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Three-Variable Conway Potential Function of Links

Dedicated to Professor Hiroshi Toda on his sixtieth birthday

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J. H. Conway introduced the potential function of a link in [3], and its invariance was verified by R. Hartley in [6]. Therefore we are not interested in its detailed definition in this note. We will try to give a recursive calculation of the potential functions of multi-variables. In fact we will succeed in giving it for the case of three variables as in Main Theorem. In other words, we will get a machine which makes any link into several links by a finite sequence of replacements appearing in Conway's three Identities for the case of three variables.

When we look back upon the past, we become aware that the existence of machines, which make any link into trivial knots by a finite sequence of replacements appearing in Conway's First Identity, has recently produced new polynomial invariants of links: the Jones polynomial and the skein polynomial [2, 4, 7, 8, 12, 14]. If we will get a machine for the case of multi-variables, it is possible to get a new (component-wise) link invariant.

For an ordered and oriented link in S^3, $L=K_1\cup\cdots\cup K_\mu$, we suppose that every component K_i is labeled by $t_{j(i)}$. The potential function $\nabla_L=\nabla_L(t_1,\cdots)$ has the following characterization [3], [6].

(I) (First Identity) For three links L_+, L_- and L_0 which differ only in one place as shown in Fig. 1, the potential function satisfies

$$\nabla_{L_+}=\nabla_{L_-}+(t_i-t_i^{-1})\nabla_{L_0}\ .$$

(II) (Second Identity) For three links L_{++}, L_{--} and L_{00} which differ only in one place as shown in Fig. 2 (a) or alternatively (b), the potential function satisfies

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$$\nabla_{L_{++}}+\nabla_{L_{--}}=(t_it_j+t_i^{-1}t_j^{-1})\nabla_{L_{00}}$$

in the first case, and

$$\nabla_{L_{++}}+\nabla_{L_{--}}=(t_it_j^{-1}+t_i^{-1}t_j)\nabla_{L_{00}}$$

in the second case.

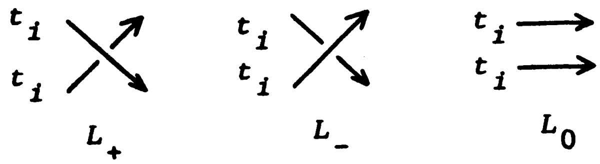

FIGURE 1

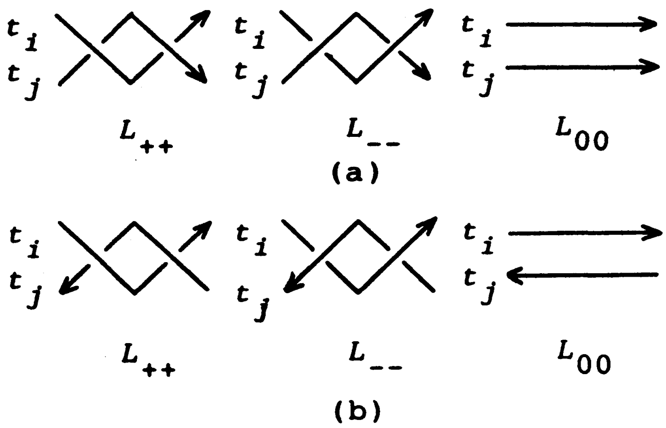

FIGURE 2

(III) (Third Identity) For four links L_1, L_2, L_3 and L_4 which differ only in one place as shown in Fig. 3, the potential function satisfies

$$\nabla_{L_1}+\nabla_{L_2}=\nabla_{L_3}+\nabla_{L_4}\ .$$

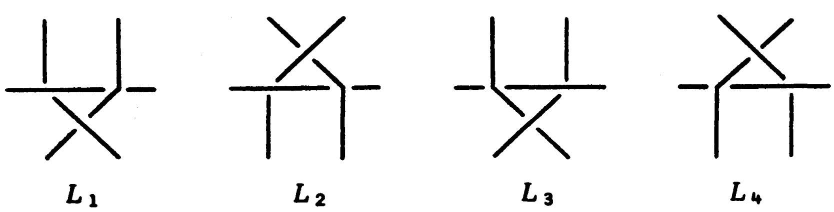

FIGURE 3

(IV) For a trivial knot K with a label t,

$$\nabla_K=1/(t-t^{-1})\ .$$

(V) For a simple positive clasp (2-component Hopf-link) L with labels

t_1 and t_2 as shown in Fig. 4 (a),

$$\nabla_L = 1 .$$

(The conditions (II), (V) and (VI) imply that $\nabla_L = -1$ for a negative clasp as in Fig. 4 (b)).

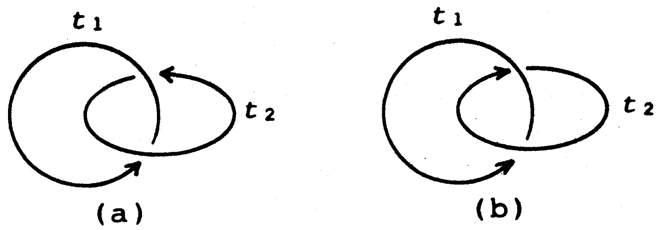

FIGURE 4

(VI) For a split link L,

$$\nabla_L = 0 .$$

(VII) For a connected sum of simple positive clasps, L, with labels t_1, t_2 and t_3 as shown in Fig. 5,

$$\nabla_L = t_2 - t_2^{-1} .$$

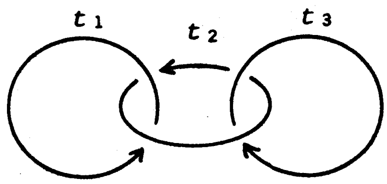

FIGURE 5

(VIII) For a 3-component Hopf-link L with labels t_1, t_2 and t_3 as shown in Fig. 6,

$$\nabla_L = -t_1 t_2 t_3 + t_1^{-1} t_2^{-1} t_3^{-1} .$$

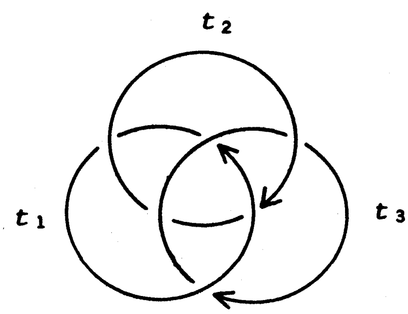

FIGURE 6

From now, we try to get a machine which changes a given link into one of several links appearing in (IV), (V), (VI), (VII) and (VIII) by a finite sequence of three replacements appearing in (I), (II) and (III).

