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On the Action of Hecke Rings on Homology Groups of Smooth Compactifications of Siegel Modular Varieties and Siegel Cusp Forms

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Introduction and notations.

Let $g \geqq 1$ and $N \geqq 3$ be rational integers. We use the same notations as in Hatada [9]. Recall

1_g = the $g \times g$ unit integral matrix; $J_g = \begin{bmatrix} 0 & -1_g \\ 1_g & 0 \end{bmatrix}$;

$\Gamma = \Gamma_g(N)$ = the principal congruence subgroup of level N of $\mathrm{Sp}(g, \boldsymbol{Z})$ $(\subset \mathrm{GL}(2g, \boldsymbol{Z}))$;

$\mathfrak{H}_g$ = the Siegel upper half plane of degree g;

$\Gamma \backslash \mathfrak{H}_g$ denotes the usual complex analytic quotient space;

$\mathrm{GSp}^+(g, \boldsymbol{R}) = \{\gamma \in \mathrm{GL}(2g, \boldsymbol{R}) \mid {}^t\gamma J_g \gamma J_g^{-1}$ is a scalar matrix whose eigenvalue is positive.$\}$;

$r(\alpha)$ = the eigenvalue of ${}^t\alpha J_g \alpha J_g^{-1}$ for $\alpha \in \mathrm{GSp}^+(g, \boldsymbol{R})$;

$\mathrm{GSp}^+(g, \boldsymbol{Z}) = \{\gamma \in \mathrm{GSp}^+(g, \boldsymbol{R}) \mid \gamma$ is an integral matrix.$\}$;

$\mathrm{GSp}^+(g, \boldsymbol{Q}) = \mathrm{GSp}^+(g, \boldsymbol{R}) \cap \mathrm{GL}(2g, \boldsymbol{Q})$;

$HR(\Gamma, \mathrm{GSp}^+(g, \boldsymbol{Z}))$ = the Hecke ring with respect to the group Γ and the monoid $\mathrm{GSp}^+(g, \boldsymbol{Z})$, cf. Hatada [8] and [9].

We consider the toroidal compactification of $\Gamma \backslash \mathfrak{H}_g$. We fix a regular and projective $\mathrm{Sp}(g, \boldsymbol{Z})$-admissible family of polyhedral cone decompositions: $\Sigma = \{\Sigma_\alpha\}_{F_\alpha:\ \text{rational components}}$ once for all. For example here we take a suitable refinement of the second Voronoi decomposition (cf. Namikawa [13], [14]). We write $(\Gamma \backslash \mathfrak{H}_g)^\sim$ for the projective smooth toroidal compactification of $\Gamma \backslash \mathfrak{H}_g$ with respect to this Σ. Write $M = (\Gamma \backslash \mathfrak{H}_g)^\sim$ for simplicity in this paper. For $\Gamma = \Gamma_g(N)$, define

$$\Gamma' = \{\xi \in \mathrm{Sp}(g, \boldsymbol{Z}) \mid \xi \ (\mathrm{mod}\ N) \text{ is a } 2g \times 2g \text{ diagonal matrix with coefficients in } \boldsymbol{Z}/N\boldsymbol{Z}.\},$$

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which is a subgroup of $\mathrm{Sp}(g, \boldsymbol{Z})$. Let Ω denote a real analytic Hodge metric on M induced from the projective space into which M is embedded. Let $\gamma \in \mathrm{Sp}(g, \boldsymbol{Z})$. By our choice of the toroidal compactification of $\Gamma\backslash\mathfrak{H}_g$, the complex analytic isomorphism

$$\gamma\colon \Gamma\backslash\mathfrak{H}_g \longrightarrow \Gamma\backslash\mathfrak{H}_g \qquad \text{given by} \quad \Gamma z \longmapsto \Gamma(\gamma z)$$

is extended to the whole of $(\Gamma\backslash\mathfrak{H}_g)^\sim$ as a unique isomorphism

$$\gamma^\sim\colon (\Gamma\backslash\mathfrak{H}_g)^\sim \longrightarrow (\Gamma\backslash\mathfrak{H}_g)^\sim ,$$

(cf. Hatada [8, Proposition 1.2]). Let $\gamma^{\sim *}\Omega$ denote the pull back of Ω by $\gamma^\sim$ (cf. Hatada [8, Definition 1.4]). This $\gamma^{\sim *}\Omega$ is also a Hodge metric on M. We have easily

LEMMA 1. *On M there is a real analytic Hodge metric Ω_0 which is invariant under any $\gamma \in \mathrm{Sp}(g, \boldsymbol{Z})$, i.e.,*

$$\gamma^{\sim *}\Omega_0 = \Omega_0 \qquad \textit{for any} \quad \gamma \in \mathrm{Sp}(g, \boldsymbol{Z}) .$$

Throughout this paper the harmonic forms on M we consider are those with respect to this Hodge metric Ω_0. Then for a harmonic form φ on M and an element γ of $\mathrm{Sp}(g, \boldsymbol{Z})$, the pull back $\gamma^{\sim *}\varphi$ of φ by $\gamma^\sim$ is also a harmonic form on M. For integers p and q, $H^{(p,q)}(M)$ denotes the space of the harmonic forms of type (p, q) on M. One sees that the factor group $\Gamma'/\Gamma \cong$ the direct product of g copies of the unit group of $(\boldsymbol{Z}/N\boldsymbol{Z})$. Write $(\Gamma'/\Gamma)^* =$ the dual group of (Γ'/Γ) $(=\mathrm{Hom}_{\boldsymbol{Z}}(\Gamma'/\Gamma, \boldsymbol{C}^\times))$. For an element $\chi \in (\Gamma'/\Gamma)^*$ write

$$H^{(p,q)}(\chi, M) = \{\varphi \in H^{(p,q)}(M) \mid \gamma^{\sim *}\varphi = \chi(\gamma \ (\mathrm{mod}\ \Gamma))\varphi \ \text{ for all } \gamma \in \Gamma'.\} .$$

This is a $\boldsymbol{C}$-subspace of $H^{(p,q)}(M)$. For simplicity write $\chi(\gamma) = \chi(\gamma \ (\mathrm{mod}\ \Gamma))$ for $\gamma \in \Gamma'$. For a positive integer s with G.C.D.$(s, N) = 1$, write

$$\mathcal{O}_{g,N}(s) = \left\{ \alpha \in \mathrm{GSp}^+(g, \boldsymbol{Z}) \;\middle|\; r(\alpha) = s.\ \ \alpha = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \text{ partitioned into blocks on dimension } g \times g, \text{ satisfies } A - 1_g \equiv B \equiv C \equiv 0 \ (\mathrm{mod}\ N). \right\} .$$

