_X|\in \mathrm{Sing}(X)$.

When the index $r(X)>1$ we are more interested in properties of a general member of $|H|$ than those of a general member of $|-K_X|$. In this respect we obtain the following theorem.

(0.6) THEOREM. *Let $X$ be a del-Pezzo 3-fold with canonical singularities and $H$ be an ample Cartier divisor such that* $-K_X\sim 2H$. *By $W$ we denote the image of $X$ by the rational map associated with* $|H|$. *Then we have*

(1) *A general member $S$ of $|H|$ has at worst rational double points as its singularities.*

(2) $d(X)>1 \Leftrightarrow \mathrm{Bs}|H|=\varnothing \Leftrightarrow \dim W=3$.

(3) $d(X)=1 \Leftrightarrow \mathrm{Bs}|H|\neq\varnothing \Leftrightarrow \mathrm{Bs}|H|=\{\textit{one point}\} \Leftrightarrow \dim W=2$.

*Moreover a general* $S\in|H|$ *is smooth at* $\mathrm{Bs}|H|$ *if* $d(X)=1$.

(0.7) COROLLARY. *Let $X$ and $H$ be the same as in* (0.6). *Then the polarized variety* $(X, \mathcal{O}_X(H))$ *has a regular ladder* (*see* (3.2)).

From [HW, Theorem 4.4] and Theorem (0.6) (1) we immediately obtain the following corollary.

(0.8) COROLLARY. *Under the same hypothesis and notation as in* (0.6), *we have*

(1) *If* $d(X)\geqq 3$ *then $H$ is very ample and $W$ is a projectively normal subvariety of degree $d(X)$ in* $\boldsymbol{P}^{d(X)+1}$. *Moreover*

(1.a) *If* $d(X)\geqq 4$, *then* $W\subset \boldsymbol{P}^{d(X)+1}$ *is a scheme-theoretic intersection of* $d(X)(d(X)-3)/2$ *quadrics.*

(1.b) *If* $d(X)=3$, *then $W$ is cubic in* $\boldsymbol{P}^4$.

(2) *If* $d(X)=2$, *then* $\Phi_{|H|}$ *gives a double cover over* $\boldsymbol{P}^3$ *whose branch divisor is a quadric hypersurface and $X$ is isomorphic to a hypersurface of degree* 4 *in* $\boldsymbol{P}(1, 1, 1, 1, 2)$. *In particular $mH$ is very ample whenever* $m\geqq 2$.

(3) *If* $d(X)=1$ *then $X$ is isomorphic to a hypersurface of degree* 6 *in* $\boldsymbol{P}(1, 1, 1, 2, 3)$. *In particular $mH$ is very ample whenever* $m\geqq 3$.

REMARK. We have another proof of (2) and (3) of Theorem (0.6) through Corollary (0.8).

When $H$ is very ample and $\Delta(V, H)=1$, $V$ is a subvariety of degree $d$ in $\boldsymbol{P}^{d+1}$. Such varieties are studied in [F3].

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## § 1. Preliminary.

First we recall some basic results.

(1.1) PROPOSITION. *Let $V$ be an arbitrary variety and $D$ a Cartier divisor on $V$. If $H^1(V, \mathcal{O}_V(-D))=0$, then any member of $|D|$ is connected. Conversely if $D$ is reduced and connected, then $H^1(V, \mathcal{O}_V(-D))=0$ provided that $H^1(V, \mathcal{O}_V)=0$.*

For instance in our case this proposition guarantees that every member of $|mH|$ is connected for any positive integer $m$.