(1) When the number of labels is just one, L. H. Kauffman [10, 11], J. Hoste [8] and W. B. R. Lickorish and K. C. Millet [12] showed the existence of machines which change a given link into a trivial knot (or a split link with the same label) by a finite sequence of replacements appearing in (I). (However, some of them had another purpose to get the Jones polynomial or the skein polynomial.) We remarked that a several component link with the same label can be deformed into knots by the replacements in (I) from L_0 to L_+ and L_- as shown in Fig. 7.

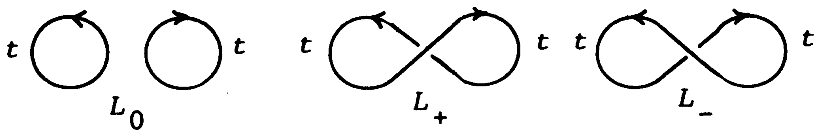

FIGURE 7

(2) When the number of labels is just two, we can deform a given link into 2-component links one of whose components is trivial through the above machine. We regard the trivial component as an axis and the other component as a string winding around the axis as shown in Fig. 8, where we watch the link in the direction of the axis.

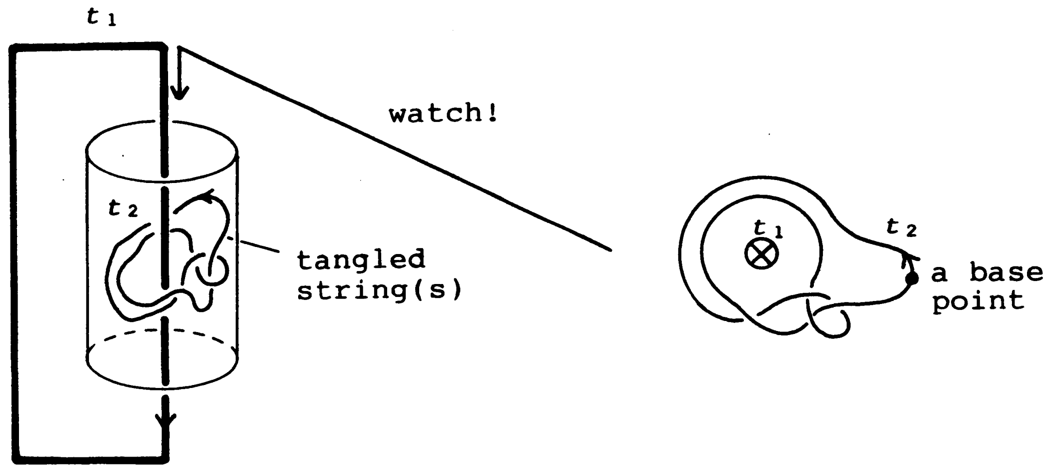

FIGURE 8

1st Step. On each string, we choose the farthest point from the axis to be a base point. We start from the farthest base point and go along the string by its orientation. At every time we meet the crossing of the strings, if necessary, we operate the replacement in (I) from $L_\pm$

