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On Minimum Genus Heegaard Splittings of Some Orientable Closed 3-Manifolds

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Dedicated to Professor Fujitsugu Hosokawa on his 60th birthday

Abstract. In this paper we deal with all 3-manifolds which are obtained by glueing the boundaries of two Seifert fibered spaces over a disk with two exceptional fibers. We will give a necessary and sufficient condition for those 3-manifolds to admit Heegaard splittings of genus two. Moreover we will evaluate the numbers of Heegaard splittings of genus two, up to isotopy, of those 3-manifolds. In fact, we will see that the numbers are at most four.

§0. Introduction.

Let M be an orientable closed 3-manifold. Then it is well-known that M can be splitted into two handlebodies. The splitting is called a Heegaard splitting, and denoted by $(V_1, V_2; F)$, where V_i is a handlebody $(i=1, 2)$, $M=V_1\cup V_2$ and $V_1\cap V_2=\partial V_1=\partial V_2=F$. Then F is called a Heegaard surface and the genus of F is called the genus of the Heegaard splitting. Two Heegaard splittings $(V_1, V_2; F)$ and $(W_1, W_2; G)$ of the same genus of M are called homeomorphic if there exists an auto-homeomorphism f of M with $f(F)=G$, and are called isotopic if the homeomorphism f is isotopic to the identity on M.

By $D(2)$, we denote the family of all Seifert fibered spaces over a disk with two exceptional fibers. For any element S of $D(2)$, S is oriented and ∂S has the orientation induced from that of S. For a fiber h in ∂S and the boundary loop c of a cross section of S, h and c are oriented so that the algebraic intersection number of h and c (in this order) is 1.

Let S_1 and S_2 be two elements of $D(2)$, and let $f\colon \partial S_2\to\partial S_1$ be a

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homeomorphism. Then we have an orientable closed 3-manifold $M=S_1\cup_f S_2$ by glueing ∂S_1 and ∂S_2 by f.

We denote a fiber in ∂S_i by h_i $(i=1, 2)$. We denote an orientable twisted I-bundle over a Klein bottle by KI, and denote a $(2, n)$-torus knot exterior in S^3 by $E_{2,n}$ for an odd integer $n>1$. If $S_i=KI$, then by u_i we denote a fiber in ∂S_i as a circle bundle over a Möbius band $(i=1, 2)$. If $S_i=E_{2,n}$, then by m_i we denote a meridian loop in $\partial E_{2,n}$ $(i=1, 2)$. Note that if $S_i=KI$ ($E_{2,n}$ resp.) then u_i (m_i resp.) is the boundary loop of a cross section of S_i $(i=1, 2)$. For two oriented loops x and y in a torus, we denote the algebraic intersection number of x and y by $I(x, y)$.

In this paper, we regard an oriented loop as an element of the first homology group. Then $\{h_i, c_i\}$ is a basis of $H_1(\partial S_i)$ $(i=1, 2)$, and f is represented by a matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ with $ad-bc=\pm 1$ such that $\begin{bmatrix} f(h_2) \\ f(c_2) \end{bmatrix}=\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} h_1 \\ c_1 \end{bmatrix}$, where c_i is the boundary loop of a cross section of S_i $(i=1, 2)$. Then we have:

PROPOSITION 1. *$M=S_1\cup_f S_2$ admits a Heegaard splitting of genus three.*

THEOREM 1. *$M=S_1\cup_f S_2$ admits a Heegaard splitting of genus two if and only if one of the following conditions holds*:

(1) $\begin{bmatrix} f(h_2) \\ f(c_2) \end{bmatrix}=\begin{bmatrix} a & \varepsilon \\ c & d \end{bmatrix}\begin{bmatrix} h_1 \\ c_1 \end{bmatrix}$ *with* $ad-\varepsilon c=\pm 1$ *and* $\varepsilon=\pm 1$,

(2) $S_1=E_{2,\alpha}$, $S_2=KI$ *and* $\begin{bmatrix} f(h_2) \\ f(u_2) \end{bmatrix}=\begin{bmatrix} \varepsilon & b \\ 0 & \delta \end{bmatrix}\begin{bmatrix} h_1 \\ m_1 \end{bmatrix}$ *with* $\varepsilon\delta=\pm 1$ *or*

(3) $S_1=KI$, $S_2=E_{2,\beta}$ *and* $\begin{bmatrix} f(h_2) \\ f(m_2) \end{bmatrix}=\begin{bmatrix} \varepsilon & b \\ 0 & \delta \end{bmatrix}\begin{bmatrix} h_1 \\ u_1 \end{bmatrix}$ *with* $\varepsilon\delta=\pm 1$.

REMARK 1. The condition (1) of Theorem 1 is equivalent to the condition $I(h_1, f(h_2))=\pm 1$. The condition (2) ((3) resp.) of Theorem 1 is equivalent to the condition $I(m_1, f(u_2))=0$ ($I(u_1, f(m_2))=0$ resp.).

REMARK 2. In the case when M is not a Seifert fibered space, the above result has been showed in Theorem of [7]. In the case when M is a Seifert fibered space, the above result has been showed in Theorem 1.1 of [3]. Theorem 1 therefore is obtained by combining these results. In this paper, by improving the argument of the proof of Theorem of [7], we will give a proof which is not influenced by whether M is a Seifert fibered space or not.

REMARK 3. For the details of 3-manifolds obtained from two twisted I-bundles over a Klein bottle, see [9].

THEOREM 2. *$M=S_1\cup_f S_2$ admits at most four non-isotopic Heegaard splittings of genus two.*

In section 5, we will give a more detailed evaluation of the numbers of Heegaard splittings of genus two, up to isotopy, of $M=S_1\cup_f S_2$. See Table 5.2.

By $S(b;\beta_1/\alpha_1,\beta_2/\alpha_2,\beta_3/\alpha_3,\beta_4/\alpha_4)$ we denote a Seifert fibered space over a 2-sphere with four exceptional fibers, where β_i/α_i is the Seifert invariant of the exceptional fiber ($1\leqq i\leqq 4$) and b is an integer representing the obstruction class (cf. [13] or [17]). Then, by Theorem 1 and Table 5.2, we have the following corollaries.

COROLLARY 1 (cf. Theorem 1.1 of [3]). *Let M be a Seifert fibered space over a 2-sphere with four exceptional fibers. Then M admits a Heegaard splitting of genus two if and only if M is homeomorphic to $S(0;1/2,1/2,-1/2,-a/(2a+1))$ for some positive integer a.*

Moreover $S(0;1/2,1/2,-1/2,-a/(2a+1))$ admits exactly one Heegaard splitting of genus two up to isotopy.

REMARK 4. The first half of the above corollary has been already obtained by using another method in Theorem 1.1 of [3].

COROLLARY 2. *Let M be an orientable Seifert fibered space over a projective plane with two exceptional fibers. Then M admits at most two non-isotopic Heegaard splittings of genus two.*

By the proof of Theorem 2, we will see that in almost cases M admits at most two non-isotopic Heegaard splittings of genus two. In particular, we will see that the 3-manifolds which may admit four non-isotopic Heegaard splittings of genus two are only $M=E_{2,\alpha}\cup_f E_{2,\beta}$ with $\begin{bmatrix} f(h_2) \\ f(m_2)\end{bmatrix}=\begin{bmatrix} 0 & \varepsilon \\ \delta & 0\end{bmatrix}\begin{bmatrix} h_1 \\ m_1\end{bmatrix}$ $(\varepsilon\delta=\pm 1)$. We denote this manifold by $M_{\alpha,\beta,\varepsilon\delta}$. By Theorem 8 of [1], a Heegaard splitting of genus two of an orientable closed 3-manifold corresponds to a 6-plat representation of a 3-bridge knot or link in S^3. Then the four 6-plat representations of the 3-bridge knots or links corresponding to the four Heegaard splittings of genus two of $M_{\alpha,\beta,1}$ are those ones illustrated in Figure 0.1, where $\alpha=2a+1$ $(a>0)$ and $\beta=2b+1$ $(b>0)$. Since the four knots in Figure 0.1 are all equivalent, we denote the knot by $K_{a,b,1}$. If $a=1$ or $b=1$, then by Proposition 5.3, two Heegaard splittings corresponding to the 6-plat representations $(K_{a,b,1},S_1)$ and $(K_{a,b,1},S_2)$ are mutually isotopic. If $a>1$ and $b>1$, then it seems that the four Heegaard splittings of genus two of $M_{\alpha,\beta,1}$ corresponding to the four 6-plat representations of $K_{a,b,1}$ are all mutually non-isotopic. The author, however, has no proof. For $M_{\alpha,\beta,-1}$, we have the knot $K_{a,b,-1}$ similar to $K_{a,b,1}$ by substituting the tangle T_b in the diagram of $K_{a,b,1}$ for the tangle T_{-b} illustrated in Figure 0.2.

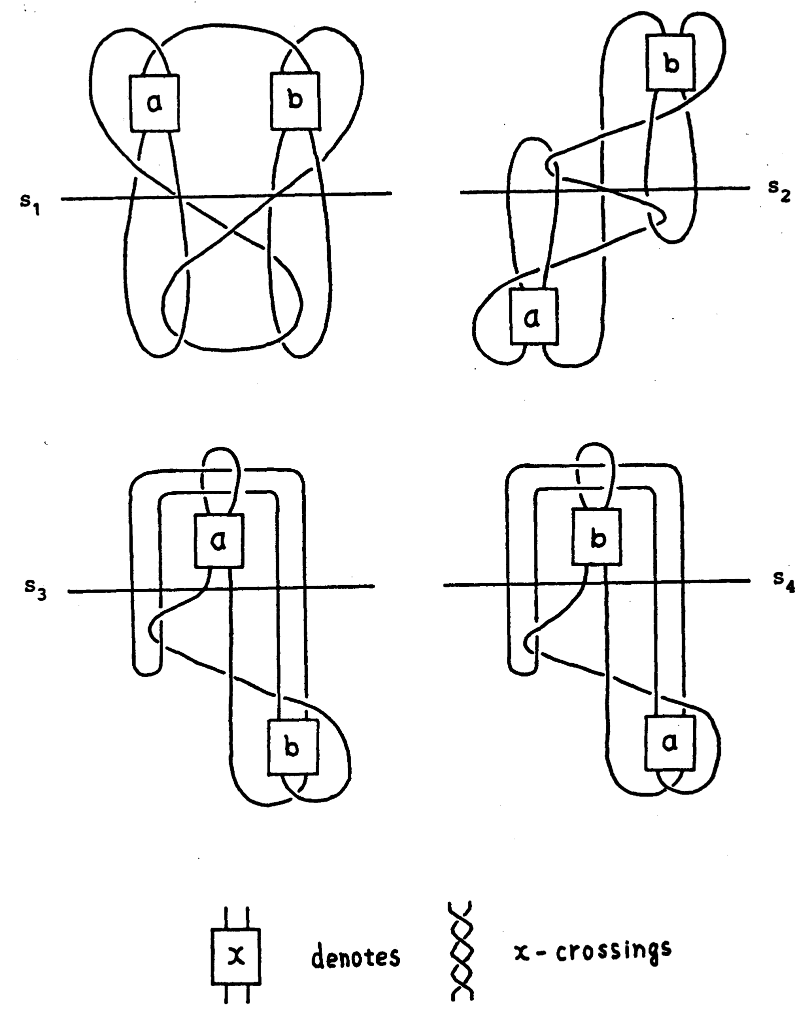

FIGURE 0.1

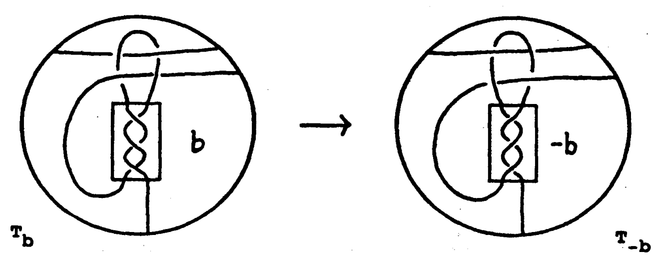

FIGURE 0.2

Now, we will prove the above theorems as follows:

First, by improving the argument of the proof of Theorem of [7], we will show the following lemma.

LEMMA 1.1. *Let M be an orientable closed 3-manifold which admits a Heegaard splitting $(V_1, V_2; F)$ of genus two.*

Suppose that M contains a family Σ consisting of finitely many mutually disjoint incompressible tori. Then Σ is ambient isotopic to a family of tori which intersects V_i in essential annuli $(i=1, 2)$.

By applying Lemma 1.1 to $M=S_1\cup_f S_2$ and by careful consideration, we will obtain Lemma 1.5, which minutely analyzes the intersections of the torus ∂S_1 $(=\partial S_2)$ and the handlebodies of the Heegaard splitting.

Then Theorem 1 will be proved immediately by Lemma 1.5 and using the argument similar to the proof of Theorem of [7].

REMARK 5. Lemma 1.1 does not hold in general if the genus of the Heegaard splitting is greater than 2. See the introduction of [8] (cf. Lemma 3.1 of [10]).

Next, we will introduce several families of Heegaard surfaces of genus two of $M=S_1\cup_f S_2$, which are described in section 3. Then by Lemma 1.5, we can see that any Heegaard surface of genus two of M is ambient isotopic to a Heegaard surface belonging to one of the families (Proposition 3.1).

To prove Theorem 2 we have to evaluate the numbers, up to isotopy, of Heegaard surfaces of each family. For this purpose, we will show the following two theorems.

We say that an orientable closed 3-manifold is a lens space if it admits a Heegaard splitting of genus one (cf. [5]). Let L be a lens space and K a knot in L. We say that K is a core of L if $\mathrm{Cl}(L-N(K))$ is a solid torus, where $N(K)$ is a regular neighborhood of K in L, K is a torus knot in L if there exists a torus in L which contains K and splits L into two solid tori, and K is a trivial knot if K bounds a disk in L.