By Hatada [8, Proposition 4.2],

$$\mathcal{O}_{g,N}(s) = \bigcup_{i=1}^{\nu(s)} \Gamma \alpha_i \Gamma \qquad \text{(a finite disjoint union)} .$$

Then we define

$$T(s)=\sum_{i=1}^{\nu(s)} \Gamma\alpha_i\Gamma \in HR(\Gamma, \mathrm{GSp}^+(g, \boldsymbol{Z})) .$$

Let f_n denote the ring homomorphism: $HR(\Gamma, \mathrm{GSp}^+(g, \boldsymbol{Z})) \to \mathrm{End}_{\boldsymbol{Z}} H_n(M, \boldsymbol{Z})$ given by Hatada [8, Theorem 1] for each integer $n \geqq 0$. For an element $Y \in HR(\Gamma, \mathrm{GSp}^+(g, \boldsymbol{Z}))$, let $f_n(Y) \otimes_{\boldsymbol{Z}} \mathrm{id.}$ denote the $\boldsymbol{C}$-linear endomorphism of $H_n(M, \boldsymbol{C})$ induced from the isomorphism: $H_n(M, \boldsymbol{C}) \cong H_n(M, \boldsymbol{Z}) \otimes_{\boldsymbol{Z}} \boldsymbol{C}$, and let ${}^t(f_n(Y) \otimes_{\boldsymbol{Z}} \mathrm{id.})$ denote the transposed endomorphism of $H^n(M, \boldsymbol{C})$ with respect to the Kronecker index of complete duality: $H^n(M, \boldsymbol{C}) \times H_n(M, \boldsymbol{C}) \to \boldsymbol{C}$.

In this paper first we give:

THEOREM 1. *Let p and q be integers. Then we obtain:*

(1): $H^{(p,q)}(M) = \bigoplus_{\chi \in (\Gamma'/\Gamma)^*} H^{(p,q)}(\chi, M)$; *and*

(2): *Each space $H^{(p,q)}(\chi, M)$ is invariant under the operators ${}^t(f_{p+q}(T(n)) \otimes_{\boldsymbol{Z}} \mathrm{id.})$ for all positive $n \in \boldsymbol{Z}$ with* G.C.D.$(n, N)=1$.

Using Theorem 1 we give our main

THEOREM 2. *Assume $g=2$. (Hence $\Gamma=\Gamma_2(N)$ and $M=(\Gamma\backslash\mathfrak{H}_2)^{\sim}$.) Let n be an integer with $n \geqq 2$ and* G.C.D.$(N, n)=1$. *Let λ_n be an eigenvalue of the $\boldsymbol{C}$-linear endomorphism $f_3(T(n)) \otimes_{\boldsymbol{Z}} \mathrm{id.}$ of $H_3(M, \boldsymbol{C})$. Then we obtain*

$$|\lambda_n| \lneqq \textit{the number of the left cosets in } \Gamma\backslash\mathcal{O}_{2,N}(n) \ (=\sharp(\Gamma\backslash\mathcal{O}_{2,N}(n))$$

for any archimedean absolute value $|\cdot|$ with $|2|=2$.

COROLLARY OF THEOREM 2. *Notations being as in Theorem 2, we obtain $|\lambda_l| \lneqq (1+l)(1+l^2)$ for any prime number l with $l \nmid N$.*

In §2 we give a proposition asserting that

$$\dim_{\boldsymbol{C}} H_3((\Gamma_2(N)\backslash\mathfrak{H}_2)^{\sim}, \boldsymbol{C}) \to +\infty \qquad \text{when} \quad N \to +\infty .$$

As to the eigenvalues of Hecke operators on the spaces of Siegel cusp forms, we give:

THEOREM 3. *Let $g \geqq 1$ and $w \geqq 0$ be rational integers, and let $S_{g+w+1}(\Gamma)=$ the space of the holomorphic Siegel cusp forms of weight $g+w+1$ with respect to $\Gamma=\Gamma_g(N)$. Let n be an integer with $n \geqq 2$ and* G.C.D.$(n, N)=1$, *and let $T_{g+1+w}(n)$ be the usual Hecke operator acting on $S_{g+1+w}(\Gamma)$. (For the definition of the $T_{g+1+w}(n)$, see Hatada [8, Remark 3.2, p. 392] and use $T_{g+1+w}(n)(F)=\sum_{i=1}^{\nu(n)} F|_{g+1+w}[\Gamma\beta_i\Gamma]$ where $\mathcal{O}_{g,N}(n)=\bigcup_{i=1}^{\nu(n)} \Gamma\beta_i\Gamma$ (disjoint).) Let $\lambda(n)$ be an eigenvalue of the $T_{g+1+w}(n)$ on $S_{g+w+1}(\Gamma)$. Then we obtain*

$$|\lambda(n)| \lneqq n^{gw/2} \times (\textit{the number of the left cosets in } \Gamma\backslash\mathcal{O}_{g,N}(n))$$

for any archimedean absolute value $|\cdot|$ *with* $|2|=2$. (cf. Drinfeld [2] for the case of $g=1$, $w=0$ and n: prime number with $n\equiv 1 \pmod{N}$.)

In Freitag [4, Hilfssatz 4.8, p. 269] it was proved that

$$|\lambda(n)|\leqq n^{gw/2}\times(\text{the number of the left cosets in } \Gamma\backslash\mathcal{O}_{g,1}(n))$$

in the case of $\Gamma=\mathrm{Sp}(g, \boldsymbol{Z})$. Therefore our Theorem 3 is an improvement of that result of Freitag even in the case of $\Gamma=\mathrm{Sp}(g, \boldsymbol{Z})$.

COROLLARY OF THEOREM 3. *Notations being as in Theorem* 3, *we obtain*

$$|\lambda(l)|\lneqq l^{gw/2}\left(\prod_{u=1}^{g}(1+l^u)\right) \textit{ for any prime number } l \textit{ with } l\nmid N\,.$$

We give proofs of Theorems 1, 2 and 3 in §1, §2 and §3 respectively.

§1. On Theorem 1.

First we prove Lemma 1. Let $\Gamma\backslash\mathrm{Sp}(g, \boldsymbol{Z})=\cup_{j=1}^{a}\Gamma G_j$ (a disjoint union). For the Hodge metric Ω on M explained in the introduction put

$$\Omega_0=\sum_{j=1}^{a} G_j^{\sim *}\Omega$$

which is a Hodge metric on M satisfying the required property of Lemma 1.

We consider harmonic forms on M with respect to this Ω_0.

(1) of Theorem 1 is directly derived from the well known theorem on the representation of abelian groups.