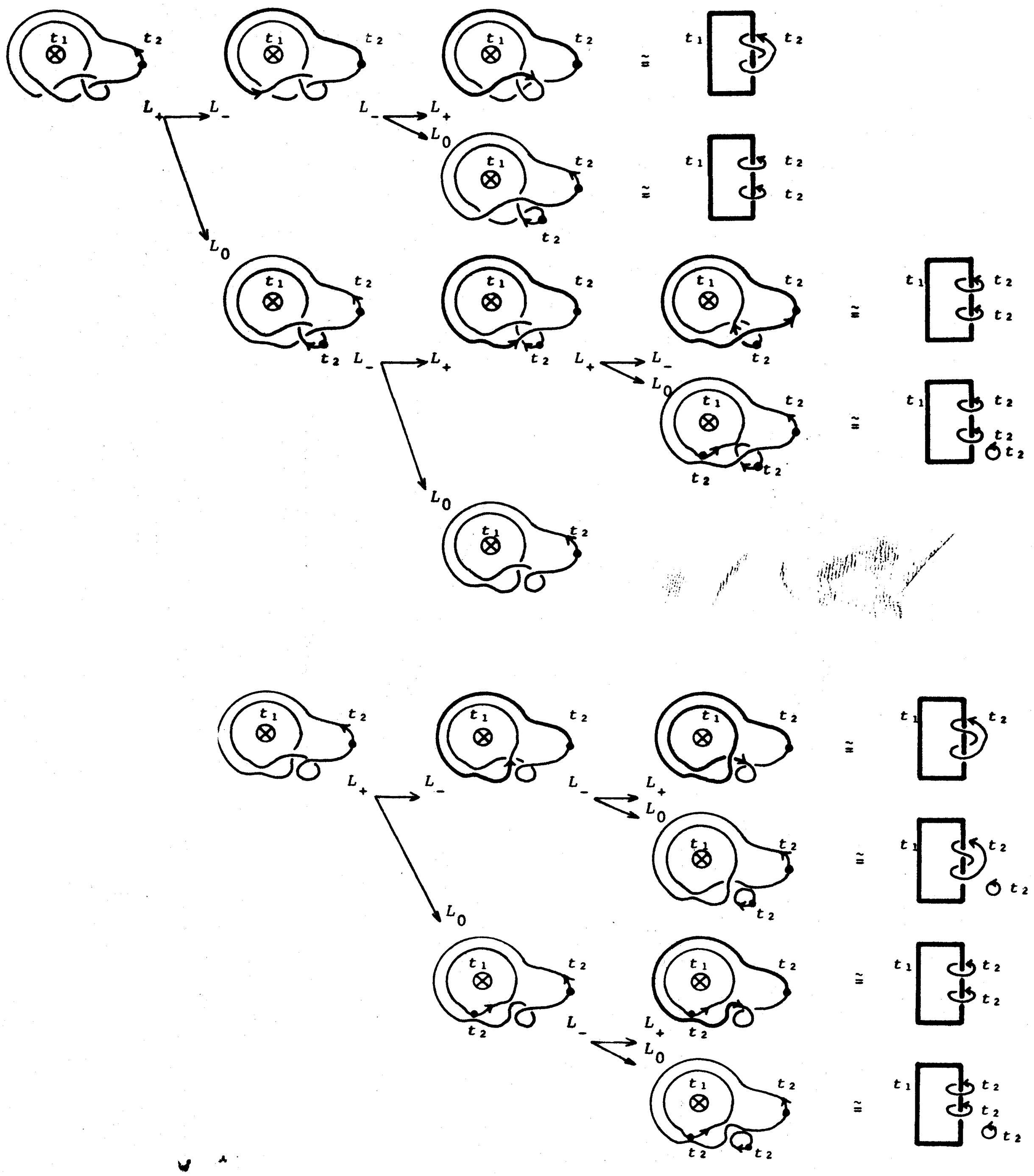

FIGURE 9

to $L_{\mp}$ to change the strings monotonely descending. On the other hand, the replacement from $L_{\pm}$ to L_0 decreases the number of crossings of the strings. On this branch, we return to the beginning of the 1st Step. When we return to the base point, we start again from the farthest base point among the remaining. We perform such operations successively, and we obtain (component-wise splitting) strings whose each component has one maximal point and one minimal point in the direction of the axis at every end of branches of replacements as in Fig. 9.

2nd Step. The link composed of the axis and the strings is a connected sum of torus links of type (2, $2k$). For each factor (2, $2k$)-torus knot, we operate the replacement in (II) from $L_{\pm\pm}$ to $L_{\mp\mp}$ and L_{00} and obtain a (2, $2k\mp2$)-torus link and a (2, $2k\mp4$)-torus link (see Fig. 10).

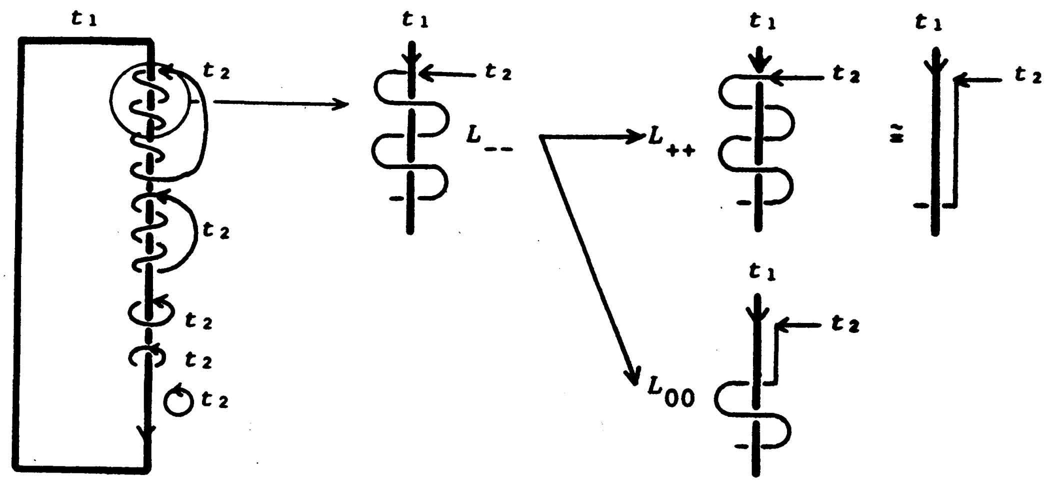

FIGURE 10

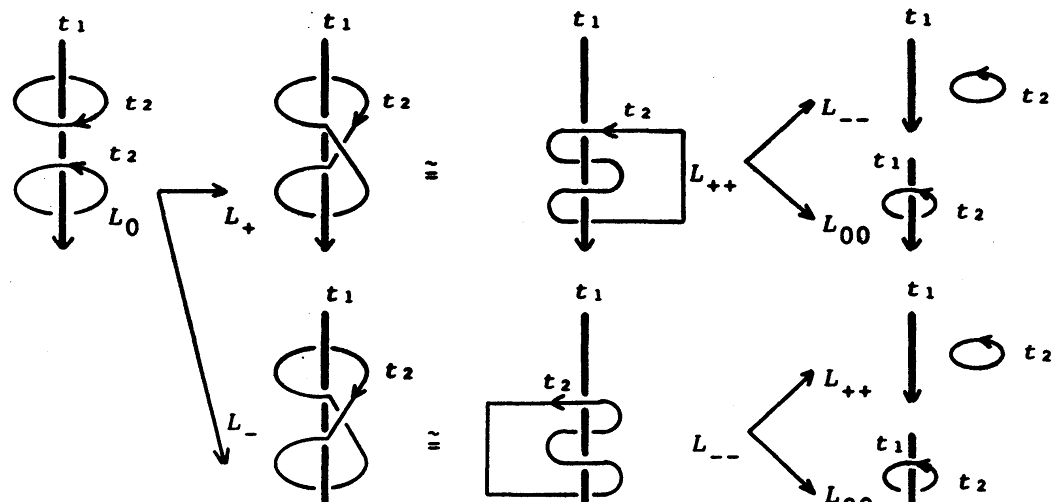

FIGURE 11

By induction on $|k|$, we obtain a connected sum of simple positive clasps and a 2-component trivial links.