THEOREM 3. *Let L be a lens space and K a 1-bridge knot in L, and let $(V_1, V_2; G)$ be a Heegaard splitting of genus one of L which gives a 1-bridge representation of K i.e., $\alpha_i=V_i\cap K$ is a single trivial arc in V_i $(i=1, 2)$.*

Suppose that K is a non-trivial torus knot and is not a core of L. Then for $i=1, 2$, there exists a disk Δ_i in V_i such that $\partial V_i\cap\Delta_i=\beta_i$ is *an arc in* ∂V_i, $\partial \Delta_i = \alpha_i \cup \beta_i$ *and* $\beta_1 \cap \beta_2 = \partial\beta_1 = \partial\beta_2$.

NOTE. The important point of this theorem is the last assertion $\beta_1 \cap \beta_2 = \partial\beta_1 = \partial\beta_2$.

Theorem 3 says that any 1-bridge representation of a torus knot in a lens space is trivial. The next theorem says that 2-bridge representations of a $(2, n)$-torus knot in S^3 are unique up to ambient isotopy rel. the knot.

THEOREM 4. *Let K be a non-trivial $(2, n)$-torus knot in S^3, and let S_1 and S_2 be 2-spheres in S^3 each of which gives a 2-bridge representation of K.*

Suppose $S_1 \cap K = S_2 \cap K$ (=4-points). Then there exists an ambient isotopy f_t $(0 \leqq t \leqq 1)$ of S^3 such that $f_0 = \mathrm{id.}$, $f_1(S_2) = S_1$ and $f_t | K$ is the identity on K $(0 \leqq t \leqq 1)$.

NOTE. The important point of this theorem is the last condition that $f_t | K$ is the identity on K $(0 \leqq t \leqq 1)$.

Then, by combining these results, we will show Theorem 2.

Concerning the numbers of Heegaard splittings of genus two, M. Boileau and J. P. Otal proved in [2] that any Seifert fibered space over a 2-sphere with three exceptional fibers admits at most three non-isotopic Heegaard splittings of genus two. And J. Hass proved in [4] that any orientable closed hyperbolic 3-manifold admits finitely many non-isotopic Heegaard splittings of genus two. These facts, however, do not hold in general. Recently M. Sakuma proved in [15] that there exist infinitely many orientable closed 3-manifolds each of which admits infinitely many non-isotopic Heegaard splittings of genus two. But the author does not know whether there exists a 3-manifold which admits infinitely many non-homeomorphic Heegaard splittings of genus two.

This paper is organized as follows. In section 1, Lemmas 1.1 and 1.5 will be proved. In section 2, we will prove Proposition 1 and Theorem 1. In section 3, we will describe several families of Heegaard surfaces of genus two and show Proposition 3.1. In section 4, Theorems 3 and 4 will be proved. Then, by combining these results, we will prove Theorem 2 and Corollaries 1, 2 in section 5.

Throughout this paper we will work in the piecewise linear category. For the definitions of the standard terms in 3-manifold topology and knot theory, we refer, [5], [6] and [14].

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§1. Some lemmas to prove Theorems 1 and 2.

We say that a surface F properly embedded in a compact 3-manifold M is ∂-parallel if F is isotopic to a surface in ∂M rel. ∂F, and F is essential in M if F is incompressible and is not ∂-parallel. For a given manifold X and a submanifold Y, $N(Y)$ denotes a regular neighborhood of Y in X.

PROOF OF LEMMA 1.1. We may assume that each component of $\Sigma \cap V_1$ is a disk and that $\#(\Sigma \cap V_1)$ is minimal among all families consisting of tori which are ambient isotopic to Σ and intersect V_1 in disks, where $\#(\Sigma \cap V_1)$ is the number of components of $\Sigma \cap V_1$.

Put $\Sigma_1 = \Sigma \cap V_1$ and $\Sigma_2 = \Sigma \cap V_2$.

CLAIM 1. *Σ_2 is incompressible in V_2.*

Since M admits a Heegaard splitting of genus two and contains incompressible tori, M is irreducible. Then Claim 1 follows from the irreducibility of M, the incompressibility of Σ and the minimality of $\#(\Sigma \cap V_1)$.

Let $E = E_1 \cup E_2$ be a complete meridian disk system of V_2, i.e. E_1, E_2 are mutually disjoint disks properly embedded in V_2 $(i=1, 2)$ and $\mathrm{Cl}(V_2 - N(E_1 \cup E_2))$ is a 3-ball. By Claim 1, we may assume that Σ_2 intersects E in arcs.

Let a be an outermost arc component of $E \cap \Sigma_2$ in E. If a is an inessential arc in Σ_2, i.e. a cuts off a disk in Σ_2, then by using this disk, we can exchange E for another complete meridian disk system E' so that $\#(E' \cap \Sigma_2) < \#(E \cap \Sigma_2)$. Hence as in Ch. II of [6], at each stage by exchanging complete meridian disk systems if necessary, we have a sequence of isotopies of type A at arcs a_i $(1 \leqq i \leqq n)$ each of which is an essential arc properly embedded in Σ_2^{i-1}, where $\Sigma_2^0 = \Sigma_2$, $\Sigma_2^i = \mathrm{Cl}(\Sigma_2^{i-1} - N(a_i))$ and Σ_2^n consists of disks. For the definition of an isotopy of type A, see Ch. II of [6]. Furthermore we may assume that each a_i is an essential arc properly embedded in Σ_2 and that $a_i \cap a_j = \emptyset$ $(i \neq j)$. Then each a_i is one of the following three types.

We say that a_i is of type 1 if a_i connects distinct components of $\partial \Sigma_2$, a_i is of type 2 if a_i meets a single component of $\partial \Sigma_2$ and is a separating arc in Σ_2, and a_i is of type 3 if a_i meets a single component of $\partial\Sigma_2$ and is a non-separating arc in Σ_2. Moreover we say that a_i is a d-arc if a_i is of type 1 and there exists a component c of $\partial\Sigma_2$ which meets a_i such that c does not meet a_j for any $j<i$. See Figure 1.1.

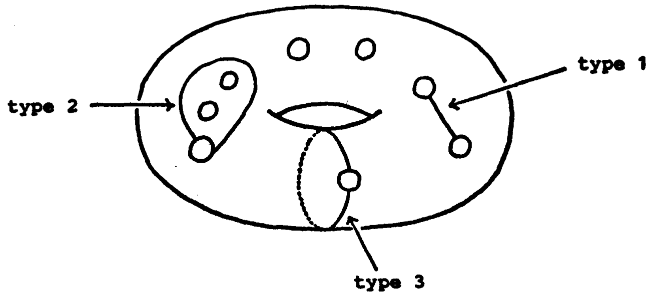

FIGURE 1.1

CLAIM 2. *Each a_i is not a d-arc.*

If an arc a_k is a d-arc, then by the inverse operation of an isotopy of type A defined in [12], Σ is ambient isotopic to Σ' which intersects V_1 in disks with $\#(\Sigma' \cap V_1) < \#(\Sigma \cap V_1)$. This is a contradiction.

CLAIM 3. *Each a_i is not of type 2.*

If there exists an arc of type 2, then by noting that each a_i is essential in Σ_2, we can find a d-arc. This is contradictory to Claim 2.

Put $\Sigma^{(0)}=\Sigma$, and let $\Sigma^{(i)}$ be the image of $\Sigma^{(i-1)}$ after an isotopy of type A at a_i $(1\leqq i\leqq n)$. Then we have $\Sigma_2^i=\Sigma^{(i)}\cap V_2$ $(0\leqq i\leqq n)$. Put $\Sigma_1^i=\Sigma^{(i)}\cap V_1$ $(0\leqq i\leqq n)$. By performing an isotopy of type A at a_i, a band in V_1 is produced. We denote the band by b_i.

Now, let $\Sigma_1=D_1\cup D_2\cup\cdots\cup D_r$ $(r>0)$ be disks in V_1.

Note that, by Claims 2 and 3, there are no pairs of two disks in $\{D_i\}_{i=1}^r$ which are complete meridian disk systems of V_1.

By Claims 2 and 3, a_1 is of type 3. If $r=1$, then Σ_1^1 is a single annulus, and the proof is completed.

Suppose $r>1$. By Claims 2 and 3, we may assume that a_1 and a_2 are both of type 3 and that b_1 meets D_1.

Suppose that b_2 also meets D_1. Let T be the component of $\Sigma^{(2)}$ containing D_1, and put $T'=T\cap V_1$. Since $r>1$, b_1 and b_2 meet D_1 in the same side. Then T' is contained in a solid torus obtained by cutting V_1 by D_1. This is contradictory to that T is incompressible. Hence we may assume that b_2 meets D_2, and we can put $\Sigma_1^2=A_1\cup A_2\cup D_3\cup\cdots\cup D_r$, where A_i is an annulus $(i=1, 2)$.

If $r=2$, then the proof is completed. Suppose $r>2$. If a_3 is of type 1, then by Claim 2, a_3 connects ∂D_1 and ∂D_2. Then, by noting the existence of the disk D_3, we can push the band b_2 into V_2 missing b_3. See Figure 1.2 and Lemmas 3.2, 3.4 and 3.5 of [7].

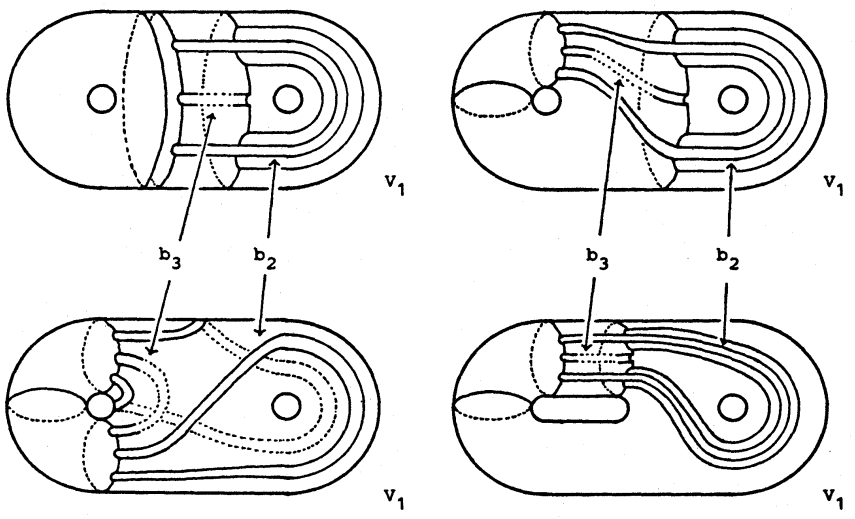

FIGURE 1.2

By performing this operation, we can change the order of a_2 and a_3, and we have a d-arc. This is contradictory to Claim 2. Thus a_3 is of type 3. If b_3 meets D_1 or D_2, then we have a compressible component of Σ similarly to the above, and a contradiction. Hence b_3 meets D_3 and we have $\Sigma_1^3 = A_1 \cup A_2 \cup A_3 \cup D_4 \cup \cdots \cup D_r$, where A_i is an annulus ($i=1, 2, 3$). By continuing these procedures, we complete the proof of Lemma 1.1. $\square$

The following two lemmas follow from Theorem VI. 34 of [6] and the uniqueness of the characteristic Seifert pairs, see Ch. IX of [6].

LEMMA 1.2. *Suppose that $M = S_1 \cup_f S_2$ is a Seifert fibered space. Then the base space of M is one of a 2-sphere with four exceptional points, a projective plane with two exceptional points or a Klein bottle without exceptional points.*

LEMMA 1.3. (1) *Any separating incompressible torus in $M = S_1 \cup_f S_2$ splits M into two 3-manifolds belonging to $D(2)$.*

(2) *If $M = S_1 \cup_f S_2$ contains a non-separating torus, then M is a torus bundle over a circle such that the torus is a fiber.*

LEMMA 1.4. *Let P be a projective plane with two holes. Then there exist exactly two different simple loops, up to ambient isotopy, each of which bounds a Möbius band in P.*

PROOF. This can be easily proved by noting that P is a Möbius band with one hole. □

For an integer n ($\geqq 0$), by $P(n)$ we denote the family consisting of all orientable Seifert fibered spaces over a projective plane with n exceptional fibers.

LEMMA 1.5. *Suppose that $M=S_1\cup_f S_2$ admits a Heegaard splitting $(V_1, V_2; F)$ of genus two, and put $T'=\partial S_1=f(\partial S_2)$.*

Then T' is ambient isotopic to a torus T which satisfies one of the following three conditions. (See Figure 1.3, and see also Lemmas 3.2, 3.4 and 3.5 of [7].)