(1.2) PROPOSITION. *Let $V$ be an $n$-dimensional Fano variety with canonical singularities. Then*

(1) $H^i(V, \mathcal{O}_V(mH))=0$ *for any integers $m$ and $i$ such that $0<i<n$,*

(2) $H^0(V, \mathcal{O}_V(mH))=0$ *for $m<0$, and*

(3) $H^n(V, \mathcal{O}_V(mH))=0$ *for $m>-r(V)$.*

PROOF. The proof immediately follows from Serre duality and [KMM, Theorem 1-2-5]. □

(1.3) PROPOSITION. *Let $V$ be a 3-dimensional Fano variety with canonical singularities. Then*

$$h^0(mH)=\frac{m(m+r)(2m+r)}{12}d+\frac{2m}{r}+1 .$$

PROOF. Applying Proposition (1.2) to Riemann-Roch formula,

$$h^0(mH)=\frac{m(m+r)(2m+r)}{12}d+\left(\frac{2m}{r}+1\right)\frac{c_1\cdot c_2}{24} .$$

When $m=0$, from this we obtain $1=\chi(\mathcal{O}_V)=c_1\cdot c_2/24$. Hence we obtain the assertion. □

## § 2. Proof of (0.5).

The next lemma, which is a slight generalization of [SD, Proposition 8.1], is the key to the proof of (1.5).

(2.1) Lemma. *Let $S$ be a nonsingular $K3$-surface and $D$ be a numerically effective divisor on $S$ such that $D^2>0$. Then either of the followings holds.*

(1) $\mathrm{Bs}|D|=\varnothing$.

(2) *There exist a nonsingular elliptic curve $E$, a $(-2)$-curve $\Gamma$ on $S$ and an integer $k>1$ such that $D\sim kE+\Gamma$.*

Proof. Letting $F$ denote the fixed part of $|D|$, we have $D\sim M+F$, where $|M|$ is the movable part. The following properties hold.

(a) Every component $\Delta$ of $F$ is a $(-2)$-curve, i.e. $\Delta\cong \boldsymbol{P}^1$ and $\Delta\cdot\Delta=-2$ (see [SD, 2.7.1]).

(b) $M$ is linearly equivalent to

(b.1) a prime divisor, or

(b.2) $kE$ for a smooth elliptic curve $E$ and an integer $k\geqq 2$.

Furthermore, $\mathrm{Bs}|M|$ is empty in the both cases (see [SD, Proposition 2.6]).

(c) When $M\sim kE$ for an elliptic curve $E$ and an integer $k\geqq 2$, there exists a component $\Gamma$ of $F$ which intersects with $E$ effectively. Actually this immediately follows from (1.1) and [KMM, Theorem 1-2-5].

Furthermore such a $\Gamma$ is unique and $\Gamma\cdot E=1$ (see [SD, 2.7.1]).

Let $L$ be $M$ in the case (b.1) or $M+\Gamma$ in the case (b.2). By Riemann-Roch formula and (1.1), we get

$$\begin{array}{c} h^0(L)=\dfrac{L^2}{2}+2 \\ \| \\ h^0(D)=\dfrac{D^2}{2}+2 \end{array}$$

and so $2L\cdot G+G\cdot G=0$, where $D\sim L+G$. On the other hand from numerically effectiveness of $D$, $L\cdot G+G\cdot G=0$, therefore $D\cdot G=G\cdot G=0$. This implies $G=0$ by the Hodge index theorem. □

The next corollary immediately follows from the above lemma.

(2.2) Corollary. *Under the same condition as in* (2.1), $\mathrm{Bs}|mD|=\varnothing$ *for* $m\geqq 2$.

(2.3) PROOF OF (0.5). Theorem (0.4) shows a general member of $|-K_X|$ is an irreducible normal Gorenstein surface with $\mathcal{O}_S(K_S)\cong\mathcal{O}_S$ whose singularities are at worst rational double points.

By the vanishing theorem, $H^0(X, \mathcal{O}_X(-K_X))\to H^0(S, \mathcal{O}_S(-K_X))$ is surjective, hence $\mathrm{Bs}|-K_X|$ coincides with $\mathrm{Bs}|(-K_X)\cdot S|$. Let $\mu\colon T\to S$ be the minimal resolution of $S$. $T$ is a smooth $K3$-surface. $D=\mu^*(-K_X)$ is numerically effective and $D^2>0$.