3rd Step. For split components, we operate the replacements in (I) from L_0 to L_+ and L_- as in Fig. 7. For two simple positive clasps, we operate the replacements in (I) and (II) as in Fig. 11. By induction on the number of components, we obtain a simple positive clasp or a 2-component trivial link at every end of branches of replacements. Hence, we get a machine which changes a given link with two labels to links appearing in (V) and (VI) by a finite sequence of replacements appearing in (I) and (II).

(3) When the number of labels is just three, we can deform a given link into 3-component links two of whose components are a simple positive clasp or a 2-component trivial link through the above machine. We regard the two components as parallel axes (which may or may not be linking in the deep bottom) and the other component as a string winding around the axes similarly as in the above (2). (See Fig. 12, where we watch the link in the direction of the axes.)

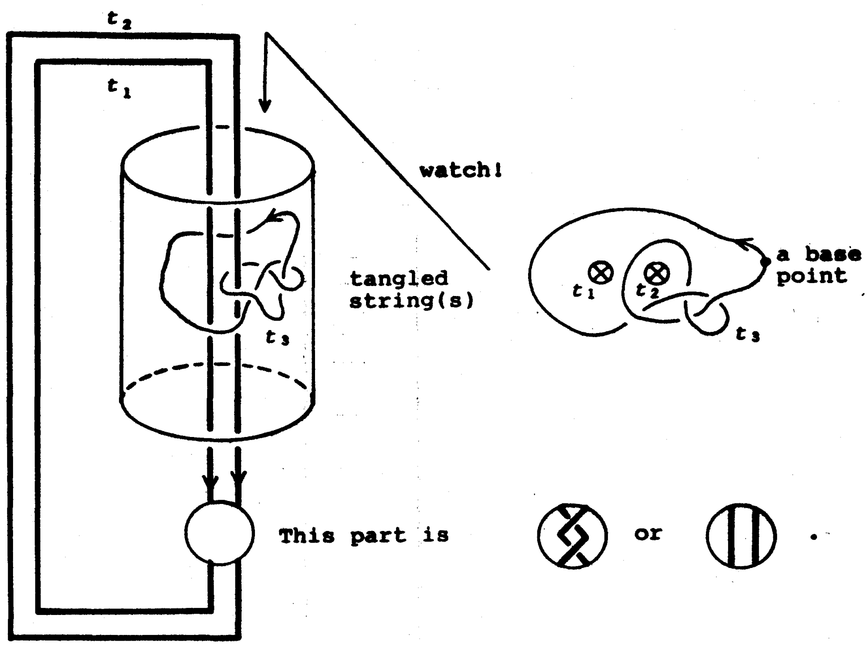

FIGURE 12

1st Step. By the same method as in the above (2); 1st Step, we obtain (component-wise splitting) strings whose each component has one maximal point and one minimal point in the direction of the axes at every end of branches of replacements.

2nd Step. Now, the link composed of the axes and the strings is a sum of 3-braids of special type, each one of which can be presented by $\sigma_1^{\delta_1}\sigma_2^{\delta_2}\sigma_1^{\delta_3}\sigma_2^{\delta_4}\cdots\sigma_1^{\delta_{2p-1}}\sigma_2^{\delta_{2p}}$ using generators of the 3-braid-group as shown in Fig. 13, where all integers δ_i's are even.