Proof of (2) of Theorem 1. Let $\tau\in\Gamma'=\Gamma_g(N)'$. We have $\tau^{-1}\mathcal{O}_{g,N}(n)\tau=\mathcal{O}_{g,N}(n)$. Write $\mathcal{O}_{g,N}(n)=\cup_{i=1}^{\mu}\Gamma\alpha_i$ (a disjoint union). Then $\cup_{i=1}^{\mu}\Gamma\alpha_i\tau=\cup_{i=1}^{\mu}\tau\Gamma\alpha_i=\cup_{i=1}^{\mu}\Gamma\tau\alpha_i$ since $\tau\Gamma\tau^{-1}=\Gamma$. Let $\varphi\in H^{(p,q)}(\chi, M)$. Let i be an integer with $1\leqq i\leqq\mu$. Let j denote the integer with $\Gamma\alpha_i\tau=\Gamma\tau\alpha_j$. We have the following commutative diagram (1.1). cf. Hatada [8, (2.2.1)].

$$\begin{array}{ccccc}
(\Gamma_g(n^2N)\backslash\mathfrak{H}_g)^{\sim} & \xrightarrow{\tau^{\sim}} & (\Gamma_g(n^2N)\backslash\mathfrak{H}_g)^{\sim} & & \\
\downarrow{\scriptstyle \pi_j} & & \downarrow{\scriptstyle \pi_i} & & \\
(\tau^{-1}\alpha_i^{-1}\Gamma_g(nN)\alpha_i\tau\backslash\mathfrak{H}_g)^{\sim} & \xrightarrow{\tau^{\sim}} & (\alpha_i^{-1}\Gamma_g(nN)\alpha_i\backslash\mathfrak{H}_g)^{\sim} & \xrightarrow{\tilde{\alpha_i}} & (\Gamma_g(nN)\backslash\mathfrak{H}_g)^{\sim} \\
\downarrow{\scriptstyle \pi^{(j)}} & & \downarrow{\scriptstyle \pi^{(i)}} & & \downarrow{\scriptstyle [\pi]} \\
M=(\Gamma\backslash\mathfrak{H}_g)^{\sim} & \xrightarrow{\tau^{\sim}} & M=(\Gamma\backslash\mathfrak{H}_g)^{\sim} & & M=(\Gamma\backslash\mathfrak{H}_g)^{\sim}
\end{array} \qquad (1.1)$$

In (1.1), $\Gamma=\Gamma_g(N)$, $\pi=\pi^{(i)}\circ\pi_i=\pi^{(j)}\circ\pi_j$, and the vertical lines denote the canonical morphisms. Note $(\tau^{-1}\alpha_i^{-1}\Gamma_g(nN)\alpha_i\tau\backslash\mathfrak{H}_g)^\sim=(\alpha_j^{-1}\Gamma_g(nN)\alpha_j\backslash\mathfrak{H}_g)^\sim$. We use Definition 1.4 in Hatada [8]. We write $\langle\varphi\rangle=\sum_{i=1}^{\mu}\pi_i^*\circ\alpha_i^{\sim*}\circ[\pi]^*(\varphi)$ now. By Hatada [8, Lemma 3.1], there exists a unique (p, q)-form $\xi^\sim$ on M with $\langle\varphi\rangle=\pi^*(\xi^\sim)$. By (1.1),

$$\begin{aligned}\pi^*\circ\tau^{\sim*}(\xi^\sim)&=\tau^{\sim*}\circ\pi^*(\xi^\sim)=\tau^{\sim*}(\langle\varphi\rangle)\\&=\sum_{i=1}^{\mu}\tau^{\sim*}\circ\pi_i^*\circ\alpha_i^{\sim*}\circ[\pi]^*(\varphi)\\&=\sum_{i=1}^{\mu}\pi_{j(i)}^*\circ\tau^{\sim*}\circ\alpha_i^{\sim*}\circ[\pi]^*(\varphi)\ .\end{aligned}$$

Note that $x^{\sim*}\circ[\pi]^*(\varphi)=[\pi]^*(\varphi)$ for all $x\in\Gamma$ and that the following diagram is commutative.

$$\begin{array}{ccc}(\Gamma_g(nN)\backslash\mathfrak{H}_g)^\sim & \xrightarrow{\ \tau^\sim\ } & (\Gamma_g(nN)\backslash\mathfrak{H}_g)^\sim\\ \big\downarrow{\scriptstyle[\pi]} & & \big\downarrow{\scriptstyle[\pi]}\\ M & \xrightarrow{\ \tau^\sim\ } & M\end{array}$$

Then we obtain

$$\begin{aligned}\tau^{\sim*}\circ\alpha_i^{\sim*}\circ[\pi]^*(\varphi)&=(\alpha_i\tau)^{\sim*}\circ[\pi]^*(\varphi)\\&=(\gamma'\tau\alpha_j)^{\sim*}\circ[\pi]^*(\varphi)\qquad\text{for some}\quad\gamma'\in\Gamma\\&=\alpha_j^{\sim*}\circ\tau^{\sim*}\circ[\pi]^*(\varphi)\\&=\alpha_j^{\sim*}\circ[\pi]^*\circ\tau^{\sim*}(\varphi)\\&=\chi(\tau)\alpha_j^{\sim*}\circ[\pi]^*(\varphi)\ .\end{aligned}$$

Hence

$$\begin{aligned}\pi^*\circ\tau^{\sim*}(\xi^\sim)&=\chi(\tau)\sum_{j=1}^{\mu}\pi_j^*\circ\alpha_j^{\sim*}\circ[\pi]^*(\varphi)\\&=\chi(\tau)\langle\varphi\rangle\\&=\chi(\tau)\pi^*(\xi^\sim)\\&=\pi^*(\chi(\tau)\xi^\sim)\ .\end{aligned}$$

Hence

$$\tau^{\sim*}(\xi^\sim)=\chi(\tau)\xi^\sim\qquad\text{for any}\quad\tau\in\Gamma'\ . \tag{1.2}$$

Recall the orthogonal projection in the potential theory: id. $=\boldsymbol{H}+d\delta\boldsymbol{G}+\delta d\boldsymbol{G}$. Here $\boldsymbol{G}$ denotes the Green's operator on M. By Hatada [8, Theorem 8 (ii)],

$$^t(f_{p+q}(T(n))\otimes_Z \text{id.})(\varphi)=\boldsymbol{H}\xi^{\sim} .$$

We express $\boldsymbol{H}$ as the integral operator by the theory of de Rham [15, p. 132]. We quote the lines 12–17 at p. 132 of the book. "Let $h_1, h_2, \cdots, h_d$ be an orthonormal base of the vector space of harmonic forms, so that $(h_i, h_j)=\delta^i_j$, and put

$$h(x, y)=\sum_i h_i(x)h_i(y) .$$

Then

$$\int_y h(x, y)\wedge *_y T(y)=\sum_i (h_i, T)h_i(x)$$

is a harmonic form which is exactly HT. The double form $h(x, y)$ is thus the *metric kernel* of H." Apply this to our case. Then

$$\begin{aligned}\tau^{\sim *}(\boldsymbol{H}\xi^{\sim})&=\int_{y\in M} h(\tau^{\sim}(x), y)\wedge *_y\xi^{\sim}(y)\\ &=\int_{y\in M} h(\tau^{\sim}(x), \tau^{\sim}(y))\wedge(*_y\xi^{\sim})(\tau^{\sim}(y)) .\end{aligned}$$