(1) *For $i=1, 2$, $V_i\cap T$ consists of a single separating essential annulus.*

(2) *$V_1\cap T$ (or $V_2\cap T$ resp.) consists of two disjoint non-separating essential annuli satisfying the following condition: there exists a complete meridian disk system (D_1, D_2) of V_1 (or V_2 resp.) such that $D_1\cap(V_1\cap T)=\emptyset$ (or $D_1\cap(V_2\cap T)=\emptyset$ resp.) and $D_2\cap(V_1\cap T)$ (or $D_2\cap(V_2\cap T)$ resp.) consists of two arcs each of which is an essential arc properly embedded in each*

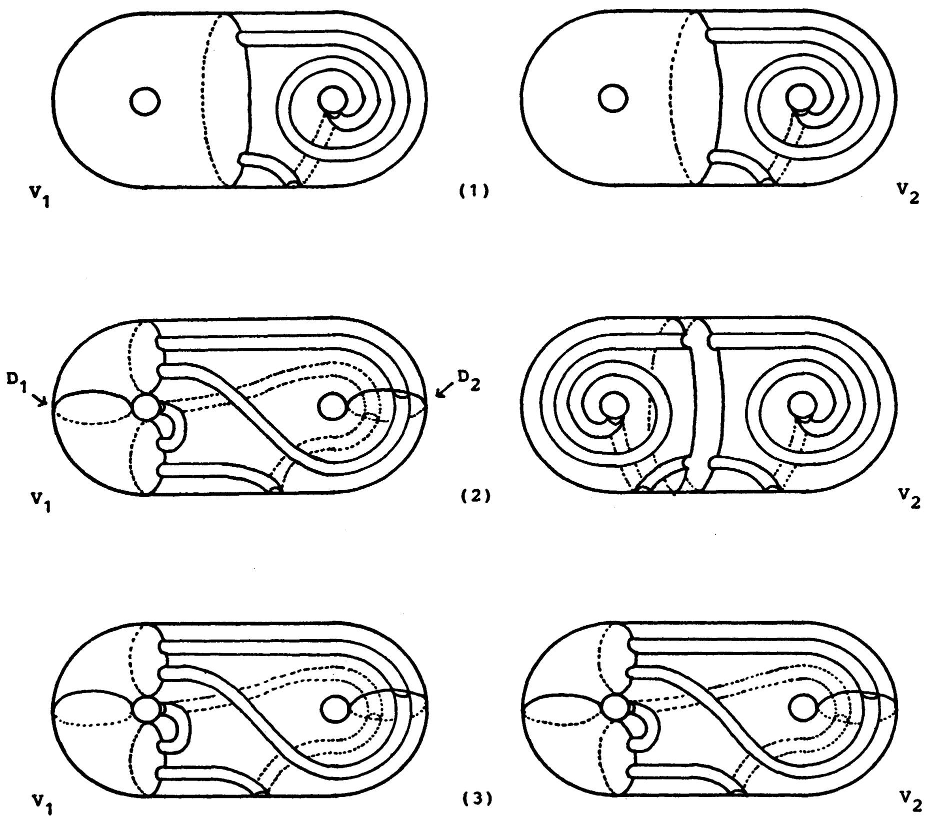

FIGURE 1.3

annulus of $V_1\cap T$ *(or* $V_2\cap T$ *resp.), and* $V_2\cap T$ *(or* $V_1\cap T$ *resp.) consists of two disjoint non-parallel separating essential annuli.*

(3) *For* $i=1, 2$, $V_i\cap T$ *consists of two disjoint non-separating essential annuli satisfying the same condition as that of* (2).

PROOF. By Lemma 1.1, T' is ambient isotopic to a torus T which intersects V_i in essential annuli ($i=1, 2$). Put $\Sigma_i=V_i\cap T$ ($i=1, 2$). Then we have the following three cases.

Case 1: Both Σ_1 and Σ_2 consist of separating annuli.

The case when all annuli of Σ_1 are mutually parallel. Since, by Lemma 3.2 of [7], there exists exactly one component of $\mathrm{Cl}(\partial V_1-N(\Sigma_1))$ which is a torus with two holes, all annuli of Σ_2 also are mutually parallel. Let A_i be a component of Σ_i which cuts off a torus with two holes G_i in ∂V_i with $G_i\cap\Sigma_i=\partial A_i$ ($i=1, 2$), see Figure 1.4. Since G_2 is identified with G_1 in M, ∂A_2 is identified with ∂A_1 in M. This shows $T=A_1\cup A_2$, and the conclusion (1) holds.

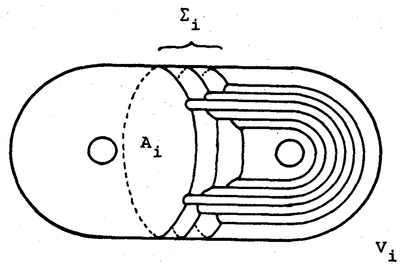

FIGURE 1.4

The case when Σ_1 contains non-parallel annuli. Since, by Lemmas 3.4 and 3.5 of [7], there exists exactly one component of $\mathrm{Cl}(\partial V_1-N(\Sigma_1))$ which is a sphere with four holes, Σ_2 also contains non-parallel annuli. Let A_i and B_i be the components of Σ_i which cut off a sphere with four holes G_i in ∂V_i with $G_i\cap\Sigma_i=\partial(A_i\cup B_i)$ ($i=1, 2$). Put $W_i\cup U_i\cup R_i=\mathrm{Cl}(V_i-N(A_i\cup B_i))$, where W_i is a genus two handlebody and U_i and R_i are solid tori. Since $\partial V_2\cap(U_2\cup R_2)$ is identified with $\partial V_1\cap(U_1\cup R_1)$, $\partial(U_1\cup R_1\cup U_2\cup R_2)$ consists of two tori. Then $W_1\cup W_2$ is a 2-bridge link exterior in S^3 and $A_1\cup B_1\cup A_2\cup B_2$ is two tori. This is a contradiction.

Case 2: One of Σ_1 or Σ_2 contains a non-separating annulus and the other consists of separating annuli.

In this case we may assume that Σ_1 contains a non-separating annulus. Since there exists exactly one component of $\mathrm{Cl}(\partial V_1-N(\Sigma_1))$ which is a sphere with four holes, Σ_2 contains non-parallel annuli. Let A_1 and B_1 (A_2 and B_2 resp.) be non-separating (separating resp.) essential annuli in

V_1 (V_2 resp.) which cut off a sphere with four holes in ∂V_1 (∂V_2 resp.) disjoint from Σ_1 (Σ_2 resp.). Note A_i and B_i are not components of Σ_i ($i=1, 2$). See Figure 1.5.

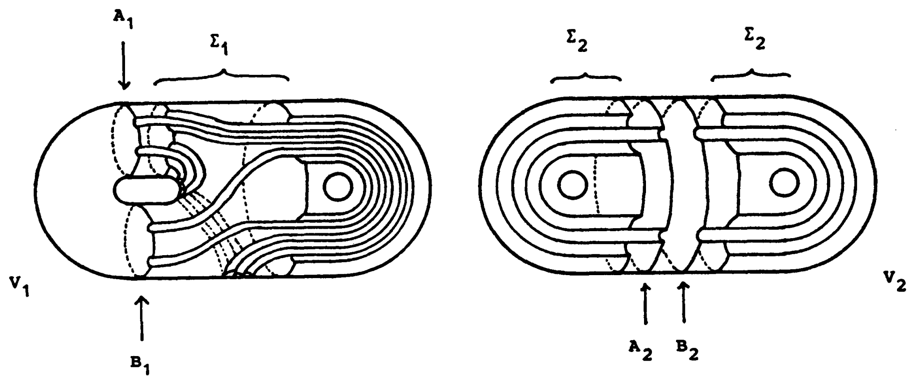

FIGURE 1.5

Since there exists exactly one component G_i of $\mathrm{Cl}(\partial V_i - N(\Sigma_i))$ which is a sphere with four holes ($i=1, 2$), G_2 is identified with G_1 in M. Then by noting that $\partial(A_i \cup B_i)$ is ambient isotopic to ∂G_i in G_i ($i=1, 2$), we may assume that $\partial(A_2 \cup B_2)$ is identified with $\partial(A_1 \cup B_1)$. Put $W_1 \cup U_1 = \mathrm{Cl}(V_1 - N(A_1 \cup B_1))$ and $W_2 \cup U_2 \cup R_2 = \mathrm{Cl}(V_2 - N(A_2 \cup B_2))$, where W_1 and W_2 are genus two handlebodies and U_1, U_2 and R_2 are solid tori. Then, by the above argument, $\partial V_2 \cap W_2$ is identified with $\partial V_1 \cap W_1$, and $\partial V_2 \cap (U_2 \cup R_2)$ is identified with $\partial V_1 \cap U_1$. Put $N_1 = W_1 \cup W_2$ and $N_2 = U_1 \cup U_2 \cup R_2$ in M. Then, by [7, §6 Case 2.2.2], N_2 is a Seifert fibered space over a disk with two or three exceptional fibers, and N_1 is a 2-bridge knot exterior in S^3.

Suppose that N_2 has three exceptional fibers. If N_1 is not a solid torus, then ∂N_1 is a separating incompressible torus which bounds N_2. This is contradictory to Lemma 1.3. If N_1 is a solid torus, then, since a meridian loop in ∂N_1 as a 2-bridge knot exterior and a fiber in ∂N_2 are identified in M, M is a Seifert fibered space over a sphere with three exceptional fibers. This is contradictory to Lemma 1.2. Hence N_2 has two exceptional fibers. Since T is contained in N_2, T is ambient isotopic to $\partial N_2 = A_1 \cup B_1 \cup A_2 \cup B_2$, and the conclusion (2) holds.

Case 3: Both Σ_1 and Σ_2 contain non-separating annuli.

Let A_i and B_i be non-separating annuli in V_i ($i=1, 2$) such as A_1 and B_1 in V_1 of Case 2. Then, by the same argument as the proof of Case 2, we may assume that $\partial(A_2 \cup B_2)$ is identified with $\partial(A_1 \cup B_1)$ in M. Put $W_i \cup U_i = \mathrm{Cl}(V_i - N(A_i \cup B_i))$ ($i=1, 2$), where W_i is a genus two handlebody and U_i is a solid torus. Put $N_1 = W_1 \cup W_2$ and $N_2 = U_1 \cup U_2$ in M. Then N_1 is a 2-bridge knot or link exterior in S^3. If N_1 is a 2-bridge link exterior, then a component of ∂N_1, say T', is a non-separating torus in M. Since $T' \cap T = \emptyset$, and by Lemma 1.3, T is ambient isotopic to T'. This is contradictory to that T is a separating torus. Thus N_1 is a 2-bridge knot exterior, and N_2 is a Seifert fibered space over a Möbius band with 0, 1 or 2 exceptional fibers. If N_2 has no exceptional fibers, then, since T is contained in N_2, T is ambient isotopic to $\partial N_2 = A_1 \cup B_1 \cup A_2 \cup B_2$, and the conclusion (3) holds. If N_2 has one exceptional fiber, then by Lemma 1.3, N_1 is a solid torus. Since a meridian loop in ∂N_1 as a 2-bridge knot exterior in S^3 and a fiber in ∂N_2 are identified in M, M belongs to $P(1)$. This is contradictory to Lemma 1.2.

Suppose that N_2 has two exceptional fibers. Then N_1 is a solid torus and M belongs to $P(2)$. By Lemma 1.4, T is ambient isotopic to one of the two tori T_1 or T_2 indicated in Figure 1.6.

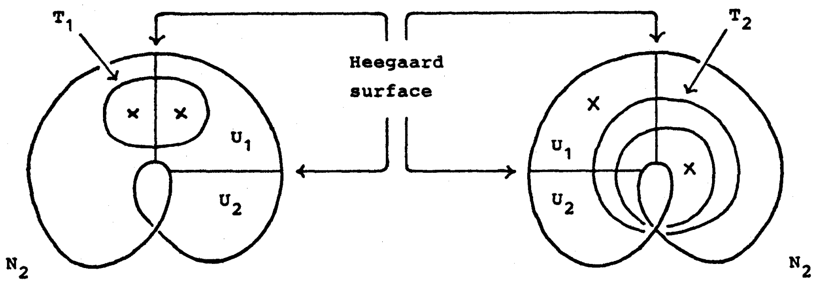

FIGURE 1.6

Since T_1 satisfies the condition (1), the proof is completed if T is ambient isotopic to T_1.

Suppose that T is ambient isotopic to T_2. Put $T_2 \cap V_i = R_i \cup S_i$ $(i=1, 2)$. We may assume that both R_i and S_i are parallel to A_i in V_i $(i=1, 2)$. Then A_i, B_i, R_i and S_i are four annuli illustrated in Figure 1.7 $(i=1, 2)$.

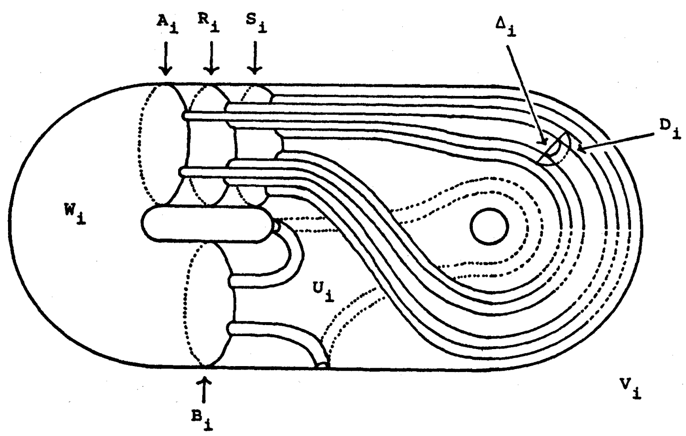

FIGURE 1.7

Put $\partial A_i = a_i \cup a'_i$, $\partial B_i = b_i \cup b'_i$, $\partial R_i = r_i \cup r'_i$ and $\partial S_i = s_i \cup s'_i$ $(i=1, 2)$, where $a_i, a'_i, \cdots, s'_i$ are boundary components of those annuli. Since $A_1 \cup B_1 \cup A_2 \cup B_2$ is a single torus, we may assume that a_1 (a'_1, b_1 and b'_1 resp.) is identified with a_2 (b_2, a'_2 and b'_2 resp.) in M. Then, by the fact that $W_1 \cup W_2 = N_1$ is a trivial 2-bridge knot exterior in S^3 and the uniqueness of 2-bridge representations of a trivial knot (i.e. Schubert's normal form theorem of [16]), we have a disk Δ_i in V_i $(i=1, 2)$ with $\Delta_1 \cap \Delta_2 = \emptyset$ such that $\partial \Delta_i$ is a union of an arc in ∂V_i and an essential arc $(= \Delta_i \cap A_i = \partial \Delta_i \cap A_i)$ in A_i, see Figure 1.7. Let D_i be a disk in V_i containing Δ_i $(i=1, 2)$ such that ∂D_i is a union of an arc in ∂V_i and an essential arc $(= D_i \cap R_i = \partial D_i \cap R_i)$ in R_i. Then by $\Delta_1 \cap \Delta_2 = \emptyset$, we may assume $D_1 \cap D_2 = \emptyset$. Hence we can perform the isotopies of type A along D_1 and D_2 simultaneously. Note here that the arc $D_1 \cap \partial V_1$ ($D_2 \cap \partial V_2$ resp.) connects r_2 and s_2 (r_1 and s_1 resp.) because the arc $\Delta_1 \cap \partial V_1$ ($\Delta_2 \cap \partial V_2$ resp.) connects a_2 and b_2 (a_1 and b_1 resp.). Let $\tilde{T}_2$ be the image of T_2 after the isotopies. Then by the above note, we can see that $\tilde{T}_2 \cap V_i$ is a separating essential annulus properly embedded in V_i $(i=1, 2)$. Thus $\tilde{T}_2$ satisfies the condition (1), and this completes the proof of Lemma 1.5. □

We say that an arc α properly embedded in a compact 3-manifold M is trivial if there exists an arc β in ∂M with $\alpha \cap \beta = \partial \alpha = \partial \beta$ such that $\alpha \cup \beta$ bounds a disk in M. Let L be a lens space and K a knot in L. We say that K is a 1-bridge knot in L if there exist two solid tori V_1 and V_2 in L such that $L = V_1 \cup V_2$, $V_1 \cap V_2 = \partial V_1 = \partial V_2$ and $V_i \cap K$ is a trivial arc in V_i $(i=1, 2)$.