If $r(X)>1$ then $\mathrm{Bs}|D|$ must be empty by (2.2). Thus we obtain (1).

If $\mathrm{Bs}|-K_X|$ is not empty, it must be a smooth rational curve or exactly one point. We claim that $\mathrm{Bs}|-K_X|\cap\mathrm{Sing}(S)=\varnothing$ if $\dim\mathrm{Bs}|-K_X|=1$ and that $\mathrm{Bs}|-K_X|\in\mathrm{Sing}(S)$ if $\dim\mathrm{Bs}|-K_X|=0$. Actually if the claim were false, $|D|$ would have more than two fixed components, which contradicts (2.1).

Therefore we only have to check $\mathrm{Bs}|-K_X|\in\mathrm{Sing}(X)$ when $\dim\mathrm{Bs}|-K_X|$ consists of a point. Assuming $\mathrm{Bs}|-K_X|\in X_{\mathrm{reg}}$, we shall derive a contradiction.

We take the monoidal transform $\pi\colon Y\to X$ with center $\mathrm{Bs}|-K_X|$. When we write $\pi^*(-K_X)\sim M+mE$ where $E$ is the exceptional divisor of $\pi$ and $mE$ is the fixed part of $|\pi^*(-K_X)|$, we have $m=2$. Let $T$ be a general irreducible member of $|M|$ which dominates $S$. We have

$$h^0(T, \mathcal{O}_T(M))=h^0(Y, \mathcal{O}_Y(M))-1=h^0(X, \mathcal{O}_X(-K_X))-1\ .$$

On the other hand

$$h^0(T, \mathcal{O}_T(\pi^*(-K_X))=h^0(S, \mathcal{O}_S(-K_X))=h^0(X, \mathcal{O}_X(-K_X))-1\ .$$

Thus it follows that $2E\cdot T=|\pi^*(-K_X)\cdot T|_{\mathrm{fix}}$, which contradicts (2.1). □

## §3. On the theory of polarized varieties.

In this section we collect some theorems from the theory of $\Delta$-genera by T. Fujita. First of all we define some notation.

(3.1) DEFINITION. A *prepolarized variety* $(V, L)$ is defined to be a pair of a variety $V$ and a line bundle $L$ on $V$. For an $n$-dimensional Gorenstein prepolarized variety $(V, L)$ we define the following symbols and call these values the *degree, sectional genus* and *$\Delta$-genus* of $(V, L)$, respectively:

$$d(V, L)=L^n\ ,$$
$$g(V, L)=\frac{(K_V+(n-1)L)\cdot L^{n-1}}{2}+1\ ,$$

$$\Delta(V, L)=n+d(V, L)-h^0(V, L) .$$

Furthermore if $L$ is ample, we call $(V, L)$ a *polarized variety*.

(3.2) DEFINITION. (1) Let $(V, L)$ be an $n$-dimensional prepolarized variety. When a sequence $(D_n, \cdots, D_1)$ of subvarieties of $V$ satisfies the conditions below, we call the sequence a *ladder* of $(V, L)$.
(a) $D_n=V$.
(b) For $j>1$, $D_{j-1}$ is an irreducible reduced member of $|L_j|$, where $L_j$ is a restriction of $L$ to $D_j$.

(2) For a prepolarized variety $(V, L)$, its ladder $(D_n, \cdots, D_1)$ is *regular* whenever natural homomorphisms $H^0(D_j, L_j)\to H^0(D_{j-1}, L_{j-1})$ for $n\geqq j\geqq 2$ are all surjective.