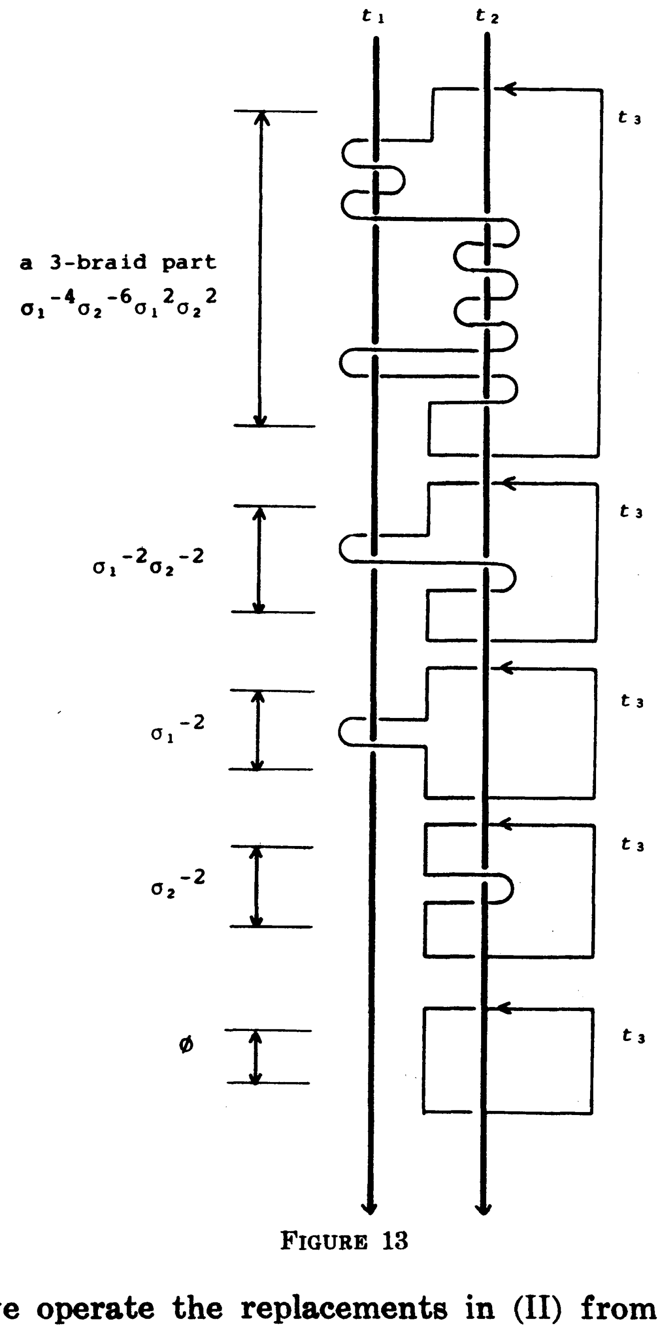

FIGURE 13

When $|\delta_i|>2$, we operate the replacements in (II) from $L_{\pm\pm}$ to $L_{\mp\mp}$ and L_{00} similarly as in Fig. 10 and obtain $|\delta_i|\leqq 2$ at every end of branches of replacements.

3rd Step. If $\delta_i=0$, we can ignore δ_i, i.e.

$$\cdots \sigma_j^{\delta_{i-1}}\sigma_k^{0}\sigma_j^{\delta_{i+1}} \cdots = \cdots \sigma_j^{\delta_{i-1}+\delta_{i+1}} \cdots$$

and return to the beginning of the 2nd Step. Therefore, we can assume that each string is presented by

$$\sigma_1^{2\varepsilon_1}\sigma_2^{2\varepsilon_2}\sigma_1^{2\varepsilon_3}\sigma_2^{2\varepsilon_4}\cdots\sigma_k^{2\varepsilon_q} \quad \text{or} \quad \sigma_2^{2\varepsilon_1}\sigma_1^{2\varepsilon_2}\sigma_2^{2\varepsilon_3}\sigma_1^{2\varepsilon_4}\cdots\sigma_k^{2\varepsilon_q}$$
$$(k=1 \text{ or } 2;\ q\leqq 2p;\ \varepsilon_i=\pm 1)\,.$$

4th Step. When $q\geqq 4$, if necessary, we operate the replacements in (II) from $L_{\pm\pm}$ to $L_{\mp\mp}$ and L_{00} and obtain a sequence:

$$\sigma_1^{-2}\sigma_2^{-2}\sigma_1^{2}\sigma_2^{2}\cdots\sigma_k^{2\varepsilon_r} \quad \text{or} \quad \sigma_2^{-2}\sigma_1^{-2}\sigma_2^{2}\sigma_1^{2}\cdots\sigma_k^{2\varepsilon_r}$$

($k=1$ or 2; $r\leqq q$; $\varepsilon_i=\pm 1$). (Of course, for each branch on which ε_i becomes 0, we return to the beginning of the 2nd Step.)

5th Step. For $\sigma_1^{-2}\sigma_2^{-2}\sigma_1^{2}\sigma_2^{2}\cdots\sigma_k^{2\varepsilon_r}$, we operate the replacements in (II) and (III) as shown in Fig. 14. Though the axes may link, we get a shorter sequence. We return to the beginning of the 2nd Step. Similarly for $\sigma_2^{-2}\sigma_1^{-2}\sigma_2^{2}\sigma_1^{2}\cdots\sigma_k^{2\varepsilon_r}$, we perform such operations.