The $*$-operator is $\mathrm{Sp}(g, \boldsymbol{Z})$ invariant since the Hodge metric Ω_0 is $\mathrm{Sp}(g, \boldsymbol{Z})$ invariant (cf. Kodaira and Morrow [11, p. 93]). Hence

$$(*_y\xi^{\sim})(\tau^{\sim}(y))=*_y(\xi^{\sim}(\tau^{\sim}(y))) .$$

We also note that $\tau^{\sim *}h_1, \tau^{\sim *}h_2, \cdots, \tau^{\sim *}h_d$ are also an orthonormal basis of the vector space of the harmonic forms if so are $h_1, h_2, \cdots, h_d$. Therefore

$$\begin{aligned}\tau^{\sim *}(\boldsymbol{H}\xi^{\sim})&=\int_{y\in M} h(\tau^{\sim}(x), \tau^{\sim}(y))\wedge *_y(\tau^{\sim *}\xi^{\sim})(y)\\ &=\boldsymbol{H}(\tau^{\sim *}\xi^{\sim}) .\end{aligned} \tag{1.3}$$

By (1.2), $\tau^{\sim *}(\boldsymbol{H}\xi^{\sim})=\boldsymbol{H}(\chi(\tau)\xi^{\sim})=\chi(\tau)\boldsymbol{H}(\xi^{\sim})$. Namely

$$\tau^{\sim *}\circ{}^t(f_{p+q}(T(n))\otimes_Z \text{id.})(\varphi)=\chi(\tau)^t(f_{p+q}(T(n))\otimes_Z \text{id.})(\varphi)$$

for all $\varphi\in H^{(p,q)}(\chi, M)$ and all $\tau\in\Gamma'$. (2) of Theorem 1 is proved.

§2. On Theorem 2.

In this section we assume $g=2$. Write $M=(\Gamma\backslash\mathfrak{H}_2)^{\sim}$. For $Z\in\mathfrak{H}_2$, write

$$Z=\begin{pmatrix} z_1 & z_2 \\ z_2 & z_3 \end{pmatrix}=\begin{pmatrix} x_1 & x_2 \\ x_2 & x_3 \end{pmatrix}+\sqrt{-1}\begin{pmatrix} y_1 & y_2 \\ y_2 & y_3 \end{pmatrix}$$

with real coefficients $x_1, x_2, x_3, y_1, y_2, y_3$. The differential form

$$\frac{dx_1\wedge dx_2\wedge dx_3\wedge dy_1\wedge dy_2\wedge dy_3}{(y_1 y_3-y_2^2)^3}$$

on $\mathfrak{H}_2$ is invariant under the action of $\mathrm{Sp}(g, \boldsymbol{R})$ (cf. Maass [12]). The volumes we treat in §2 are measured with respect to this volume form.

LEMMA 2.1. $H^1(M, \boldsymbol{C})=\{0\}$.

PROOF. By the Hodge decomposition

$$H^1(M, \boldsymbol{C})\cong H^{(1,0)}(M)\oplus\overline{H^{(1,0)}(M)} .$$

$H^{(1,0)}(M)\cong$ the space of the holomorphic 1-forms on M, which is $\{0\}$ by Freitag [3]. Hence $H^1(M, \boldsymbol{C})=\{0\}$.

By the Poincaré duality, $H^5(M, \boldsymbol{C})=\{0\}$. Let $P^3(M)$ be the third primitive cohomology of M defined by $\mathrm{Ker}(L: H^3(M, \boldsymbol{C})\rightarrow H^5(M, \boldsymbol{C}))$, cf. Griffiths and Harris [6, p. 111 and p. 122]. Hence in our case, $P^3(M)=H^3(M, \boldsymbol{C})$ and $P^{(p,q)}(M)=H^{(p,q)}(M)$ for non-negative integers p and q with $p+q=3$. One has:

THEOREM 2.2. *Let p and q be non-negative integers with $p+q=3$. Let $\langle\ ,\ \rangle$ denote the Hermitian form*:

$$H^{(p,q)}(M)\times H^{(p,q)}(M)\longrightarrow \boldsymbol{C} ,$$

$$(\varphi, \psi)\longmapsto \sqrt{-1}(\mathrm{Vol}(\Gamma\backslash\mathfrak{H}_2))^{-1}\int_{\Gamma\backslash\mathfrak{H}_2}\varphi\wedge\bar{\psi} .$$

This $\langle\ ,\ \rangle$ is a positive definite Hermitian form.

PROOF. Since $P^{(p,q)}(M)=H^{(p,q)}(M)$, this is a direct consequence of Hodge Signature Theorem (cf. Griffiths and Harris [6, p. 123]).

For any Γ-invariant automorphic forms θ_1 and θ_2 of type (p, q) with $p+q=3$ on $\mathfrak{H}_2$ for which the left side of the following equation is defined, we have:

$$\sqrt{-1}(\mathrm{Vol}(\Gamma\backslash\mathfrak{H}_2))^{-1}\int_{\Gamma\backslash\mathfrak{H}_2}\theta_1\wedge\bar{\theta}_2=\sqrt{-1}(\mathrm{Vol}(\Gamma_1\backslash\mathfrak{H}_2))^{-1}\int_{\Gamma_1\backslash\mathfrak{H}_2}\theta_1\wedge\bar{\theta}_2$$

for any finite index subgroup Γ_1 of Γ.