LEMMA 1.6. *Let S be an element of $D(2)$. Let h be a fiber in ∂S and μ a simple loop in ∂S with $I(\mu, h) = \pm 1$. Then S is a 1-bridge knot exterior in some lens space such that μ is a meridian loop of the knot.*

PROOF. Let V be a solid torus and m a meridian loop in ∂V. Let $\psi: \partial V \to \partial S$ be a homeomorphism with $\psi(m) = \mu$. Let K be a core of V. Then by $I(\mu, h) = \pm 1$, $L = S \cup_\psi V$ admits a Seifert fibration over a 2-sphere with two exceptional fibers such that K is a regular fiber. Namely L is a lens space. Let T be a torus in L containing K saturated in the Seifert fibratian which splits L into two solid tori each of which contains an exceptional fiber. Let $\tilde{T}$ be a torus intersecting K in two points obtained from T by slightly moving T. Then $\tilde{T}$ splits L into two solid tori V_1 and V_2 such that $V_i \cap K$ is a trivial arc in V_i $(i=1, 2)$. Hence K is a 1-bridge knot in L, $S = \mathrm{Cl}(L - N(K))$ and μ is a meridian loop. □

§ 2. Proof of Proposition 1 and Theorem 1.

PROOF OF PROPOSITION 1. Since S_1 and S_2 belong to $D(2)$, we can put $S_1 = V_1 \cup W_1$ and $S_2 = V_2 \cup W_2$, where V_i and W_i are solid tori and $V_i \cap W_i = \partial V_i \cap \partial W_i = A_i$ is an essential annulus in S_i $(i=1, 2)$. Let α_i be an essential arc properly embedded in A_i and N_i a regular neighborhood of α_i in S_i $(i=1, 2)$. Put $U_i = \mathrm{Cl}(S_i - N_i)$, then U_i is a genus two handlebody $(i=1, 2)$. Since we may assume $f(N_2 \cap \partial S_2) \cap (N_1 \cap \partial S_1) = \varnothing$, $H_1 = U_1 \cup_f N_2$ and $H_2 = U_2 \cup_f N_1$ are genus three handlebodies. Then $(H_1, H_2; F)$ is a genus three Heegaard splitting of M, where $F = \partial H_1 = \partial H_2$. This completes the proof of Proposition 1. □

PROOF OF THEOREM 1. Suppose that $M = S_1 \cup_f S_2$ admits a Heegaard splitting $(V_1, V_2; F)$ of genus two. Put $T = \partial S_1 = f(\partial S_2)$. Then by Lemma 1.5, we may assume that T satisfies one of the three conditions of Lemma 1.5. In the following proof, note that if two elements of $D(2)$ are homeomorphic, then the homeomorphism is isotopic to a fiber preserving homeomorphism.

Case 1: T satisfies the condition (1).

For $i=1, 2$, put $W_i \cup U_i = \mathrm{Cl}(V_i - N(T))$, where W_i is a genus two handlebody and U_i is a solid torus. Put $N_1 = W_1 \cup W_2$ and $N_2 = U_1 \cup U_2$ in M. Then N_1 is a 1-bridge knot exterior in a lens space L, and a meridian loop in ∂N_1 is identified with a fiber in ∂N_2. Let μ be a meridian loop in ∂N_1 and h_i a fiber in ∂N_i $(i=1, 2)$. If $|I(\mu, h_1)| > 1$, then L admits a Seifert fibration whose base space is a 2-sphere with three exceptional points. This is a contradiction. If $I(\mu, h_1) = 0$, then by Theorem of Ch. 1 of [13], L is a connected sum of two lens spaces. This also is a contradiction. Thus $I(\mu, h_1) = \pm 1$. Since $I(\mu, f(h_2)) = 0$, we have $I(h_1, f(h_2)) = \pm 1$. Then by Remark 1, we have the conclusion (1) of Theorem 1.

Case 2: T satisfies the condition (2).

We may assume that $T \cap V_1$ is two non-separating annuli and $T \cap V_2$ is two separating annuli. Put $W_1 \cup U_1 = \mathrm{Cl}(V_1 - N(T))$ and $W_2 \cup U_2 \cup R_2 = \mathrm{Cl}(V_2 - N(T))$, where W_1 and W_2 are genus two handlebodies and U_1, U_2 and R_2 are solid tori. Put $N_1 = W_1 \cup W_2$ and $N_2 = U_1 \cup U_2 \cup R_2$ in M. Then N_1 is a 2-bridge knot exterior in S^3 and a meridian loop in ∂N_1 is identified with a fiber in ∂N_2. Then, by the same argument as the proof of Case 1, we have $I(h_1, f(h_2)) = \pm 1$ and the conclusion (1) of Theorem 1.

Case 3: T satisfies the condition (3).

Put $W_i \cup U_i = \mathrm{Cl}(V_i - N(T))$ $(i=1, 2)$, where W_i is a genus two handlebody and U_i is a solid torus. Put $N_1 = W_1 \cup W_2$ and $N_2 = U_1 \cup U_2$. Then N_1 is a 2-bridge knot exterior in S^3, and a meridian loop in ∂N_1 is identified with a fiber in ∂N_2 as a circle bundle over a Möbius band. Since N_1 is an element of $D(2)$, by Theorem 2 of [11], N_1 is a torus knot exterior. Furthermore, since 2-bridge torus knot is a $(2, n)$-torus knot, N_1 is homeomorphic to $E_{2,n}$ for some odd integer $n>1$. Hence, $S_1=E_{2,\alpha}$, $S_2=KI$ and $I(m_1, f(u_2))=0$ if $S_1=N_1$, or $S_1=KI$, $S_2=E_{2,\beta}$ and $I(u_1, f(m_2))=0$ if $S_1=N_2$. Then by Remark 1, we have the conclusion (2) or (3) of Theorem 1.

Conversely, suppose $I(h_1, f(h_2))=\pm 1$. Then by Lemma 1.6, S_1 is a 1-bridge knot exterior in a lens space such that $f(h_2)$ is a meridian loop of the knot. Then by tracing back the above procedure of Case 1, we can construct a Heegaard splitting of genus two of M. If $S_1=E_{2,\alpha}$, $S_2=KI$ and $I(m_1, f(u_2))=0$ or $S_1=KI$, $S_2=E_{2,\beta}$ and $I(u_1, f(m_2))=0$, then by tracing back the above procedure of Case 3, we can construct a Heegaard splitting of genus two of M.

This completes the proof of Theorem 1. □

§3. Several families of Heegaard surfaces of genus two.

Let S be an element of $D(2)$, h a fiber in ∂S and μ a simple loop in ∂S with $I(\mu, h)=\pm 1$. Then by Lemma 1.6, S is a 1-bridge knot exterior in a lens space such that μ is a meridian loop of the knot, and there exists a torus with two holes properly embedded in S which gives a 1-bridge representation of the knot. We call such a punctured torus a 1-bridge representing p-torus in S w.r.t. μ. Let E be a 2-bridge knot exterior in S^3 and m a meridian loop in ∂E. Then there exists a sphere with four holes properly embedded in E which gives a 2-bridge representation of the knot. We call such a punctured sphere a 2-bridge representing p-sphere in E.

REMARK 6. Since all (non-trivial) 2-bridge knots have property P by [18], the meridian loop in E is unique up to ambient isotopy of ∂E.

Put $M=S_1\cup_f S_2$. In the following we introduce several families consisting of Heegaard surfaces of genus two of M.

Case 1: $I(h_1, f(h_2))=\pm 1$.

Let F be an orientable closed surface of genus two in M such that $F\cap S_1$ is a 1-bridge representing p-torus w.r.t. $f(h_2)$ and $F\cap S_2$ is a single essential annulus saturated in the Seifert fibration of S_2. Then, by the proof of Theorem 1, F is a genus two Heegaard surface of M. We denote the family consisting of all such genus two Heegaard surfaces by F(1-1). Similarly F(1-2) denotes the family consisting of all genus two Heegaard surfaces F such that $F\cap S_1$ is a single essential annulus saturated in the Seifert fibration of S_1 and $F\cap S_2$ is a 1-bridge representing p-torus w. r. t. $f^{-1}(h_1)$.

Case 2: $S_1=E_{2,\alpha}$ and $I(m_1, f(h_2))=0$ or $S_2=E_{2,\beta}$ and $I(h_1, f(m_2))=0$.

Suppose $S_1=E_{2,\alpha}$ and $I(m_1, f(h_2))=0$. Let F be an orientable closed surface of genus two in M such that $F\cap S_1$ is a 2-bridge representing p-sphere and $F\cap S_2$ is two disjoint essential annuli saturated in the Seifert fibration of S_2. Then, by the proof of Theorem 1, F is a genus two Heegaard surface of M. We denote the family consisting of all such genus two Heegaard surfaces by $F(2\text{-}1)$. Similarly if $S_2=E_{2,\beta}$ and $I(h_1, f(m_2))=0$, then $F(2\text{-}2)$ denotes the family consisting of all genus two Heegaard surfaces F such that $F\cap S_1$ is two disjoint essential annuli saturated in the Seifert fibration of S_1 and $F\cap S_2$ is a 2-bridge representing p-sphere.

Case 3: $S_1=E_{2,\alpha}$, $S_2=KI$ and $I(m_1, f(u_2))=0$ or $S_1=KI$, $S_2=E_{2,\beta}$ and $I(u_1, f(m_2))=0$.

Suppose $S_1=E_{2,\alpha}$, $S_2=KI$ and $I(m_1, f(u_2))=0$. Let F be an orientable closed surface of genus two in M such that $F\cap S_1$ is a 2-bridge representing p-sphere and $F\cap S_2$ is two disjoint essential annuli saturated in the fibration of S_2 as a circle bundle over a Möbius band. Then, by the proof of Theorem 1, F is a genus two Heegaard surface of M. We denote the family consisting of all such genus two Heegaard surfaces by $F(3\text{-}1)$. Similarly if $S_1=KI$, $S_2=E_{2,\beta}$ and $I(u_1, f(m_2))=0$, then $F(3\text{-}2)$ denotes the family consisting of all genus two Heegaard surfaces F such that $F\cap S_1$ is two disjoint essential annuli saturated in the fibration of S_1 as a circle bundle over a Möbius band and $F\cap S_2$ is a 2-bridge representing p-sphere.

Furthermore we put $F(1)=F(1\text{-}1)\cup F(1\text{-}2)$, $F(2)=F(2\text{-}1)\cup F(2\text{-}2)$ and $F(3)=F(3\text{-}1)\cup F(3\text{-}2)$. Then the following proposition follows from Lemma 1.5 immediately.

PROPOSITION 3.1. *Any genus two Heegaard surface of* $M=S_1\cup_f S_2$ *is ambient isotopic to a Heegaard surface belonging to one of* $F(1)$, $F(2)$ *or* $F(3)$.

§4. Proof of Theorems 3 and 4.

PROOF OF THEOREM 3. Since K is a torus knot, there exists a torus T in L which contains K and splits L into two solid tori. Then we may assume that T intersects V_1 in disks because T is ambient isotopic to a torus rel. K which intersects V_1 in disks. Furthermore we assume that $\sharp(V_1 \cap T)$ is minimal among all tori which are ambient isotopic to T rel. K and intersect V_1 in disks, where $\sharp(V_1 \cap T)$ denotes the number of components of $V_1 \cap T$.

Let N be a small regular neighborhood of K in L such that $N \cap T$ is an annulus in T. Put $\Sigma = \mathrm{Cl}(T-N)$, then, since K does not bound a disk, Σ is an incompressible annulus properly embedded in $\mathrm{Cl}(L-N)$. Put $W_i = \mathrm{Cl}(V_i - N)$ $(i=1, 2)$, then W_i is a genus two handlebody. Put $\Sigma_i = W_i \cap \Sigma$ $(i=1, 2)$. Then $\Sigma_1 = D_1 \cup D_2 \cup$ (essential disks properly embedded in W_i), where D_i is a disk which meets N as in Figure 4.1 or 4.2 $(i=1, 2)$.

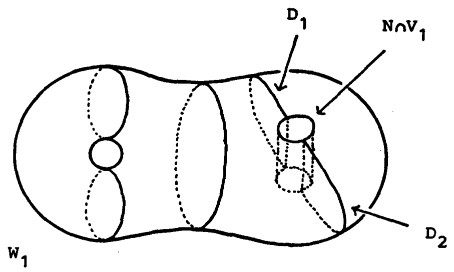

FIGURE 4.1

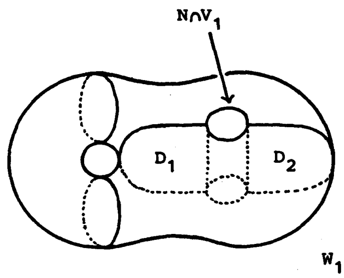

FIGURE 4.2

CLAIM 1. *Σ_2 is incompressible in W_2.*

Suppose that there exists a disk D in W_2 such that $D \cap \Sigma_2 = \partial D$ is an essential loop in Σ_2. Since Σ is incompressible in $\mathrm{Cl}(L-N)$, ∂D bounds a disk D' in Σ. Since D is contained in a solid torus cut off by T in L, $D \cup D'$ bounds a 3-ball. Then we can remove at least one component of Σ_1. This is contradictory to the minimality of $\sharp(V_1 \cap T)$. Thus Σ_2 is incompressible in W_2.