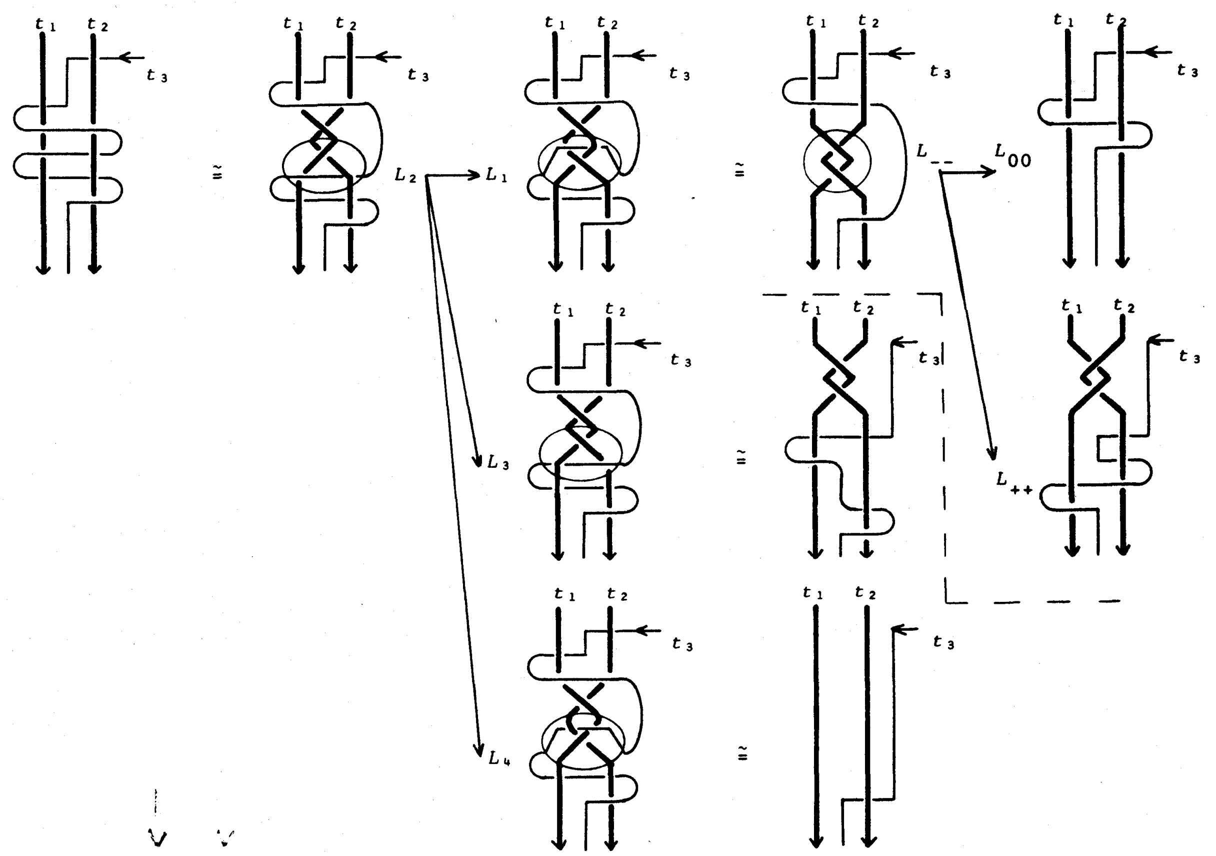

FIGURE 14

6th Step. By induction on r, we get a sequence

$$\varnothing\,,\quad \sigma_j^{2\varepsilon_1}\,,\quad \sigma_j^{2\varepsilon_1}\sigma_k^{2\varepsilon_2}\,,\quad \text{or}\quad \sigma_j^{2\varepsilon_1}\sigma_k^{2\varepsilon_2}\sigma_j^{2\varepsilon_3}\qquad (\{j,\,k\}=\{1,\,2\};\ \varepsilon_i=\pm 1)$$

at every end of replacements. If necessary, we operate the replacements in (II) from $L_{\pm\pm}$ to $L_{\mp\mp}$ and L_{00} and we obtain a sequence

$$\varnothing\,,\quad \sigma_1^{-2}\,,\quad \sigma_2^{-2}\,,\quad \sigma_1^{-2}\sigma_2^{-2}\,,\quad \sigma_2^{2}\sigma_1^{2}\,,\quad \sigma_1^{-2}\sigma_2^{-2}\sigma_1^{2}\,,\quad \text{or}\quad \sigma_2^{2}\sigma_1^{-2}\sigma_2^{-2}\,.$$

They are isotopic to a splitting trivial component, a simple positive (or negative) clasp, and a loop winding around the axes once positively (or negatively), respectively, as shown in Figs. 15-18. Of course, for negative one we operate the replacements in (II) from L_{--} to L_{++} and L_{00} and obtain a positive one or a splitting trivial component.

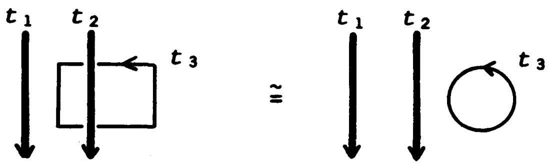

FIGURE 15

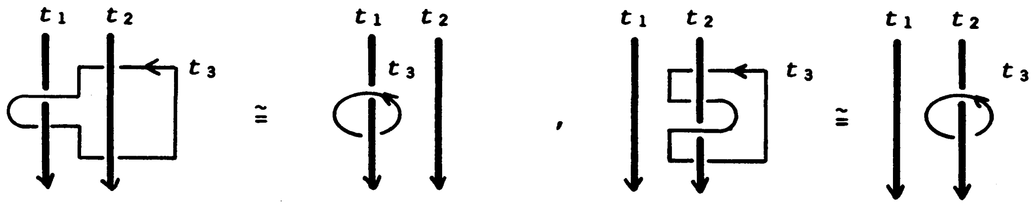

FIGURE 16

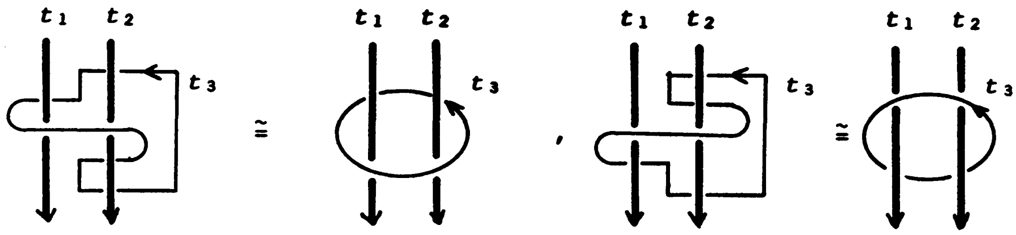

FIGURE 17

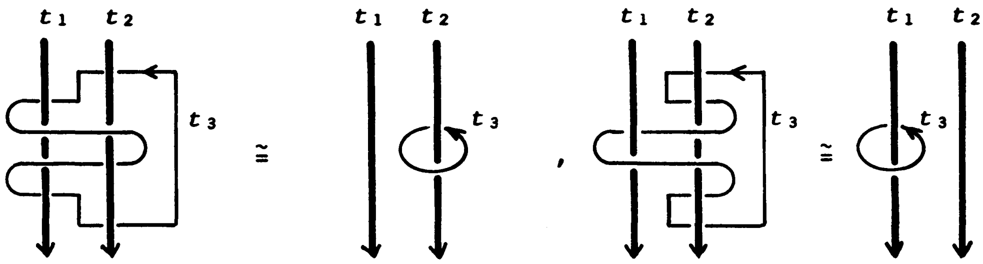

FIGURE 18

7th Step. Therefore, we get a link of the type as shown in Fig. 19 at every end of branches of replacements.