Let $\mathscr{S}$ be a subgroup of $\mathrm{Sp}(g, \boldsymbol{R})$ which is commensurable with $\mathrm{Sp}(g, \boldsymbol{Z})$. In the following manner (2.2.1) we may extend the domain of the Hermitian form $\langle\ ,\ \rangle$ in Theorem 2.2 to all the pairs (ω_1, ω_2), of $\mathscr{S}$-invariant automorphic forms of type (p, q) with $p+q=3$ on $\mathfrak{H}_2$, for that the right side of (2.2.1) is defined.

$$\langle\omega_1, \omega_2\rangle=\sqrt{-1}(\mathrm{Vol}(\mathscr{S}\backslash\mathfrak{H}_2))^{-1}\int_{\mathscr{S}\backslash\mathfrak{H}_2}\omega_1\wedge\bar{\omega}_2 \tag{2.2.1}$$

Then it should be noticed that

$$\langle\omega_1, \omega_2\rangle=\sqrt{-1}(\mathrm{Vol}((\mathscr{S}\cap\Gamma)\backslash\mathfrak{H}_2))^{-1}\int_{(\mathscr{S}\cap\Gamma)\backslash\mathfrak{H}_2}\omega_1\wedge\bar{\omega}_2\ .$$

Let $\alpha\in\mathrm{GSp}^+(g, \boldsymbol{Q})$. Then $\alpha^*\omega_1$ and $\alpha^*\omega_2$ are $\alpha^{-1}\mathscr{S}\alpha$-invariant automorphic forms. $\alpha^{-1}\mathscr{S}\alpha$ is commensurable with $\mathscr{S}$. Then we obtain

$$\langle\alpha^*\omega_1, \alpha^*\omega_2\rangle=\langle\omega_1, \omega_2\rangle \tag{2.2.2}$$

by changing the variables of the integration.

We write

$$\|\omega_1\|=\sqrt{\langle\omega_1, \omega_1\rangle} \tag{2.2.3}$$

if the integration of the right side is defined.

PROOF OF THEOREM 2. Let λ_n be an eigenvalue of the $\boldsymbol{C}$-linear endomorphism $f_3(T(n))\otimes_{\boldsymbol{Z}}\mathrm{id.}$ of $H_3(M, \boldsymbol{C})$. By Hatada [8, Theorem 2 (i)] we may assume that λ_n is an eigenvalue of ${}^t(f_3(T(n))\otimes_{\boldsymbol{Z}}\mathrm{id.})$ on $H^{(p,q)}(M)$ for some non-negative integers p and q with $p+q=3$. For simplicity write $\lambda=\lambda_n$. By Theorem 1 there exist a character $\chi\in(\Gamma'/\Gamma)^*$ and a harmonic form $\varphi\neq0$ in $H^{(p,q)}(\chi, M)$ such that ${}^t(f_3(T(n))\otimes_{\boldsymbol{Z}}\mathrm{id.})(\varphi)=\lambda\varphi$. We use the same notations in the Proof of (2) of Theorem 1 in § 1. We use also Theorem 2.2, (2.2.1), (2.2.2) and (2.2.3). We define canonical maps Π and $\Pi^\wedge$ by the composition of maps as follows.

$$\Pi:\mathfrak{H}_2\longrightarrow\Gamma\backslash\mathfrak{H}_2\hookrightarrow M\ ;\qquad \Pi^\wedge:\mathfrak{H}_2\longrightarrow\Gamma_2(n^2N)\backslash\mathfrak{H}_2\hookrightarrow(\Gamma_2(n^2N)\backslash\mathfrak{H}_2)^\sim\ .$$

We obtain:

$$\begin{aligned}\lambda\varphi&={}^t(f_3(T(n))\otimes_{\boldsymbol{Z}}\mathrm{id.})(\varphi)=\boldsymbol{H}\xi^\sim\\&=\xi^\sim-d\delta\boldsymbol{G}\xi^\sim-\delta d\boldsymbol{G}\xi^\sim=\xi^\sim-d\delta\boldsymbol{G}\xi^\sim\end{aligned}$$

since $\delta d\boldsymbol{G}\xi^\sim=0$ as a current (cf. Hatada [8, p. 391]). Recall $\pi=\pi^{(i)}\circ\pi_i$ for each $i\in[1, \mu]$. Then

$$\lambda\pi^*(\varphi)=\pi^*(\xi^\sim)-\pi^*(d\delta\boldsymbol{G}\xi^\sim)=\langle\varphi\rangle-\pi^*(d\delta\boldsymbol{G}\xi^\sim)=\sum_{i=1}^{\mu}\psi_i \tag{2.3}$$

where we have put:

$$\psi_i=\pi_i^*\circ\alpha_i^{\sim *}\circ[\pi]^*(\varphi) \qquad \text{for each} \quad i\in[1,\mu-1]\ ; \quad \text{and}$$
$$\psi_\mu=\pi_\mu^*\circ\alpha_\mu^{\sim *}\circ[\pi]^*(\varphi)-\pi^*(d\delta\boldsymbol{G}\xi^\sim) \qquad \text{for} \quad i=\mu\ .$$

These $\{\psi_i\}_{i=1}^{\mu}$ are d-closed differential forms on $(\Gamma_2(n^2N)\backslash\mathfrak{H}_2)^\sim$. We obtain that

$$\|\psi_i\|=\|\varphi\| \qquad \text{for all} \quad i\in[1,\mu]\ . \tag{2.4}$$

Let Ω_2 be a Hodge metric on $(\Gamma_2(n^2N)\backslash\mathfrak{H}_2)^\sim$. For continuous differential forms Ψ of type (p, q) on $(\Gamma_2(n^2N)\backslash\mathfrak{H}_2)^\sim$, let $\Psi=\boldsymbol{H}_2\Psi+d\delta_2\boldsymbol{G}_2\Psi+\delta d_2\boldsymbol{G}_2\Psi$ be the orthogonal projection in the potential theory on $(\Gamma_2(n^2N)\backslash\mathfrak{H}_2)^\sim$ with respect to Ω_2. Here $\boldsymbol{G}_2$ is the Green's operator on $(\Gamma_2(n^2N)\backslash\mathfrak{H}_2)^\sim$. Apply Theorem 2.2 to the harmonic forms on $(\Gamma_2(n^2N)\backslash\mathfrak{H}_2)^\sim$. We obtain $\psi_i=\boldsymbol{H}_2\psi_i+d\delta_2\boldsymbol{G}_2\psi_i$ for each $i\in[1,\mu]$. Then using Stokes' theorem we obtain:

$$\begin{aligned}\langle\psi_i,\psi_j\rangle&=\sqrt{-1}(\mathrm{Vol}(\Gamma_2(n^2N)\backslash\mathfrak{H}_2))^{-1}\int_{(\Gamma_2(n^2N)\backslash\mathfrak{H}_2)^\sim}\psi_i\wedge\bar{\psi}_j\\&=\langle\boldsymbol{H}_2\psi_i,\boldsymbol{H}_2\psi_j\rangle\end{aligned}$$

for all $i\in[1,\mu]$ and all $j\in[1,\mu]$. Now write $V=\sum_{i=1}^{\mu}\boldsymbol{C}\psi_i$. We have obtained that the sesquilinear form $\langle\ ,\ \rangle|_{V\times V}\colon V\times V\to\boldsymbol{C}$ is a positive definite Hermitian form. Then we obtain from (2.3) and (2.4):

$$|\lambda|\,\|\varphi\|=\|\lambda\pi^*(\varphi)\|=\left\|\sum_{i=1}^{\mu}\psi_i\right\|\leqq\sum_{i=1}^{\mu}\|\psi_i\|=\mu\|\varphi\|\ .$$

Hence $|\lambda|\leqq\mu$.