Let E_1 and E_2 be two disjoint non-parallel meridian disks in W_2 such that $E_1 \cap N = \emptyset$ and $E_2 \cap N$ is a single arc disjoint from Σ_2 as in Figure 4.3.

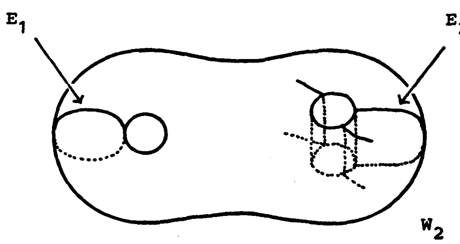

FIGURE 4.3

Put $E = E_1 \cup E_2$. By Claim 1, we may assume that Σ_2 intersects E in arcs. Note that $E \cap \Sigma_2 \neq \emptyset$. Put $N \cap \Sigma_2 = \gamma_1 \cup \gamma_2$. Since $E \cap N$ is a single arc in ∂E, we can find an outermost arc component a of $E \cap \Sigma_2$ in E which cuts off a disk Δ in E with $\Delta \cap \Sigma_2 = a$ and $\Delta \cap N = \emptyset$.

If a cuts off a disk in Σ_2 which does not contain γ_1 or γ_2, then by using the disk, we can exchange E for another complete meridian disk system E' so that $\#(E' \cap \Sigma_2) < \#(E \cap \Sigma_2)$. Thus we may assume that a does not cut off such a disk in Σ_2.

We call an inessential arc properly embedded in Σ_2 which cuts off a disk containing γ_1 or γ_2 "s-inessential." See Figure 4.4. Then as in Ch. II of [6], at each stage by exchanging complete meridian disk systems if necessary, we have a sequence of isotopies of type A, rel. N, at arcs a_i $(1 \leqq i \leqq n)$ each of which is an essential arc or an s-inessential arc properly embedded in Σ_2^{i-1}, where $\Sigma_2^0 = \Sigma_2$, $\Sigma_2^i = \mathrm{Cl}(\Sigma_2^{i-1} - N(a_i))$ and Σ_2^n consists of disks. Furthermore we may assume that each a_i is an arc properly embedded in Σ_2 and that $a_i \cap a_j = \emptyset$ $(i \neq j)$. Then for an essential arc a_i, we have the following four types.

We say that a_i is of type 1 if a_i connects two distinct components of $\partial \Sigma_2$ and at least one of the two components is a component of $\partial(\Sigma_1 - (D_1 \cup D_2))$, a_i is of type 2 if a_i meets one component, say c, of $\partial \Sigma_2$ and there exists a component e of $c - a_i$ such that $e \cup a_i$ bounds a disk in Σ, a_i is of type 3 if a_i meets one component, say c, of $\partial \Sigma_2$ and $e \cup a_i$ is an essential loop in Σ for each component e of $c - a_i$, a_i is of type 4 if a_i connects ∂D_1 and ∂D_2. See Figure 4.4.

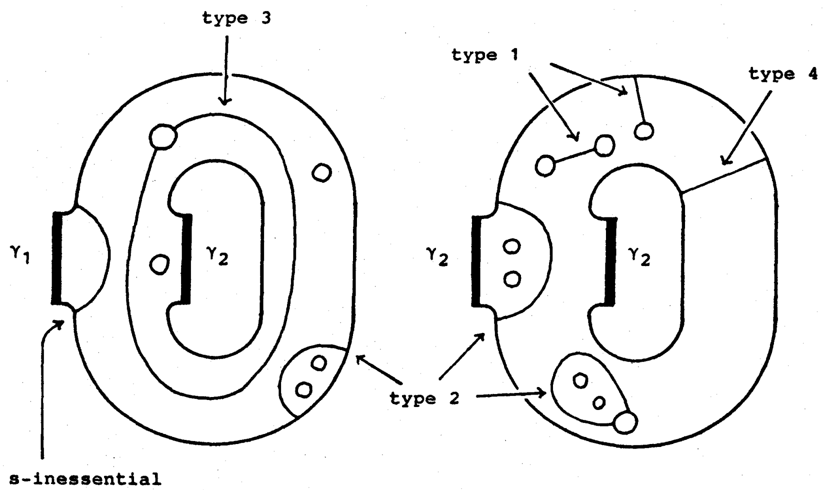

FIGURE 4.4

Moreover we say that a_i is a d-arc if a_i is of type 1 and there exists a component c of $\partial(\Sigma_1 - (D_1 \cup D_2))$ which meets a_i such that c does not meet a_j for any $j < i$.

The following two claims are proved similarly to the proof of Claims 2 and 3 of Lemma 1.1.

CLAIM 2. *Each a_i is not a d-arc.*

CLAIM 3. *Each a_i is not of type* 2.

By Claim 2 and by noting that if a_i is of type 3, then a_i is essential in Σ_2^{i-1}, we have the following claim.

CLAIM 4. *If two arcs a_i and a_j $(i\neq j)$ are both of type* 3, *then a_i and a_j meet different components of $\partial\Sigma_2$.*

Put $\Sigma^{(0)}=\Sigma$, and let $\Sigma^{(i)}$ be the image of $\Sigma^{(i-1)}$ after the isotopy of type A at a_i $(1\leqq i\leqq n)$. Then we have $\Sigma_2^i=\Sigma^{(i)}\cap W_2$. Put $\Sigma_1^i=\Sigma^{(i)}\cap W_1$. Note that $\Sigma_j^0=\Sigma_j$ $(j=1, 2)$.

CLAIM 5. *$\Sigma_1=D_1\cup D_2$ or some a_k is an s-inessential arc.*

Suppose that $\Sigma_1\neq D_1\cup D_2$ and that each a_i is an essential arc. By Claims 2 and 3, a_1 is of type 3 or 4. If a_1 is of type 4, then we can find a d-arc. This is a contradiction.

Suppose a_1 is of type 3. Since we can not have two arcs of type 3 and 4 simultaneously, each a_i is of type 1 or 3. Suppose that a_i $(1\leqq i\leqq k-1)$ is of type 3 and a_k is of type 1. Then by Claim 4, we can put $\Sigma_1^{k-1}=D_1\cup D_2\cup A_1\cup\cdots\cup A_{k-1}\cup$(disks), where A_i is an annulus in W_1 produced by the isotopy of type A at a_i. Since A_i is incompressible in W_1, the case as in Figure 4.2 does not occur. Let b_k be a core of the band in W_1 produced by the isotopy of type A at a_k. Then, since a_k is not a d-arc, b_k connects two annuli A_p and A_{p+1} or one annulus A_{k-1} and the disk D_1 or D_2. If b_k connects A_p and A_{p+1}, then by noting that A_p and A_{p+1} are mutually parallel, we can change the order of a_k and a_i for any i with $p+1\leqq i\leqq k-1$ as in the proof of Lemma 1.1 (cf. Figure 1.2). Then we have a d-arc, and a contradiction. If b_k connects A_{k-1} and the disk D_1 or D_2, then by the deformation of b_k as in Figure 4.5,

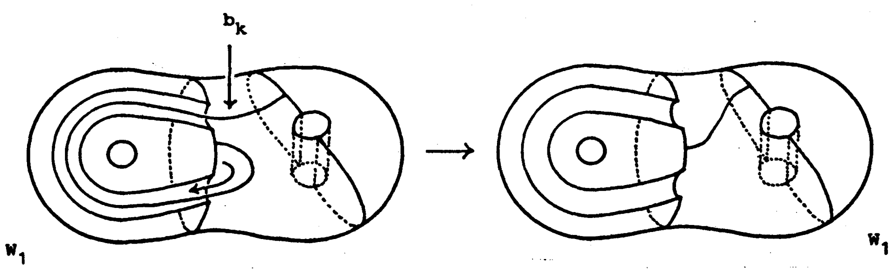

FIGURE 4.5

we can change the order of a_{k-1} and a_k. Then we have a d-arc, and a contradiction again. This completes the proof of Claim 5.

Now we show that in both cases of Claim 5 we have required disks Δ_1 and Δ_2.

The case when some a_k is an s-inessential arc. Suppose that a_i $(1 \leqq i \leqq k-1)$ is an essential arc and a_k is an s-inessential arc. If a_1 is of type 4, then by noting the proof of Claim 5 we have $\Sigma_1 = D_1 \cup D_2$. Thus, by noting the proof of Claim 5, we may assume that each a_i is of type 3, and we can put $\Sigma_1^{k-1} = D_1 \cup D_2 \cup A_1 \cup \cdots \cup A_{k-1} \cup (\text{disks})$.

Let D be a disk in Σ_2 cut off by a_k. We may assume that ∂D contains γ_1. Since a_k is an outermost arc component of $\Sigma_2^{k-1} \cap E$ in E for some complete meridian disk system E, we have a disk Δ in E with $\Delta \cap \Sigma_2^{k-1} = a_k$ and $\Delta \cap N = \emptyset$. Put $a' = \mathrm{Cl}(\partial \Delta - a_k)$. Then a' is an arc in $\mathrm{Cl}(\partial W_1 - N)$ with $\partial a' \subset \partial D_1$. Let $\tilde{a}$ be an arc in ∂D_1 cut off by a' with $\tilde{a} \cap N = \emptyset$. If $a' \cup \tilde{a}$ bounds a disk in $\mathrm{Cl}(\partial W_1 - N)$, then by noting $\partial \Delta = a_k \cup a'$, we can see that $a_k \cup \tilde{a}$ bounds a disk. This is contradictory to that A is incompressible in $\mathrm{Cl}(L-N)$. Thus $a' \cup \tilde{a}$ is an essential loop in $\mathrm{Cl}(\partial W_1 - N)$ as in Figure 4.6.

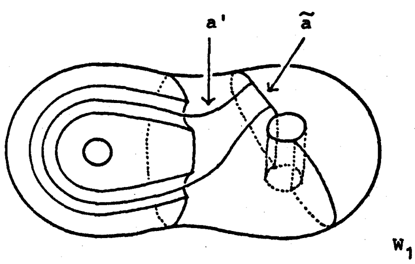

FIGURE 4.6

Let R_1 be the component of $((V_1 \cap N \cap T) - K)$ which intersects D_2, and let R_2 be the component of $((V_2 \cap N \cap T) - K)$ which intersects γ_1. Put $\Delta_1 = \mathrm{Cl}(R_1 \cup D_2)$ and $\Delta_2 = \mathrm{Cl}(R_2 \cup D \cup \Delta)$. Then Δ_i is a disk in V_i with $\Delta_i \cap K = V_i \cap K = \alpha_i$ $(i=1, 2)$. Put $\beta_i = \mathrm{Cl}(\partial \Delta_i - \alpha_i)$ $(i=1, 2)$. Then $\beta_i \subset \partial V_i$ and $\beta_1 \cap \beta_2 = \partial \beta_1 = \partial \beta_2$. This shows that the disks Δ_1 and Δ_2 are required disks.

The case when $\Sigma_1 = D_1 \cup D_2$. In this case a_1 is an s-inessential arc or is of type 4. If a_1 is an s-inessential arc, then we have required disks similarly to the above.

Suppose a_1 is of type 4. Let T_1 be the image of T after the isotopy of type A at a_1, and put $A_i = V_i \cap T_1$ $(i=1, 2)$, i.e. $A_i = \Sigma_i^1 \cup (V_i \cap N \cap T)$ is an annulus properly embedded in V_i. If the case as in Figure 4.2 occurs, then A_i is compressible in V_i and K is a core. This is a contradiction.

Thus only the case as in Figure 4.1 occurs, and A_i is an incompressible annulus properly embedded in V_i $(i=1, 2)$. Since any incompressible annuli properly embedded in a solid torus are ∂-parallel, A_i is isotopic to an annulus in ∂V_i rel. ∂A_i $(i=1, 2)$, say B_i. Let U_i be a solid torus in V_i bounded by $A_i \cup B_i$ $(i=1, 2)$. Put $C_i = \mathrm{Cl}(\partial V_i - B_i)$ $(i=1, 2)$, and let $\psi: \partial V_2 \to \partial V_1$ be an attaching homeomorphism so that $L = V_1 \cup_\psi V_2$. See Figure 4.7.

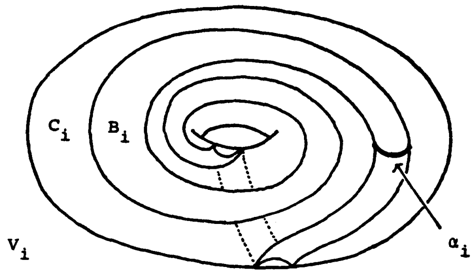

FIGURE 4.7

Since $\psi(\partial A_2) = \partial A_1$, we have the following two cases.

The case when $\psi(B_2) = C_1$. Let β_i be an arc in B_i such that $\beta_i \cap \alpha_i = \partial \beta_i = \partial \alpha_i$ and $\alpha_i \cup \beta_i$ bounds a disk Δ_i in U_i $(i=1, 2)$. Then by $\psi(\beta_2) \subset C_1$, Δ_1 and Δ_2 are required disks.