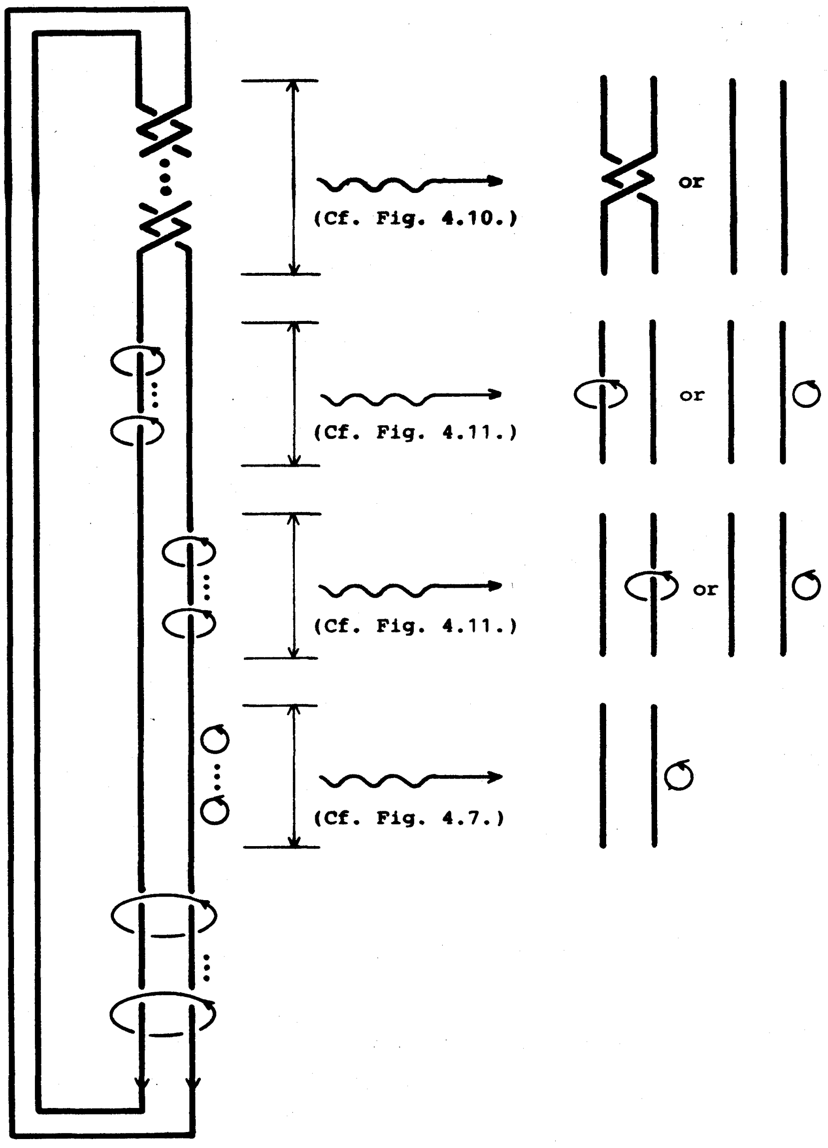

FIGURE 19

For linking of the axes, we operate the replacements in (II) from $L_{\pm\pm}$ to $L_{\mp\mp}$ and L_{00}, and obtain a linking 0 or $+1$ (cf. Fig. 10). For positive clasps, we perform the replacements like as the 3rd Step in (2) and we obtain only one clasp or only one splitting trivial component (cf. Fig. 11). For splitting trivial components, we operate the replacements in (I) from L_0 to L_+ and L_-, and obtain one splitting trivial component (cf. Fig. 7). For loops winding around the axes positively, we perform the replacements like as in the following 8th Step. For a splitting trivial component, a simple positive clasp, and/or a loop winding around the axes positively, we perform the replacements like as in the following 9th Step. By induction on the number of components, we obtain only one string which is a splitting trivial component, a simple positive clasp, or a loop winding around the axes positively at every end of branches of replacements. Then, we get a 3-component trivial link, a split-sum of trivial knot and a simple positive clasp, a connected sum of simple positive clasps, or a 3-component Hopf-link which is appearing in (VI), (VII), or (VIII).

8th Step. For two loops winding around the axes positively, we operate the replacements in (I) from L_0 to L_+ and L_- as in Fig. 20.

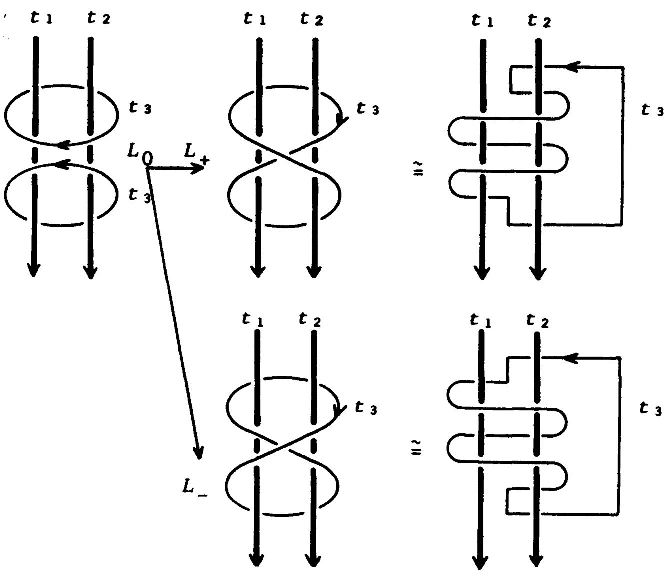

FIGURE 20

Therefore, we get a 3-braid presented by $\sigma_2{}^2\sigma_1{}^2\sigma_2{}^2\sigma_1{}^2$ or $\sigma_1{}^{-2}\sigma_2{}^{-2}\sigma_1{}^{-2}\sigma_2{}^{-2}$ without consideration on the orientation of the string. We perform the replacements like as in the 2nd Step - 6th Step and we get a splitting trivial component, a simple positive clasp, or a loop winding around the axes positively at every end of branches of replacements. By induction on the number of loops winding around the axes positively, we obtain at most one loop winding around the axes positively, several splitting components and several simple positive clasps at every end of branches of replacements. We return to the 7th Step.

9th Step. For two simple positive clasps of distinct types, we operate the replacements in (I) and (II) as shown in Fig. 21 and decrease the number of strings.