Now assume that $|\lambda|=\mu$ here. Then from (2.3) and (2.4) we obtain:

$$\psi_1=\psi_i \quad \text{for all} \quad i\in[1,\mu]\ ; \quad \text{and}$$
$$\lambda\pi^*(\varphi)=\mu\psi_1=\mu\pi_1^*\circ\alpha_1^{\sim *}\circ[\pi]^*(\varphi)\ .$$

Hence

$$\lambda\Pi^*(\varphi)=\lambda\Pi^{\wedge *}\circ\pi^*(\varphi)=\mu\Pi^{\wedge *}\circ\pi_1^*\circ\alpha_1^{\sim *}\circ[\pi]^*(\varphi)\ . \tag{2.5}$$

By (2.5),

$$\lambda\Pi^*(\varphi)=\mu(\Pi^*(\varphi))\circ\alpha_1$$

where $\circ\alpha_1$ denotes the pull back by α_1. We may write $\alpha_1=\sigma\begin{pmatrix}n1_g & 0\\ 0 & 1_g\end{pmatrix}$ with some $\sigma \in \mathrm{Sp}(g, \boldsymbol{Z})$ satisfying $\sigma \equiv \begin{pmatrix} n^{-1}1_g & 0 \\ 0 & n1_g \end{pmatrix}$ (mod N) by Hatada [8, Corollary 4.4 and Lemma 4.5, pp. 394-395]. Hence

$$\lambda \Pi^*(\varphi) = \mu \chi(\sigma \ (\mathrm{mod}\ \Gamma))^{-1}(\Pi^*(\varphi)) \circ \begin{pmatrix} n1_g & 0 \\ 0 & 1_g \end{pmatrix} .$$

Hence

$$\Pi^*(\varphi) = (\mu\lambda^{-1}\chi(\sigma \ (\mathrm{mod}\ \Gamma))^{-1})^k (\Pi^*(\varphi)) \circ \left(\begin{pmatrix} n1_g & 0 \\ 0 & 1_g \end{pmatrix}^k \right) \tag{2.6}$$

for all the integers $k \geqq 1$. Let us consider a system $\{\zeta_1, \zeta_2, \zeta_3\}$ of local parameters at a point $\in M$ over $\begin{pmatrix} \sqrt{-1}\infty & * \\ * & \sqrt{-1}\infty \end{pmatrix} \in$ (Satake Compactification of $\Gamma \backslash \mathfrak{H}_2$). They are written as

$$\zeta_j = \exp(2N^{-1}[pi]\sqrt{-1}(t_{j1}z_1 + t_{j2}z_2 + t_{j3}z_3)) \qquad (j = 1, 2, 3)$$

for some rational numbers t_{ij} $(1 \leqq i \leqq 3, 1 \leqq j \leqq 3)$. Here $[pi]$ denotes the ratio of the circumference of a circle to its diameter. (cf. Ash et al. [1], Namikawa [13], [14].) Remember that the Hodge metric Ω_0 chosen by us is real analytic. The right side of (2.6) is expressed in terms of variables $\{\zeta_1^{n^k}, \zeta_2^{n^k}, \zeta_3^{n^k}, \bar{\zeta}_1^{n^k}, \bar{\zeta}_2^{n^k}, \bar{\zeta}_3^{n^k}\}$ as convergent power series for each arbitrarily given positive integer k. So is $\Pi^*(\varphi)$. This is a contradiction since the local power series expansion of φ in terms of $\{\zeta_1, \zeta_2, \zeta_3, \bar{\zeta}_1, \bar{\zeta}_2, \bar{\zeta}_3\}$ is unique. Hence we obtain $|\lambda| \lneqq \mu = \#(\Gamma \backslash \mathcal{O}_{2,N}(n))$. Theorem 2 is proved.

REMARK 2.7. By Hatada [8, Proposition 4.2],

$$\#(\Gamma_g(N) \backslash \mathcal{O}_{g,N}(n)) = \#(\mathrm{Sp}(g, \boldsymbol{Z}) \backslash \mathcal{O}_{g,1}(n)) \quad \text{when} \quad \mathrm{G.C.D.}(n, N) = 1 .$$

Now we show:

PROPOSITION 2.8. *Let $(\Gamma_2(N) \backslash \mathfrak{H}_2)^\sim$ denote Igusa's non-singular and projective compactification of $\Gamma_2(N) \backslash \mathfrak{H}_2$* (cf. Igusa [10], Namikawa [13], [14]). *Then one obtains:*

$$\dim_{\boldsymbol{C}} H_3((\Gamma_2(N) \backslash \mathfrak{H}_2)^\sim, \boldsymbol{C}) \to +\infty \qquad \textit{when} \quad N \to +\infty .$$

PROOF. This is a direct consequence of results in Geer [5, pp. 331-332]. We give this proof for the convenience of the reader. Write $M = (\Gamma_2(N) \backslash \mathfrak{H}_2)^\sim$ here. Write Euler(M) = the Euler number of M. By Geer [5],

$$\mathrm{Euler}(M) = \mathscr{A}(N)\zeta_{\boldsymbol{Q}}(-1)\zeta_{\boldsymbol{Q}}(-3) + 2^{-1}N\mathscr{B}(N)$$

where ζ_Q is the Riemann zeta function;

$$\mathscr{A}(N)=N^{10}\prod_{p:\text{ prime number, } p|N}((1-p^{-2})(1-p^{-4}))\ ;$$
$$\mathscr{B}(N)=N^{4}\prod_{p:\text{ prime number, } p|N}(1-p^{-4})\ .$$

Remember that $\zeta_Q(-1)\zeta_Q(-3)$ is a negative rational number. We have $N\mathscr{B}(N)/\mathscr{A}(N)\leqq N^{-3}$. Hence we obtain that

$$\mathrm{Euler}(M)\to-\infty \quad \text{when} \quad N\to+\infty\ .$$

By the definition of the Euler number,

$$\mathrm{Euler}(M)=\sum_{j=0}^{6}(-1)^j\dim_C H^j(M, C)\ .$$

By Lemma 2.1 and the Poincaré duality,

$$\mathrm{Euler}(M)=2+2\dim_C H^2(M, C)-\dim_C H^3(M, C)\ .$$

Hence

$$\dim_C H^3(M, C)=2+2\dim_C H^2(M, C)-\mathrm{Euler}(M)\geqq 2-\mathrm{Euler}(M)\ .$$

Hence

$$\dim_C H_3(M, C)=\dim_C H^3(M, C)\to+\infty \quad \text{when} \quad N\to+\infty\ .$$

§ 3. Proof of Theorem 3.