The case when $\psi(B_2) = B_1$. Let m_i be a meridian loop in ∂V_i and a_i a component of ∂A_i $(i=1, 2)$. If both $|I(m_1, a_1)|$ and $|I(m_2, a_2)|$ are greater than 1, then we can see that the torus $T_1 = A_1 \cup A_2$ bounds a Seifert fibered space over a disk with two exceptional fibers. This is contradictory to that T splits L into two solid tori. Thus we may assume $I(m_1, a_1) = \pm 1$. Then there exists a meridian disk D in V_1 with $D \cap A_1 = \alpha_1$. Put $\beta_1 = \mathrm{Cl}(\partial D - U_1)$ and $\Delta_1 = \mathrm{Cl}(D - U_1)$. Let β_2 be an arc in B_2 such that $\beta_2 \cap \alpha_2 = \partial \beta_2 = \partial \alpha_2$ and $\beta_2 \cup \alpha_2$ bounds a disk Δ_2 in U_2. Then Δ_1 and Δ_2 are required disks. This completes the proof of Theorem 3. □

Let S be an element of $D(2)$. Let ν_1 and ν_2 be mutually disjoint fibers in ∂S, and let μ_1 and μ_2 be mutually disjoint parallel simple loops

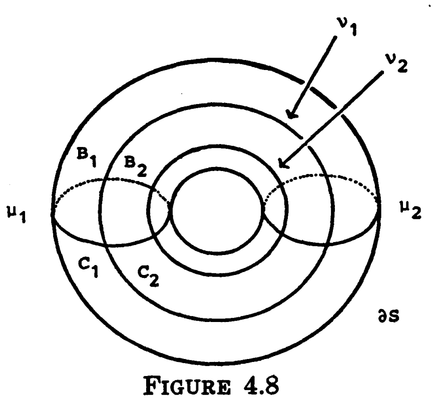

FIGURE 4.8

in ∂S each of which intersects ν_i in a single point ($i=1, 2$). Let B_1, B_2, C_1 and C_2 be the closure of the components of $\partial S-(\nu_1 \cup \nu_2 \cup \mu_1 \cup \mu_2)$ so that those are the four disks as in Figure 4.8. Then $B_1 \cap C_2 = B_2 \cap C_1$ consists of four points.

COROLLARY 4.1. *Under the above notations, fix an essential annulus A properly embedded in S which is saturated in the Seifert fibration with $\partial A=\nu_1 \cup \nu_2$.*

Let G be a torus with two holes properly embedded in S which is a 1-bridge representing p-torus with $\partial G=\mu_1 \cup \mu_2$. Then G is isotopic to one of $A \cup B_1 \cup C_2$ or $A \cup B_2 \cup C_1$. In addition the isotopy fixes ∂G setwise.

Conversely put $G_1'=A \cup B_1 \cup C_2$ and $G_2'=A \cup B_2 \cup C_1$, and let G_i be a torus with two holes obtained from G_i' by pushing Int(G_i') *into* Int(S) *($i=1, 2$). Then G_i is a 1-bridge representing p-torus in S w.r.t. μ_1.*

PROOF. Let V be a solid torus, m a meridian loop in ∂V and K a core of V. Let $\psi: \partial V \to \partial S$ be a homeomorphism with $\psi(m)=\mu_1$. Since $I(\mu_1, \nu_1)=\pm 1$, $L=S \cup_\psi V$ is a lens space, which admits a Seifert fibration containing K as a regular fiber. Then we have a torus in L containing K which is saturated in the Seifert fibration and splits L into two solid tori each of which contains an exceptional fiber. Thus K is a non-trivial torus knot in L and is not a core. Since $\psi^{-1}(\mu_i)$ is a meridian loop in ∂V ($i=1, 2$), $\psi^{-1}(\mu_i)$ bounds a disk D_i in V such that $D_1 \cap D_2=\emptyset$ and D_i intersects K in a single point. Put $\tilde{G}=G \cup_\psi D_1 \cup_\psi D_2$, then $\tilde{G}$ is a 1-bridge representing torus of K in L. Let V_1 and V_2 be the two solid tori in L which are bounded by $\tilde{G}$. Then by Theorem 3, there exists a disk $\tilde{\Delta}_i$ in V_i ($i=1, 2$) such that $\tilde{\Delta}_i \cap \tilde{G}=\partial\tilde{\Delta}_i \cap \tilde{G}=\tilde{\beta}_i$ is an arc, $\tilde{\Delta}_i \cap K=\partial\tilde{\Delta}_i \cap K= V_i \cap K=\tilde{\alpha}_i$ is an arc, $\partial\tilde{\Delta}_i=\tilde{\alpha}_i \cup \tilde{\beta}_i$ and $\tilde{\beta}_1 \cap \tilde{\beta}_2=\partial\tilde{\beta}_1=\partial\tilde{\beta}_2$.

Put $\Delta_i=\mathrm{Cl}(\tilde{\Delta}_i - V)$ and $\beta_i=\tilde{\beta}_i \cap \Delta_i$ ($i=1, 2$). Then Δ_i is a disk and β_i is an arc. See Figure 4.9.

Let P_i and Q_i be two points in μ_i ($i=1, 2$) such that $\{P_i, Q_i\}$ separates

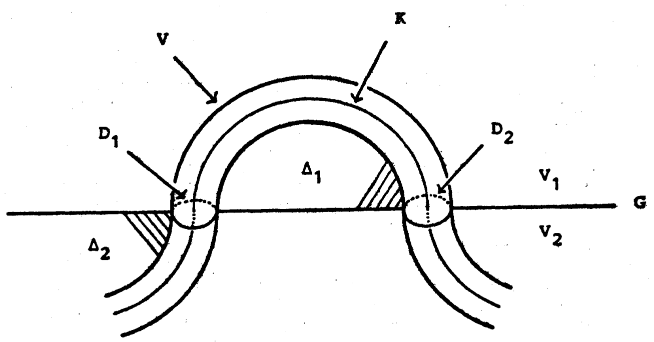

FIGURE 4.9

two points $\mu_i \cap (\beta_1 \cup \beta_2)$ as in Figure 4.10. Let E_i be a regular neighborhood of β_i in G $(i=1, 2)$ as in Figure 4.10.

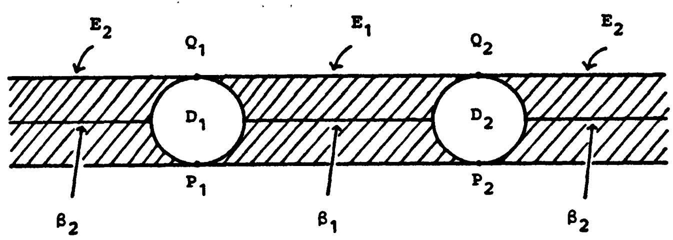

FIGURE 4.10

Put $E_3=\mathrm{Cl}(G-(E_1 \cup E_2))$, then E_3 is an annulus. By using Δ_i $(i=1, 2)$, we have an ambient isotopy h_t $(0 \leqq t \leqq 1)$ of L such that $h_0=\mathrm{id}$, $h_t|D_i=\mathrm{id}.|D_i$ and $h_1(G) \cap V=h_1(G) \cap \partial V=h_1(E_1 \cup E_2) \cap \partial V=h_1(E_1) \cup h_1(E_2)$. Put $h_1(E_i)=F_i$ $(i=1, 2, 3)$. Then it is easily seen that F_3 is an essential annulus properly embedded in S.

Since any two essential annuli properly embedded in S are mutually ambient isotopic, we have an ambient isotopy f_t $(0 \leqq t \leqq 1)$ of L such that $f_0=\mathrm{id}.$, $f_t(V)=V$, $f_t(D_i)=D_i$ $(i=1, 2)$ $(0 \leqq t \leqq 1)$ and $f_1(F_3)=A$. Then we have $f_1(F_1)=B_1$ and $f_1(F_2)=C_2$ or $f_1(F_1)=B_2$ and $f_1(F_2)=C_1$. Namely $f_1(F_1 \cup F_2 \cup F_3)=A \cup B_1 \cup C_2$ or $A \cup B_2 \cup C_1$. Thus by using ambient isotopies h_t and f_t and by noting $h_t(D_i)=D_i$ and $f_t(D_i)=D_i$ $(i=1, 2)$, we have a required isotopy of S.

On the other hand, by the above argument, the converse is clear. Thus the proof is completed. □

To prove Theorem 4 we prepare the following two lemmas.

LEMMA 4.2. *Let V be a standard solid torus in S^3, and let K be a non-trivial $(2, n)$-torus knot contained in ∂V such that K intersects a meridian loop in ∂V in two points. Let S be a 2-sphere in S^3 which gives a 2-bridge representation of K. Then there exists an ambient isotopy f_t $(0 \leqq t \leqq 1)$ of S^3 such that $f_0=\mathrm{id}.$, $f_t|K=\mathrm{id}.$ on K and $f_1(S)$ intersects V in two meridian disks.*

PROOF. Let B_1 and B_2 be the closure of the components of S^3-S. Then B_i is a 3-ball and $B_i \cap K=\alpha_i \cup \beta_i$ are two trivial arcs in B_i $(i=1, 2)$. Put $T=\partial V$. Then we may assume that T intersects B_1 in disks because T is ambient isotopic to a torus rel. K which intersects B_1 in disks. Furthermore we assume that $\#(B_1 \cap T)$ is minimal among all tori which are ambient isotopic to T rel. K and intersect B_1 in disks, where $\#(B_1 \cap T)$ denotes the number of components of $B_1 \cap T$.

Let N be a small regular neighborhood of K in S^3 such that $N \cap T$

is an annulus in T. Put $\Sigma=\mathrm{Cl}(T-N)$. Then, since K is a non-trivial knot, Σ is an incompressible annulus properly embedded in $\mathrm{Cl}(S^3-N)$. Put $W_i=\mathrm{Cl}(B_i-N)$ $(i=1, 2)$, then W_i is a genus two handlebody. Put $\Sigma_i=W_i\cap\Sigma$ $(i=1, 2)$. Then $\Sigma_1=D_1\cup D_2\cup D_3\cup D_4\cup$(separating disks), where D_i is a non-separating disk $(1\leqq i\leqq 4)$ such that both $\{D_1, D_3\}$ and $\{D_2, D_4\}$ are complete meridian disk systems of W_1 as in Figure 4.11.

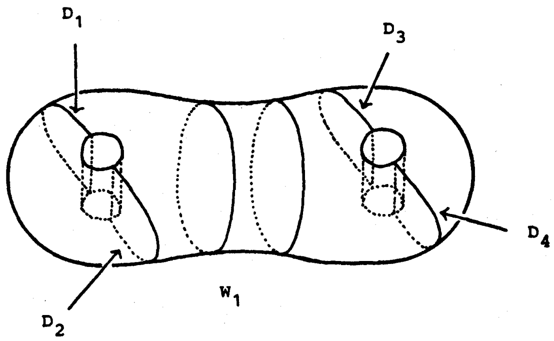

FIGURE 4.11

CLAIM 1. *Σ_2 is incompressible in W_2.*
This can be proved by the argument similar to the proof of Claim 1 of Lemma 1.1.

Let N_1 and N_2 be two components of $N\cap B_2$ and E a disk properly embedded in W_2 which separates N_1 from N_2.

CLAIM 2. $\Sigma_1=D_1\cup D_2\cup D_3\cup D_4$.
Since Σ_2 connects N_1 and N_2, $E\cap\Sigma_2$ is not empty. By Claim 1, we may assume that each component of $E\cap\Sigma_2$ is an arc. Let a_1 be an outermost arc component of $E\cap\Sigma_2$ in E and b_1 the band in W_1 produced by the isotopy of type A at a_1. Let Σ^1 (T^1 resp.) be the image of Σ (T resp.) after the isotopy, and put $\Sigma_i^1=\Sigma^1\cap W_i$ $(i=1, 2)$.

Suppose $\Sigma_1\neq D_1\cup D_2\cup D_3\cup D_4$. If b_1 meets a single component of Σ_1, then by noting Figure 4.12, there exists a component of Σ_1^1 which is a compressible annulus. Then, by the minimality of $\#(B_1\cap T)$ and the incompressibility of Σ, a_1 cuts off a disk in Σ_2 which is disjoint from $N_1\cup N_2$. Then we can exchange the disk E for another disk E' with $\#(E'\cap\Sigma_2)<\#(E\cap\Sigma_2)$. Thus we may assume that b_1 connects two distinct components of $\partial\Sigma_1$.

If $b_1\cap(\Sigma_1-(D_1\cup D_2\cup D_3\cup D_4))\neq\emptyset$, then each component of $B_1\cap T^1$ is a disk, and we have a contradiction for the minimality of $\#(B_1\cap T)$. If b_1 connects D_1 and D_2 or D_3 and D_4. Then there exists a disk Δ in $\mathrm{Cl}(\partial W_1-N)$ such that $\partial\Delta$ consists of an arc in ∂N and an arc in $\partial\Sigma_1^1$ as in Figure 4.12.

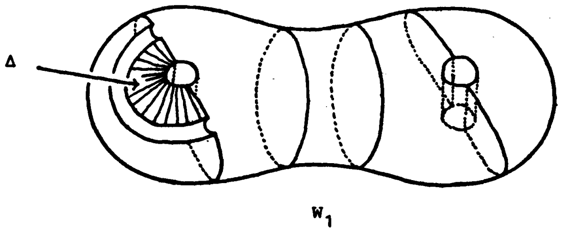

FIGURE 4.12

Then by using Δ, we can find a disk D in S^3 with $D\cap T=\partial D$ and $I(\partial D, K)=\pm 1$. This is contradictory to that K is not a trivial knot. After all we have $\Sigma_1=D_1\cup D_2\cup D_3\cup D_4$.

Now, by the argument similar to the proof of Claim 2, we may assume that b_1 connects D_1 and D_3, and Σ_1^1 consists of three disks as in Figure 4.13.

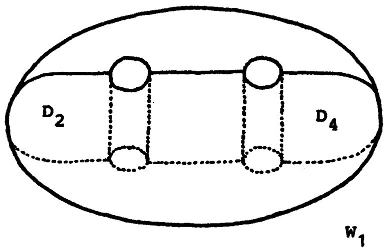

FIGURE 4.13

Since Σ_2^1 is a single disk connecting N_1 and N_2, $E\cap\Sigma_2^1$ is not empty and we have an outermost arc component a_2 of $E\cap\Sigma_2^1$ in E.