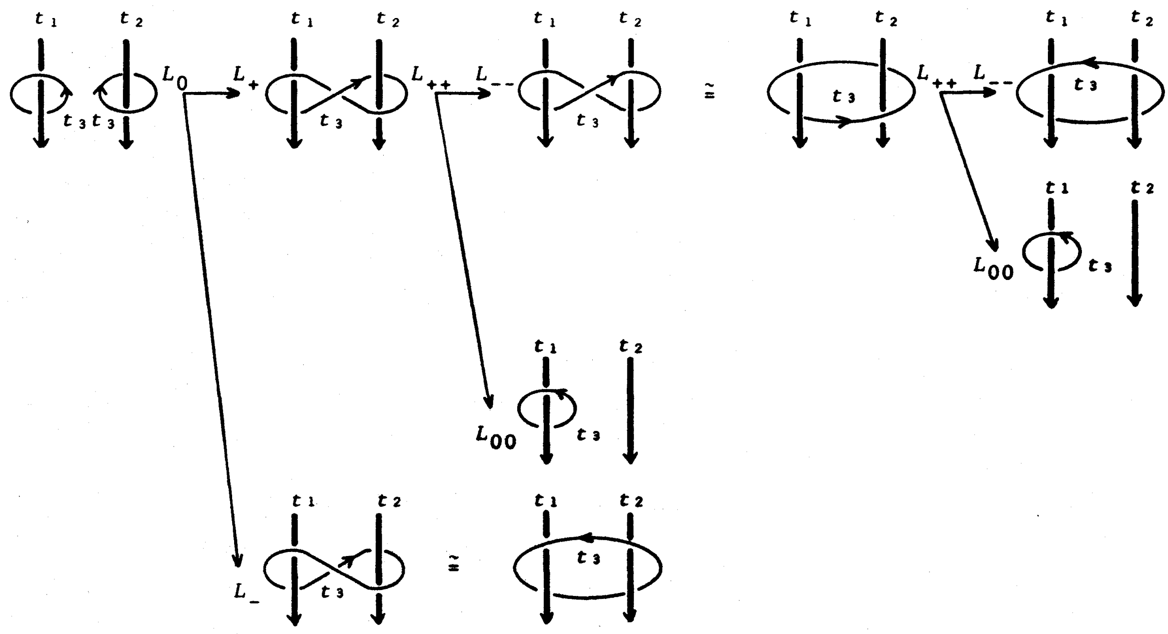

FIGURE 21

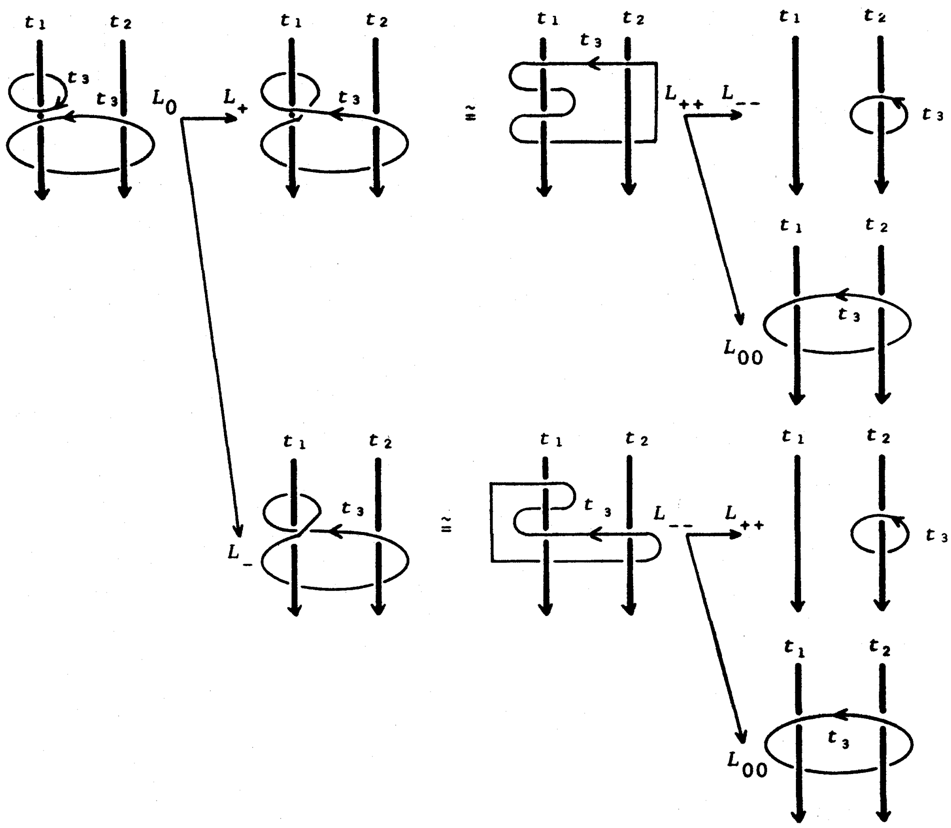

FIGURE 22

For one simple positive clasp and one loop winding around the axes positively, we operate the replacements in (I) and (II) as shown in Fig. 22 and decrease the number of strings. We return to the 7th Step.

From the above, we get a machine which changes a given link with three labels into links appearing in (VI), (VII) or (VIII) by a finite sequence of replacements appearing in (I), (II) and (III). We say it in other words as follows.

THEOREM. *There exists a recursive calculation for the potential function of three variables.*

Our recursive calculation for the potential function of two variables requires two kinds of replacements, and that of three variables requires three kinds of replacements. Then, that of μ variables requires μ kinds of replacements, is it true? The author thinks that three kind of replacements are enough to calculate the potential function recursively. But he does not know the proof yet.

REMARK. This is a part of the previous note [13]. Since it had gaps in proof at the other part and the results are independent, the author divides it into pieces.

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