(3.3) DEFINITION. Let $V$ be a variety and $L$, $M$ line bundles on $V$. We denote by $R(L, M)$ the kernel of the natural homomorphism $\Gamma(L)\otimes\Gamma(M)\to \Gamma(L\otimes M)$. We say $L$ is *simply generated* whenever natural homomorphisms $\Gamma(L)\otimes\Gamma(L^{\otimes t})\to\Gamma(L^{\otimes t+1})$ is surjective for every positive integer $t$. Furthermore $L$ is *quadratically represented*, whenever $L$ is simply generated and $R(L^{\otimes s}, L^{\otimes t})\otimes\Gamma(L)\to R(L^{\otimes s}, L^{\otimes t+1})$ is surjective for every pair of positive integers $(s, t)$.

(3.4) THEOREM ([F1, Theorem 4.1]). *Let $(V, L)$ be a prepolarized variety which has a ladder. Assume $\Delta(V, L)\leqq g(V, L)$ and $d(V, L)>0$. Then*

(1) *the ladder is regular if $d\geqq 2\Delta-1$,*
(2) $\mathrm{Bs}|L|=\varnothing$ *if $d\geqq 2\Delta$,*
(3) *$g(V, L)=\Delta(V, L)$ and $L$ is simply generated if $d\geqq 2\Delta+1$, and*
(4) *$L$ is quadratically represented if $d\geqq 2\Delta+2$.*

(3.5) THEOREM ([F2, Theorem 1.9]). *Let $(V, L)$ be a polarized variety. Then* $\dim \mathrm{Bs}|L|<\Delta(V, L)$, *where* $\dim\varnothing$ *is defined to be $-1$. In particular $\Delta(V, L)$ is not negative.*

In the case of dimension three, which is our main concern, we have the next corollary to Theorem (3.5).

(3.6) COROLLARY. *If $X$ is a 3-dimensional Fano variety with canonical singularities, then the index of $X$ does not exceed 4. Further, $d(X)=1$ if $r(X)=4$. Moreover $d(X)=2$ if $r(X)=3$.*

PROOF. By definition,

$$\begin{aligned}\Delta(X, \mathcal{O}_X(H)) &= 3 + d(X) - h^0(X, \mathcal{O}_X(H)) \\ &= 2 - \frac{(r(X)+5)(r(X)-2)}{12} d(X) - \frac{2}{r(X)} \geqq 0 .\end{aligned}$$

Solving this inequality, we obtain the assertion. □

(3.7) THEOREM ([F2, Theorem 2.1 and Theorem 2.2]). *Let $(V, L)$ be an $n$-dimensional polarized variety. Then*

(1) *if $d(V, L)=1$ and $\Delta(V, L)=0$, then $(V, L) \cong (\boldsymbol{P}^n, \mathcal{O}_{\boldsymbol{P}^n}(1))$, and*

(2) *if $d(V, L)=2$ and $\Delta(V, L)=0$, then $V$ is isomorphic to a hyperquadric in $\boldsymbol{P}^{n+1}$ and $H$ is the restriction of $\mathcal{O}_{\boldsymbol{P}^{n+1}}(1)$ to $V$.*

(3.8). Here we compute $\Delta$-genera and sectional genera of 3-dimensional Fano varieties $X$ with canonical singularities.

(1) When $r(X)=4$,

$$\Delta(X, \mathcal{O}_X(H))=0 \quad \text{and} \quad g(X, \mathcal{O}_X(H))=0 .$$

(2) When $r(X)=3$,

$$\Delta(X, \mathcal{O}_X(H))=0 \quad \text{and} \quad g(X, \mathcal{O}_X(H))=0 .$$

(3) When $r(X)=2$,

$$\Delta(X, \mathcal{O}_X(H))=1 \quad \text{and} \quad g(X, \mathcal{O}_X(H))=1 .$$

(4) When $r(X)=1$,

$$\Delta(X, \mathcal{O}_X(H))=\frac{d}{2} \quad \text{and} \quad g(X, \mathcal{O}_X(H))=\frac{d}{2}+1 .$$