Let Σ^2 (T^2 resp.) be the image of Σ^1 (T^1 resp.) after the isotopy of type A at a_2. Let b_2 be the band in W_1 produced by the isotopy. Then, by the argument similar to the proof of Claim 2, we may assume that b_2 connects D_2 and D_4. Hence $W_i\cap\Sigma^2$ consists of two disks ($i=1, 2$). Then, for $i=1, 2$, $T^2\cap B_i$ is an annulus and each component of $\partial(T^2\cap B_i)$ bounds a disk in ∂B_i which is a meridian disk of a solid torus bounded by T^2. Then by tracing back the above ambient isotopies, we have a required ambient isotopy and complete the proof of Lemma 4.2. □

LEMMA 4.3. *Let A be a Möbius band, let α and β be non-separating arcs properly embedded in A with $\partial\alpha=\partial\beta$. Then α and β are mutually ambient isotopic by an ambient isotopy fixing ∂A pointwise.*

PROOF. This can be easily proved. □

PROOF OF THEOREM 4. Let V be a standard solid torus in S^3 with

$K \subset \partial V$ such that K intersects a meridian loop in ∂V in two points. Then by Lemma 4.2, we may assume that $S_i \cap V = D_i \cup E_i$ are two meridian disks of V ($i=1, 2$). Moreover we may assume that $D_1 \cap K = D_2 \cap K$ and $E_1 \cap K = E_2 \cap K$. Let A be a Möbius band properly embedded in V with $\partial A = K$. Then by using Lemma 4.3 and noting the incompressibility of A and the irreducibility of V, we can see that D_i and E_i are ambient isotopic rel. K to two meridian disks D and E ($i=1, 2$). Let $\tilde{S}_i$ be the image of S_i after the ambient isotopy ($i=1, 2$), and put $W = \mathrm{Cl}(S^3 - V)$. Then W is a solid torus and $\tilde{S}_i \cap W$ ($i=1, 2$) is an incompressible annulus in W. Hence by noting that $\partial(\tilde{S}_1 \cap W) = \partial(\tilde{S}_2 \cap W)$, we have a required ambient isotopy and complete the proof. $\square$

By Theorem 4, we have the following corollary.

COROLLARY 4.4. *Let E be a non-trivial $(2, n)$-torus knot exterior in S^3, and let G_1 and G_2 be 2-bridge representing p-spheres properly embedded in E with $\partial G_1 = \partial G_2$. Then G_1 and G_2 are mutually ambient isotopic in E by an ambient isotopy fixing ∂E pointwise.*

§ 5. Proof of Theorem 2 and Corollaries 1, 2.

Recall the definitions of the families of Heegaard surfaces defined in § 3.

LEMMA 5.1. *If $I(h_1, f(h_2)) = \pm 1$, then any genus two Heegaard surface belonging to F(1-2) is ambient isotopic to a Heegaard surface belonging to F(1-1).*

PROOF. Let F be a genus two Heegaard surface belonging to F(1-2). Then $F \cap S_1$ is an essential annulus properly embedded in S_1 and $F \cap S_2$ is a 1-bridge representing p-torus w. r. t. $f^{-1}(h_1)$. Then by using the isotopy of Corollary 4.1, we can see that F is ambient isotopic to a surface F' such that $F' \cap \partial S_2 = F' \cap \partial S_1$ is two disks and $\mathrm{Cl}(F' \cap \mathrm{Int}(S_i))$ is an essential annulus properly embedded in S_i ($i=1, 2$). Let $\tilde{F}$ be a surface obtained from F' by pushing the two disks $F' \cap \partial S_1$ into $\mathrm{Int}(S_1)$. Then by the latter half of Corollary 4.1, $\tilde{F} \cap S_1$ is a 1-bridge representing p-torus w. r. t. $f(h_2)$. This shows that $\tilde{F}$ is a Heegaard surface belonging to F(1-1). $\square$

Let A_1 be an essential annulus properly embedded in S_1 such that $\partial A_1 = \nu_1 \cap \nu_2$ are two disjoint fibers in ∂S_1 and A_2 be an essential annulus properly embedded in S_2 such that $\partial A_2 = \mu_1 \cup \mu_2$ are two disjoint fibers in ∂S_2. Suppose $I(h_1, f(h_2)) = \pm 1$. Then we may assume that $f(\mu_i)$ intersects

ν_j in a single point $(i=1, 2)$ $(j=1, 2)$. Let B_1, B_2, C_1 and C_2 be four disks as in Corollary 4.1, see Figure 4.8. Put $F_1=A_1\cup B_1\cup C_2\cup A_2$ and $F_2=A_1\cup B_2\cup C_1\cup A_2$.

PROPOSITION 5.2. *Under the above notations, any genus two Heegaard surface F belonging to $\mathcal{F}(1)$ is ambient isotopic to F_1 or F_2 in M. Thus $\mathcal{F}(1)$ contains at most two non-isotopic Heegaard surfaces of genus two if $I(h_1, f(h_2))=\pm 1$.*

PROOF. By Lemma 5.1, we may assume that F belongs to $\mathcal{F}(1\text{-}1)$. Then we may assume that $F\cap S_2=A_2$ and $F\cap S_1$ is a 1-bridge representing p-torus w.r.t. $f(h_2)$. Then by Corollary 4.1, $F\cap S_1$ is isotopic to $A_1\cup B_1\cup C_2$ or $A_1\cup B_2\cup C_1$ in S_1. By using this isotopy, we can see that F is ambient isotopic to a surface $\tilde{F}$ in M such that $\tilde{F}\cap S_1=A_1\cup B_1\cup C_2$ or $A_1\cup B_2\cup C_1$. Since the isotopy of Corollary 4.1 fixes $\partial(F\cap S_1)$ setwise, we may assume that the above ambient isotopy fixes A_2 setwise. Hence F is ambient isotopic to $A_1\cup B_1\cup C_2\cup A_2$ or $A_1\cup B_2\cup C_1\cup A_2$. $\square$

Let S be an element of $\mathcal{D}(2)$. For a fiber h in ∂S and the boundary loop c of a cross section of S, let α/p and β/q be the Seifert invariants of two exceptional fibers. Then we denote this state by $S=D(\alpha/p, \beta/q)$ w.r.t. h and c. The following proposition was proved by M. Sakuma.

PROPOSITION 5.3. *Suppose that one of S_1 or S_2, say S_1, is $D(\pm 1/p, \pm 1/q)$ w.r.t. h_1 and c_1. If $I(c_1, f(h_2))=0$, then $\mathcal{F}(1)$ contains exactly one Heegaard surface of genus two up to isotopy.*

PROOF. Let x_1 and x_2 be two exceptional fibers of S_1, and let y be one of two exceptional fibers of S_2. Let N_i be a regular neighborhood of x_i in S_1 $(i=1, 2)$ with $N_1\cap N_2=\emptyset$, and let N be a regular neighborhood of y in S_2. Let E_1 be the cross section of S_1, i.e. E_1 is a disk with two holes properly embedded in $\mathrm{Cl}(S_1-(N_1\cup N_2))$ with $E_1\cap\partial S_1=c_1$, and let E_2 be a cross section of S_2. Then we may assume that E_1 and E_2 intersects in a single point, say P. Let a_i be an arc in E_1 connecting P and N_i $(i=1, 2)$, and let b be an arc in E_2 connecting P and N. See Figure 5.1.

Put $V_i=N_i\cup N(a_i\cup b)\cup N$ $(i=1, 2)$, where $N(a_i\cup b)$ is a regular neighborhood of $a_i\cup b$ in M. Then V_i is a genus two handlebody. Let F_1 and F_2 be two Heegaard surfaces of genus two defined in Proposition 5.2, then by changing the letters if necessary, we can see that F_i is ambient isotopic to ∂V_i $(i=1, 2)$ in M.

Now, let d_i be the component of ∂E_1-c_1 which intersects a_i $(i=1, 2)$, and put $W_i=N(d_i\cup a_i\cup b)\cup N$. Since the Seifert invariants of x_1 and x_2

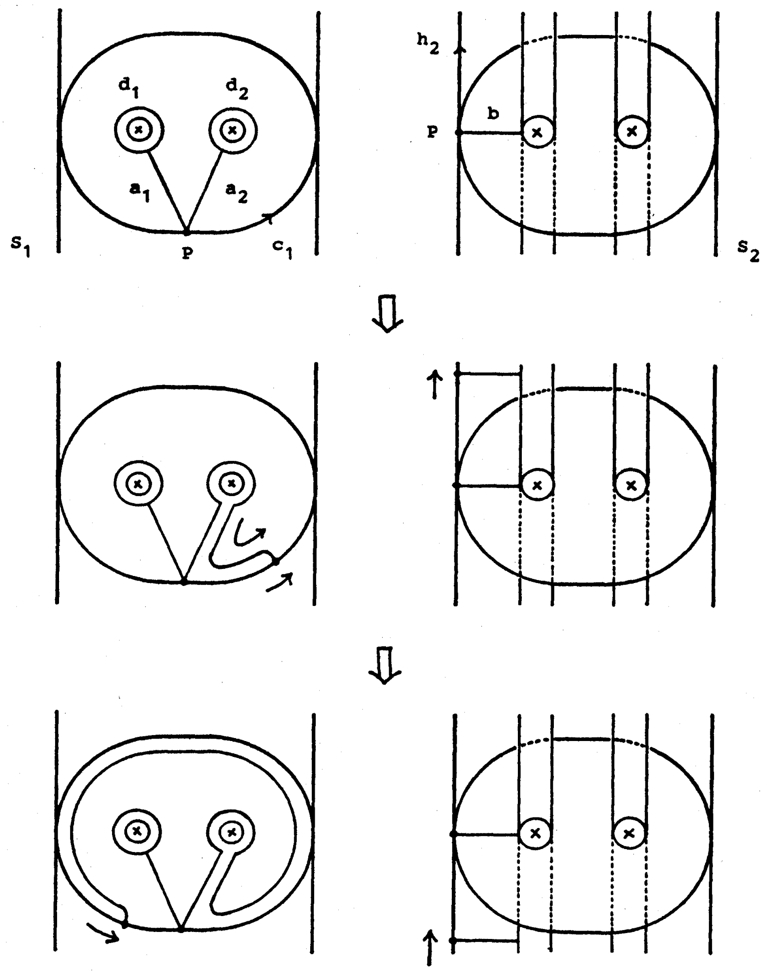

FIGURE 5.1

are $\pm 1/p$ and $\pm 1/q$, x_i is ambient isotopic to d_i $(i=1, 2)$ in S_1, and hence V_i is ambient isotopic to W_i $(i=1, 2)$. By the way, since c_1 is identified with h_2, we can do the deformation of W_2 illustrated in Figure 5.1. This shows that F_1 and F_2 are mutually ambient isotopic. Thus, together with Proposition 5.2, we complete the proof of Proposition 5.3. □

PROPOSITION 5.4. (1) *$F(2\text{-}1)$ contains exactly one Heegaard surface of genus two up to isotopy if $S_1=E_{2,\alpha}$ and $I(m, f(h_2))=0$.*

(2) *$F(2\text{-}2)$ contains exactly one Heegaard surface of genus two up to isotopy if $S_2=E_{2,\beta}$ and $I(h_1, f(m_2))=0$.*

PROOF. Suppose $S_1=E_{2,\alpha}$ and $I(m_1, f(h_2))=0$. Let F_1 and F_2 be two Heegaard surfaces belonging to $F(2\text{-}1)$. Since $F_i \cap S_2$ are two essential annuli properly embedded in S_2 saturated in the Seifert fibration $(i=1, 2)$, F_2 is ambient isotopic to a surface F_2' with $F_2' \cap S_2 = F_1 \cap S_2$. Put $G_1=F_1 \cap S_1$

and $G'_2=F'_2\cap S_1$. Then G_1 and G'_2 are 2-bridge representing p-sphere in $S_1=E_{2,\alpha}$ with $\partial G_1=\partial G'_2$. Then by Corollary 4.4, G'_2 is ambient isotopic to G_1 in S_1 rel. ∂S_1. Hence F_2 is ambient isotopic to F_1 in M.

If $S_2=E_{2,\beta}$ and $I(h_1, f(m_2))=0$, then we can prove (2) similarly. □

The next proposition is proved similarly to Proposition 5.4.

PROPOSITION 5.5. (1) F(3-1) *contains exactly one Heegaard surface of genus two up to isotopy if* $S_1=E_{2,\alpha}$, $S_2=KI$ *and* $I(m_1, f(u_2))=0$.

(2) F(3-2) *contains exactly one Heegaard surface of genus two up to isotopy if* $S_1=KI$, $S_2=E_{2,\beta}$ *and* $I(u_1, f(m_2))=0$.

PROOF OF THEOREM 2. We divide the proof into several cases. Let μ be the number of Heegaard splittings of genus two of $M=S_1\cup_f S_2$ up to isotopy. In the following proof, note that $E_{2,n}=D(1/2, -k/(2k+1))$ w.r.t. h and m, where $n=2k+1$ $(k>0)$, and $KI=D(-1/2, 1/2)$ w.r.t. h and u.

Case (1): $S_1\neq E_{2,\alpha}$ and $S_2\neq E_{2,\beta}$.

Case (1-a): $\begin{bmatrix} f(h_2) \\ f(c_2) \end{bmatrix}=\begin{bmatrix} a & \varepsilon \\ c & d \end{bmatrix}\begin{bmatrix} h_1 \\ c_1 \end{bmatrix}$ with $ad-\varepsilon c=\pm 1$ and $\varepsilon=\pm 1$. In this case, by Proposition 3.1 and the definitions of $F(1)$, $F(2)$ and $F(3)$, any genus two Heegaard surface of M is ambient isotopic to a Heegaard surface belonging to $F(1)$.

Case (1-a-1): $S_1=D(\pm 1/p, \pm 1/q)$ w.r.t. h_1 and c_1 and $a=0$ or $S_2=D(\pm 1/p, \pm 1/q)$ w.r.t. h_2 and c_2 and $d=0$. In this case by Proposition 5.3, μ is 1.

Case (1-a-2): M does not belong to Case (1-a-1). In this case, μ is at most 2.

In other cases, by Theorem 1, μ is 0.

Case (2): $S_1=E_{2,\alpha}$ and $S_2\neq KI$ nor $E_{2,\beta}$.