Let $\Gamma=\Gamma_g(N)$, and let Γ' be the subgroup of $\mathrm{Sp}(g, \boldsymbol{Z})$ defined in the introduction. For a Siegel cusp form $F(Z)\in S_{g+1+w}(\Gamma)$ and $\alpha=\begin{pmatrix}A & B\\ C & D\end{pmatrix}\in$ $\mathrm{GSp}^+(g, \boldsymbol{Q})$, partitioned into blocks on dimension $g\times g$, set $F|_{g+1+w}[\alpha](Z)=$ $F((AZ+B)(CZ+D)^{-1})(\det(CZ+D))^{-g-1-w}$. For a character $\chi\in(\Gamma'/\Gamma)^*$, write $\mathfrak{S}_{g+1+w}(N, \chi)=\{F(Z)\in S_{g+1+w}(\Gamma)\mid F|_{g+1+w}[\gamma]=\chi(\gamma)F$ for any $\gamma\in\Gamma'.\}$. This is a $\boldsymbol{C}$-subspace of $S_{g+1+w}(\Gamma)$. Then by the same argument as in the proof of (1) of Theorem 1, we obtain:

THEOREM 3.1.1. $S_{g+1+w}(\Gamma)=\bigoplus_{\chi\in(\Gamma'/\Gamma)^*}\mathfrak{S}_{g+1+w}(N, \chi)$.

Write $\mathscr{O}_{g,N}(n)=\bigcup_{i=1}^{\mu}\Gamma\alpha_i$ (disjoint). Let $\tau\in\Gamma'$. Recall $\tau^{-1}\mathscr{O}_{g,N}(n)\tau=$ $\mathscr{O}_{g,N}(n)$ and $\mathscr{O}_{g,N}(n)\tau=\bigcup_{i=1}^{\mu}\Gamma\tau\alpha_i$ (disjoint). Now let $F(Z)\in\mathfrak{S}_{g+1+w}(N, \chi)$. Recall $((T_{g+1+w}(n))F)(Z)=n^{g(g+1+w)-g(g+1)/2}(\sum_{i=1}^{\mu}F|_{g+1+w}[\alpha_i](Z))$, (cf. Hatada [8, p. 392]). Then

$$\begin{aligned}(T_{g+1+w}(n)F)|_{g+1+w}[\tau](Z) \\ &= n^{g(g+1+w)-g(g+1)/2}\Big(\sum_{i=1}^{\mu} F|_{g+1+w}[\alpha_i\tau](Z)\Big) \\ &= n^{g(g+1+w)-g(g+1)/2}\Big(\sum_{i=1}^{\mu} F|_{g+1+w}[\tau\alpha_i](Z)\Big) \\ &= \chi(\tau \ (\mathrm{mod}\ \Gamma))(T_{g+1+w}(n)F)(Z)\ .\end{aligned}$$

Hence we obtain:

THEOREM 3.1.2. *Each space* $\mathfrak{S}_{g+1+w}(N, \chi)$ *is invariant under all the* $T_{g+1+w}(n)$ *with* G.C.D.$(n, N)=1$.

Write $M(n^2N)_w$ = the projective manifold $(\Gamma_g(n^2N) \ltimes (n\boldsymbol{Z})^{2gw}\backslash\mathfrak{H}_g \times \boldsymbol{C}^{gw})^{\sim}$ defined in Hatada [8, pp. 377-378], and $d=\dim_{\boldsymbol{C}} M(n^2N)_w=g(g+1)/2+gw$ for simplicity. By the same argument as in Hatada [7], we obtain:

THEOREM 3.2. *The space* $S_{g+1+w}(\Gamma_g(n^2N))$ *(resp.* $S_{g+1+w}(\Gamma)$*) is naturally identified with the space* $H^{(d,0)}(M(n^2N)_w)$ *(resp.* $H^{(d,0)}(M(N)_w)$*).* (cf. Theorem and Lemma 3 in Hatada [7].)

Under the notations of Hatada [7, p. 505], put

$$\Theta=(\bigwedge_{1\le i\le j\le g} dz_{i,j})\wedge(\bigwedge_{\substack{1\le i\le w\\ 1\le j\le g}} du_{i,j})$$

which is a differential form on $\mathfrak{H}_g \times \boldsymbol{C}^{gw}$. Let $\lambda(n)$ be an eigenvalue of the $T_{g+1+w}(n)$ on $S_{g+1+w}(\Gamma)$. Then there exist some $\chi \in (\Gamma'/\Gamma)^*$ and some non zero $F_0 \in \mathfrak{S}_{g+1+w}(N, \chi)$ such that $(T_{g+1+w}(n))F_0=\lambda(n)F_0$. By Theorem 3.2 and Hatada [8, Lemma 2.1], $F_0(Z)\Theta$ (resp. $n^{g(g+1+w)-g(g+1)/2}F_0|_{g+1+w}[\alpha_i](Z)\Theta$) is regarded uniquely as a differential form ω (resp. ω_i) on $M(N)_w$ and $M(n^2N)_w$ (resp. on $M(n^2N)_w$ for each $i \in [1, \mu]$). Use the commutative diagram (2.2.1) and the notations in Hatada [8, p. 380] replacing c by n. Write $[\pi]$ for the canonical morphism: $(\Gamma_g(nN) \ltimes (n\boldsymbol{Z})^{2gw}\backslash\mathfrak{H}_g \times \boldsymbol{C}^{gw})^{\sim} \to (\Gamma \ltimes \boldsymbol{Z}^{2gw}\backslash\mathfrak{H}_g \times \boldsymbol{C}^{gw})^{\sim}$ given in Hatada [8, Lines 6 and 7 from the bottom of p. 381] replacing c by n. Then we obtain $\omega_i=\pi_i^* \circ (\alpha_i, 0)^{\sim *} \circ [\pi]^*(\omega)$ for each $i \in [1, \mu]$. From Hatada [8, Theorem 2] we have:

LEMMA 3.3. (i) *The differential form* $\sum_{i=1}^{\mu} \omega_i$ *on* $M(n^2N)_w$ *is regarded as a differential form on* $M(N)_w$*; and*
(ii)

$$^t(f_d(T(n))\otimes_{\boldsymbol{Z}} \mathrm{id.})(\omega)=\sum_{i=1}^{\mu} \omega_i \qquad on \quad M(N)_w\ . \tag{3.3}$$

Write, for simplicity, $T(n)(\omega)$ = the left side of (3.3).