Applying (3.6) to our case, we obtain the following

(3.9) THEOREM. *Let $X$ be a 3-dimensional Fano variety with canonical singularities. Then*

(1) *if $r(X)=4$, $X$ is isomorphic to $\boldsymbol{P}^3$, and*

(2) *if $r(X)=3$, $X$ is isomorphic to a hyperquadric in $\boldsymbol{P}^4$.*

## §4. Proof of (0.6).

(4.1) PROOF OF THEOREM (0.6). Applying (3.5) to a polarized variety $(X, \mathcal{O}_X(H))$, we have $\dim \mathrm{Bs}|H| < \Delta(X, \mathcal{O}_X(H))=1$.

Denoting the rational map associated with the linear system $|H|$ by $\Phi: X \to \Phi(X) = W \subset \boldsymbol{P}|H|$, $\Phi$ cannot be a pencil because $\dim \mathrm{Bs}|H| \leqq 0$, hence $\dim W \geqq 2$.

Let $\pi: Y \to X$ be a proper birational morphism that satisfies the following conditions (a)-(c). Define integers $r_j$ and $a_j$ as satisfying

$$\pi^*H \sim M + \sum_{j=1}^{n} r_j E_j \quad \text{and} \quad K_Y \sim \pi^* K_X + \sum_{j=1}^{n} a_j E_j ,$$

respectively, where the $E_j$ are all the exceptional divisors of $\pi$.

(a) $Y$ is nonsingular.

(b) The movable part $M$ of $|\pi^*H|$ is base point free.

(c) If $\dim \pi(E_j)=1$, then $a_j=0$.

Because $\dim W \geqq 2$ we can take an irreducible member $S$ of $|H|$. Let $T$ be an irreducible smooth member of $|M|$ that is dominating $S$. We claim that $H^1(T, \mathcal{O}_T) = H^2(T, \mathcal{O}_T) = 0$.

$H^2(T, \mathcal{O}_T) = 0$ follows from Serre duality and the fact that $-K_S$ is ample.

By definition,

$$h^0(S, \mathcal{O}_S(H)) = h^0(T, \mathcal{O}_T(M)) \leqq h^0\Big(T, \mathcal{O}_T\Big(M + \sum_{j=1}^{n} a_j E_j\Big)\Big) .$$

On the other hand by Riemann-Roch for surfaces and [KMM, 1-2-5], we have

$$h^0\Big(T, \mathcal{O}_T\Big(M + \sum_{j=1}^{n} a_j E_j\Big)\Big) = \pi^*H \cdot \Big(M + \sum_{j=1}^{n} a_j E_j\Big) \cdot T - h^1(T, \mathcal{O}_T) + 1 .$$

Combining these inequality and equality, we obtain

$$H^3 + 1 \leqq \pi^*H \cdot \Big(M + \sum_{j=1}^{n} a_j E_j\Big) \cdot T - h^1(T, \mathcal{O}_T) + 1 ,$$

hence

$$\begin{aligned} h^1(T, \mathcal{O}_T) &\leqq \pi^*H \cdot \Big\{\Big(M + \sum_{j=1}^{n} a_j E_j\Big) \cdot M - \Big(\sum_{j=1}^{n} r_j E_j + M\Big)^2\Big\} \\ &= \pi^*H \cdot \Big\{\sum_{j=1}^{n} (a_j - r_j) E_j \cdot M - \sum_{j=1}^{n} r_j E_j \cdot \pi^*H\Big\} . \end{aligned}$$

If $\dim \pi(E_j) = 0$, then $\pi^*H \cdot E_j \approx 0$. On the other hand, when $\dim \pi(E_j) = 1$ it follows that $r_j = 0$ from $\dim \mathrm{Bs}|H| \leqq 0$ and further $a_j = 0$ follows from the construction of $\pi$. Finally from the above inequality we get $h^1(T, \mathcal{O}_T) = 0$.