Case (2-a): $\begin{bmatrix} f(h_2) \\ f(c_2) \end{bmatrix}=\begin{bmatrix} 0 & \varepsilon \\ \delta & d \end{bmatrix}\begin{bmatrix} h_1 \\ m_1 \end{bmatrix}$ with $\varepsilon\delta=\pm 1$. In this case, any genus two Heegaard surface of M is ambient isotopic to a Heegaard surface belonging to $F(1)\cup F$(2-1).

Case (2-a-1): $\alpha=3$ or $S_2=D(\pm 1/p, \pm 1/q)$ w.r.t. h_2 and c_2 and $d=0$. In this case, by Propositions 5.4 and 5.3, μ is at most two.

Case (2-a-2): M does not belong to Case (2-a-1). In this case, by Propositions 5.2 and 5.4, μ is at most 3.

Case (2-b): $\begin{bmatrix} f(h_2) \\ f(c_2) \end{bmatrix}=\begin{bmatrix} a & \varepsilon \\ c & d \end{bmatrix}\begin{bmatrix} h_1 \\ m_1 \end{bmatrix}$ with $ad-\varepsilon c=\pm 1$, $\varepsilon=\pm 1$ and $a\neq 0$. In this case, any genus two Heegaard surface of M is ambient isotopic to a Heegaard surface belonging to $F(1)$.

Case (2-b-1): $\alpha=3$ and $\varepsilon a=-1$ or $S_2=D(\pm 1/p, \pm 1/q)$ w.r.t. h_2 and

c_2 and $d=0$. By noting that $D(1/2, -a/(2a+1))$ w.r.t. h_1 and m_1 = $D(-1/2, -a/(2a+1))$ w.r.t. h_1 and $h_1^{-1}m_1$, we can see that μ is 1 similarly to Case (1-a-1).

Case (2-b-2): M does not belong to Case (2-b-1). In this case, μ is at most 2 similarly to Case (1-a-2).

In other cases, by Theorem 1, μ is 0.

Case (2'): $S_1 \neq KI$ nor $E_{2,\alpha}$ and $S_2=E_{2,\beta}$. This case can be substituted for Case (2).

Case (3): $S_1=E_{2,\alpha}$ and $S_2=KI$.

Case (3-a): $\begin{bmatrix} f(h_2) \\ f(u_2) \end{bmatrix} = \begin{bmatrix} 0 & \varepsilon \\ \delta & d \end{bmatrix} \begin{bmatrix} h_1 \\ m_1 \end{bmatrix}$ with $\varepsilon\delta=\pm 1$.

Case (3-a-1): $\alpha=3$ or $d=\pm 1$ or 0. By noting that $D(-1/2, 1/2)$ w.r.t. h and u $=D(-1/2, -1/2)$ w.r.t. h and $h^{-1}u$ $=D(1/2, 1/2)$ w.r.t. h and hu, we can see that μ is at most 2 similarly to Case (2-a-1).

Case (3-a-2): $\alpha>3$ and $|d|>1$. In this case, we can see that μ is at most 3 similarly to Case (2-a-2).

Case (3-b): $\begin{bmatrix} f(h_2) \\ f(u_2) \end{bmatrix} = \begin{bmatrix} \varepsilon & b \\ 0 & \delta \end{bmatrix} \begin{bmatrix} h_1 \\ m_1 \end{bmatrix}$ with $\varepsilon\delta=\pm 1$.

Case (3-b-1): $b=\pm 1$. In this case, any genus two Heegaard surface of M is ambient isotopic to a Heegaard surface belonging to $F(1) \cup F(3\text{-}1)$. Furthermore we can see that $F(1)$ contains exactly one genus two Heegaard surface up to isotopy similarly to Case (3-a-1). Hence by Proposition 5.5, μ is at most 2.

Case (3-b-2): $b \neq \pm 1$. In this case, any genus two Heegaard surface of M is ambient isotopic to a Heegaard surface belonging to $F(3\text{-}1)$. Thus μ is 1.

Case (3-c): $\begin{bmatrix} f(h_2) \\ f(u_2) \end{bmatrix} = \begin{bmatrix} a & \varepsilon \\ c & d \end{bmatrix} \begin{bmatrix} h_1 \\ m_1 \end{bmatrix}$ with $ad-\varepsilon c=\pm 1$, $\varepsilon=\pm 1$ and $ac \neq 0$.

Case (3-c-1): $\alpha=3$ and $\varepsilon a=-1$ or $d=\pm 1$ or 0. In this case, by Proposition 5.3, μ is 1.

Case (3-c-2): M does not belong to Case (3-c-1). By Proposition 5.2, μ is at most 2.

In other cases, by Theorem 1, μ is 0.

Case (3'): $S_1=KI$ and $S_2=E_{2,\beta}$.

This case can be substituted for Case (3).

Case (4): $S_1=E_{2,\alpha}$ and $S_2=E_{2,\beta}$.

Case (4-a): $\begin{bmatrix} f(h_2) \\ f(m_2) \end{bmatrix} = \begin{bmatrix} 0 & \varepsilon \\ \delta & 0 \end{bmatrix} \begin{bmatrix} h_1 \\ m_1 \end{bmatrix}$ with $\varepsilon\delta=\pm 1$. In this case, any genus two Heegaard surface of M is ambient isotopic to a Heegaard surface belonging to $F(1) \cup F(2)$.

Case (4-a-1): $\alpha=3$ or $\beta=3$. By Propositions 5.3 and 5.4, μ is at most 3.

Case (4-a-2): $\alpha>3$ and $\beta>3$. By Propositions 5.2 and 5.4, μ is at most 4.

Case (4-b): $\begin{bmatrix} f(h_2) \\ f(m_2) \end{bmatrix} = \begin{bmatrix} a & \varepsilon \\ \delta & 0 \end{bmatrix} \begin{bmatrix} h_1 \\ m_1 \end{bmatrix}$ with $\varepsilon\delta=\pm 1$ and $a\neq 0$. In this case, any genus two Heegaard surface of M is ambient isotopic to a Heegaard surface belonging to $F(1)\cup F(2\text{-}2)$.

Case (4-b-1): $\alpha=3$ and $\varepsilon a=-1$. In this case, μ is at most 2.

Case (4-b-2): $\alpha>3$ or $\varepsilon a\neq -1$. In this case, μ is at most 3.

Case (4-c): $\begin{bmatrix} f(h_2) \\ f(m_2) \end{bmatrix} = \begin{bmatrix} 0 & \varepsilon \\ \delta & d \end{bmatrix} \begin{bmatrix} h_1 \\ m_1 \end{bmatrix}$ with $\varepsilon\delta=\pm 1$ and $d\neq 0$.

Case (4-c-1): $\beta=3$ and $\varepsilon d=-1$. In this case, μ is at most 2.

Case (4-c-2): $\beta>3$ or $\varepsilon d\neq -1$. In this case, μ is at most 3.

Case (4-d): $\begin{bmatrix} f(h_2) \\ f(m_2) \end{bmatrix} = \begin{bmatrix} a & \varepsilon \\ c & d \end{bmatrix} \begin{bmatrix} h_1 \\ m_1 \end{bmatrix}$ with $ad-\varepsilon c=\pm 1$, $\varepsilon=\pm 1$ and $ad\neq 0$.

Case (4-d-1): $\alpha=3$ and $\varepsilon\alpha=-1$ or $\beta=3$ and $\varepsilon d=-1$. In this case, μ is 1 similarly to Case (1-a-1).

Case (4-d-2): M does not belong to Case (4-d-1). In this case, μ is at most 2 similarly to Case (1-a-2).

In other cases, by Theorem 1, μ is 0.

This completes the proof of Theorem 2. □

Cases	n	F(1)	F(2-1)	F(2-2)	F(3-1)	F(3-2)
Case(1-a-1)	1	1	—	—	—	—
Case(1-a-2)	2	2	—	—	—	—
Case(2-a-1)	2	1	1	—	—	—
Case(2-a-2)	3	2	1	—	—	—
Case(2-b-1)	1	1	—	—	—	—
Case(2-b-2)	2	2	—	—	—	—
Case(3-a-1)	2	1	1	—	—	—
Case(3-a-2)	3	2	1	—	—	—
Case(3-b-1)	2	1	—	—	1	—
Case(3-b-2)	1	—	—	—	1	—
Case(3-c-1)	1	1	—	—	—	—
Case(3-c-2)	2	2	—	—	—	—
Case(4-a-1)	3	1	1	1	—	—
Case(4-a-2)	4	2	1	1	—	—
Case(4-b-1)	2	1	—	1	—	—
Case(4-b-2)	3	2	—	1	—	—
Case(4-c-1)	2	1	1	—	—	—
Case(4-c-2)	3	2	1	—	—	—
Case(4-d-1)	1	1	—	—	—	—
Case(4-d-2)	2	2	—	—	—	—

TABLE 5.2

In Table 5.2, we summarize the evaluation of the numbers of Heegaard splittings of genus two of M up to isotopy. In Table 5.2, n denotes the upper bound of μ, namely $1 \leqq \mu \leqq n$, and "—" means that M does not contain a Heegaard surface belonging to the family $F(1)$, $F(2\text{-}1)$, etc.

REMARK 7. By Table 5.2, it seems that M does not contain a Heegaard surface belonging to $F(3\text{-}2)$. But this occurs by the reason why the Case (3′) is substituted for the Case (3). In fact, in Case (3′-b), M contains a Heegaard surface belonging to $F(3\text{-}2)$.

PROOF OF COROLLARY 1. Let T be an incompressible torus in M saturated in the Seifert fibration. Then T splits M into two Seifert fibered spaces S_1 and S_2 belonging to $D(2)$. Let h_i be a fiber in ∂S_i $(i=1, 2)$ and $f\colon \partial S_2 \to \partial S_1$ the attaching homeomorphism. Since the fibration of ∂S_2 extends to the fibration of S_2, we have $I(h_1, f(h_2))=0$.

Suppose that M admits a Heegaard splitting of genus two. Then, by Theorem 1 and $I(h_1, f(h_2))=0$, we may assume that $S_1=E_{2,\alpha}$, $S_2=KI$ and $\begin{bmatrix} f(h_2) \\ f(u_2) \end{bmatrix} = \begin{bmatrix} \varepsilon & 0 \\ 0 & \delta \end{bmatrix} \begin{bmatrix} h_1 \\ m_1 \end{bmatrix}$ with $\varepsilon\delta=\pm 1$. Furthermore, since KI admits an orientation reversing auto-homeomorphism, we may assume $\varepsilon=1$ and $\delta=-1$.

By taking the meridian loop m_1 (the fiber u_2 resp.) for the boundary loop of a cross section of $E_{2,\alpha}$ (KI resp.), we may assume that the Seifert invariants of the exceptional fibers of $E_{2,\alpha}$ are $1/2$ and $-a/(2a+1)$ with $\alpha=2a+1$ $(a>0)$, and that the Seifert invariants of the exceptional fibers of KI are $1/2$ and $-1/2$. Then M is homeomorphic to $S(0; 1/2, 1/2, -1/2, -a/(2a+1))$. This completes the proof of the first half.

Since M belongs to the Case (3-b-2), by Table 5.2 we can see that M admits exactly one Heegaard splitting of genus two up to isotopy. This completes the proof of Corollary 1. □

PROOF OF COROLLARY 2. We may assume that $M=S_1 \cup_f KI$ and $\begin{bmatrix} f(h_2) \\ f(u_2) \end{bmatrix} = \begin{bmatrix} \alpha & \varepsilon \\ \delta & 0 \end{bmatrix} \begin{bmatrix} h_1 \\ c_1 \end{bmatrix}$ with $\varepsilon\delta=\pm 1$. Then M belongs to one of the cases (1-a-1), (3-a-1) or (3-c-1). Then by Table 5.2, we can see that M admits at most two non-isotopic Heegaard splittings of genus two. □

EXAMPLES. (I) Let α and β be odd integers larger than 1, and put $\varepsilon\delta=\pm 1$. Put $M_{\alpha,\beta,\varepsilon\delta}=E_{2,\alpha} \cup_f E_{2,\beta}$ with $\begin{bmatrix} f(h_2) \\ f(m_2) \end{bmatrix} = \begin{bmatrix} 0 & \varepsilon \\ \delta & 0 \end{bmatrix} \begin{bmatrix} h_1 \\ m_1 \end{bmatrix}$. Then by the proof of Case (4) of Theorem 2, $M_{\alpha,\beta,\varepsilon\delta}$ may admit four non-isotopic

Heegaard splittings of genus two if $\alpha>3$ and $\beta>3$. Then the 6-plat representations of the 3-bridge knots in S^3 corresponding to the four Heegaard splittings of genus two of $M_{\alpha,\beta,\varepsilon\delta}$ are those representations illustrated in Figure 0.1.

(II) For a matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ with $ad-bc=\pm 1$, let $K\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be a sapphire space of type $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ defined in [9], i.e. $K\begin{bmatrix} a & b \\ c & d \end{bmatrix}=KI\cup_f KI$ with $\begin{bmatrix} f(h_2) \\ f(u_2) \end{bmatrix}=\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} h_1 \\ u_1 \end{bmatrix}$. Then by Theorem 1, $K\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ admits a Heegaard splitting of genus two if and only if $b=\pm 1$ (Theorem 3 of [9]), and, by the proof of Case (1) of Theorem 2, $K\begin{bmatrix} a & \pm 1 \\ c & d \end{bmatrix}$ admits at most two non-isotopic Heegaard splittings of genus two. Moreover, by the proof of Case (1-a-1) and the note of Case (3-a-1) of Theorem 2, if $acd=0$ then $K\begin{bmatrix} a & \pm 1 \\ c & d \end{bmatrix}$ admits exactly one Heegaard splitting of genus two up to isotopy.

Hence, as a special case, $K\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ admits exactly one Heegaard splitting of genus two up to isotopy. Note that $K\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ is a 2-fold branched covering space of S^3 branched along a Borromean rings, and is also a 3-fold cyclic branched covering space of S^3 branched along a figure eight knot.

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