LEMMA 3.4. *The map* $\langle\!\langle\ ,\ \rangle\!\rangle: H^{(d,0)}(M(n^2N)_w)\times H^{(d,0)}(M(n^2N)_w)\to \boldsymbol{C}$ *given by* $(\eta_1, \eta_2)\mapsto\sqrt{-1}^d\int_{M(n^2N)_w}\eta_1\wedge\bar{\eta}_2$, *is a positive definite Hermitian form.*

For the proof, see e.g. Griffiths and Harris [6, p. 124].

Write $\|\eta_1\|=\sqrt{\langle\!\langle\eta_1, \eta_1\rangle\!\rangle}$ for $\eta_1\in H^{(d,0)}(M(n^2N)_w)$ in this §3. By (3.3),

$$\|T(n)(\omega)\|=\Big\|\sum_{i=1}^{\mu}\omega_i\Big\|\leqq\sum_{i=1}^{\mu}\|\omega_i\|\ . \tag{3.5}$$

LEMMA 3.6. *Notations being as above,*

$$\|\omega_i\|=n^{gw/2}\|\omega\|\qquad \textit{for each}\quad i\in[1, \mu]\ .$$

PROOF. Recall Hatada [8, Lemma 2.1]. We compute as follows.

$$\begin{aligned}\|\omega_i\|^2&=\sqrt{-1}^d((\alpha_i, 0)^{-1}(\Gamma\ltimes \boldsymbol{Z}^{2gw})(\alpha_i, 0):\Gamma_g(n^2N)\ltimes(n\boldsymbol{Z})^{2gw})\\ &\qquad\times\int_{(\alpha_i,0)^{-1}(\Gamma\ltimes \boldsymbol{Z}^{2gw})(\alpha_i,0)\backslash\mathfrak{H}_g\times \boldsymbol{C}^{gw}}\omega_i\wedge\bar{\omega}_i\\ &=\sqrt{-1}^d((\alpha_i, 0)^{-1}(\Gamma\ltimes \boldsymbol{Z}^{2gw})(\alpha_i, 0):\Gamma_g(n^2N)\ltimes(n\boldsymbol{Z})^{2gw})\int_{M(N)_w}\omega\wedge\bar{\omega}\\ &=\frac{((\alpha_i, 0)^{-1}(\Gamma\ltimes \boldsymbol{Z}^{2gw})(\alpha_i, 0):\Gamma_g(n^2N)\ltimes(n\boldsymbol{Z})^{2gw})}{(\Gamma\ltimes \boldsymbol{Z}^{2gw}:\Gamma_g(n^2N)\ltimes(n\boldsymbol{Z})^{2gw})}\|\omega\|^2\\ &=\frac{((\alpha_i, 0)^{-1}(\Gamma\ltimes \boldsymbol{Z}^{2gw})(\alpha_i, 0):\Gamma_g(nN)\ltimes \boldsymbol{Z}^{2gw})}{(\Gamma\ltimes \boldsymbol{Z}^{2gw}:\Gamma_g(nN)\ltimes \boldsymbol{Z}^{2gw})}\|\omega\|^2\\ &=\frac{(\Gamma:\alpha_i\Gamma_g(nN)\alpha_i^{-1})(\det\alpha_i')^w}{(\Gamma:\Gamma_g(nN))}\|\omega\|^2\\ &=n^{gw}\|\omega\|^2\end{aligned}$$

where $\alpha_i'=r(\alpha_i)\alpha_i^{-1}$. Lemma 3.6 is proved.

By (3.5) and Lemma 3.6,

$$|\lambda(n)|\ \|\omega\|=\|\lambda(n)\omega\|\leqq\sum_{i=1}^{\mu}\|\omega_i\|=\mu n^{gw/2}\|\omega\|\ .$$

Hence $|\lambda(n)|\leqq\mu n^{gw/2}$.

Now furthermore assume that $n\geqq 2$ and $|\lambda(n)|=\mu n^{gw/2}$ in this inequality. Then by Lemmas 3.4 and 3.6 and (3.3),

$$\begin{aligned}&\omega_1=\omega_i &&\text{for all } i\in[1, \mu]\ ;\ \text{and}\\ &\lambda(n)\omega=\mu\omega_1 &&\text{as a differential form on } M(n^2N)_w\ .\end{aligned}$$

Here we may assume that $\alpha_1=\sigma\begin{pmatrix}n1_g & 0\\ 0 & 1_g\end{pmatrix}$, $\sigma\in\mathrm{Sp}(g, \boldsymbol{Z})$ with $\sigma\equiv\begin{pmatrix}n^{-1}1_g & 0\\ 0 & n1_g\end{pmatrix}$

(mod N) and $\omega_1=\pi_1^*\circ(\alpha_1, 0)^{\sim *}\circ[\pi]^*(\omega)$ using the notations in Hatada [8, (2.2.1)]. Recall Hatada [7, Lemma 2]. We obtain:

$$\lambda(n)F_0(Z)=\mu n^{g(g+1+w)-g(g+1)/2}(\chi(\sigma \pmod{\Gamma}))^{-1}F_0(nZ)\ .$$

Put $c_n=\lambda(n)^{-1}\mu n^{gw+g(g+1)/2}(\chi(\sigma \pmod{\Gamma}))^{-1}\in \boldsymbol{C}$. Then we have:

$$F_0(Z)=c_nF_0(nZ)=c_n^kF_0(n^kZ) \qquad \text{for any integer } k\geqq 1\ .$$

This contradicts the uniqueness of the Fourier expansion of $F_0(Z)$ at

$$Z=\begin{pmatrix} \sqrt{-1}\infty & & & \\ & \sqrt{-1}\infty & & \\ & & \ddots & \\ & & & \sqrt{-1}\infty \end{pmatrix}$$

in terms of $\{\exp(2[pi]\sqrt{-1}\,\mathrm{Tr}((T/N)Z))\}_T$ where T runs through $g\times g$ semi-integral symmetric matrices. Hence we obtain $|\lambda(n)|\lneqq\mu n^{gw/2}$. Theorem 3 is proved.

We raise:

PROBLEM 3.7. (i) Give a better estimate for $|\lambda(l)|$ where $\lambda(l)$ is any eigenvalue of $T_{g+1+w}(l)$ on $S_{g+1+w}(\Gamma)$ in Theorem 3.

(ii) Is it true or false that for all the prime numbers l with $l\nmid N$, every eigenvalue $\lambda(l)$ of $T_{g+1+w}(l)$ on $S_{g+1+w}(\Gamma)$ satisfies

$$|\lambda(l)|\leqq 2^g l^{d/2}$$

where $d=g(g+1)/2+gw$? (The case of $g=1$ in this (ii), which had been called Ramanujan Conjecture, was positively answered by P. Deligne before.)

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