Note that $S$ is normal because $S$ is Gorenstein and has only isolated singularities.

Since $h^2(S, \mathcal{O}_S) = h^0(S, \mathcal{O}_S(-H)) = 0$ and $h^1(T, \mathcal{O}_T) = 0$ as was seen above, we have $R^1\pi_*\mathcal{O}_T\cong 0$, hence $S$ has only rational double points as its singularities.

If $\dim W=3$, then $(X, \mathcal{O}_X(H))$ has a ladder. Note that if $d=1$ then $h^0(X, \mathcal{O}_X(H))=3$, hence $W\cong \boldsymbol{P}^2$. So if $\dim W=3$, then $\mathrm{Bs}|H|=\emptyset$ by Theorem (3.4). Thus in order to complete the proof we only have to check the assertion when $\dim W=2$.

$$\begin{tikzcd} U \arrow[r, dashed] \arrow[d, "\mu"'] & \tilde{V} \arrow[d, "\nu"] \\ S \arrow[r, "\Phi|_S"] & V\subset W \end{tikzcd}$$

Let $\mu: U\rightarrow S$ be the minimal resolution of singularities of $S$. In particular $\mu^*K_S=K_U$. Denoting $\Phi(S)$ by $V$, we have $\dim V=1$. Let $\nu$: $\tilde{V}\rightarrow V$ be the normalization. $\tilde{V}\cong \boldsymbol{P}^1$ since $h^1(U, \mathcal{O}_U)=0$. Note that $\nu$ factors $\Phi|_S$. Because

$$h^0(S, \mathcal{O}_S(H))=h^0(V, \mathcal{O}_V(1))\leqq h^0(\tilde{V}, \nu^*\mathcal{O}_V(1))\leqq h^0(S, \mathcal{O}_S(H)),$$

$\nu^*|\mathcal{O}_V(1)|$ gives a complete linear system. Thus $\nu$ turns out to be an isomorphism. So we have $\mu^*H_S\sim dE+Z$ for some Cartier divisor $E$ on $U$ where $H_S$ is the restriction of $H$ to $S$ and $Z$ is the fixed part of $|\mu^*H_S|$.

$$d=(\mu^*H_S)^2=d^2E^2+dE\cdot Z+\mu^*H_S\cdot Z=d^2E^2+dE\cdot Z. \tag{4.2}$$

Since $\mu^*H$ is numerically effective and $E$ is movable, none of $E^2$ and $E\cdot Z$ is negative.

If $d>1$, we get $E^2=0$. Hence $E\cdot Z=1$.

If $d=1$ and if $S$ is singular at a point $\in \mathrm{Bs}|H_S|=\mathrm{Bs}|H|$, then the fixed part $Z$ is not zero. So by (1.1), $E\cdot Z>0$, hence $E^2=0$ and $E\cdot Z=1$.

In the above two cases, we can compute the genus of $E$ as follows:

$$2p_g-2=E\cdot(K_U+E)=E\cdot(\mu^*K_S+E)=-E\cdot((d-1)E+Z)=-1.$$

This contradicts the fact that $p_g$ is a positive integer.

Consequently we have $d=1$ and $S$ is smooth at $\mathrm{Bs}|H|$ if $\dim W=2$. Moreover observing (4.2) again we have $E^2=1$, hence $\mathrm{Bs}|H|$ consists of exactly one point. This completes the proof. □

(4.3) PROOF OF (0.8). Let $S$ be a general member of $|H|$. By [KMM, Theorem 1-2-5] a natural map $H^0(X, \mathcal{O}_X(tH))\rightarrow H^0(S, \mathcal{O}_S(tH))$ is surjective for $t\geqq 0$. Then we can calculate the graded algebra $R(X, H)=$

$\bigoplus_{t \geq 1} H^0(X, \mathcal{O}_X(tH))$ from $R(S, H|_S)$. Thus we obtain the assertion by [HW, Theorem 4.4]. $\square$

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