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Interpolation between Some Banach Spaces in Generalized Harmonic Analysis

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Introduction.

In [14, 15], N. Wiener established the generalized harmonic analysis for the analysis of almost periodic functions and sample paths of the Brownian motions. The classes of functions he treated are

$$W^2(\boldsymbol{R}^1)=\left\{f\in L^2_{\mathrm{loc}}(\boldsymbol{R}^1)\ :\ \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|f(x)|^2dx\ \text{exists}\right\} \tag{0.1}$$

and its subclasses. The $\boldsymbol{R}^2$ case of the generalized harmonic analysis was investigated by K. Anzai, S. Koizumi and K. Matsuoka [1] and K. Matsuoka [10, 11], and also the $\boldsymbol{R}^n$ case by T. Kawata [7].

Unfortunately, the class $W^2(\boldsymbol{R}^1)$ is not closed under addition. Hence, the following two more conventional Banach spaces were considered:

$$M^p(\boldsymbol{R}^1)=\left\{f\in L^p_{\mathrm{loc}}(\boldsymbol{R}^1)\ :\ \|f\|_{M^p(\boldsymbol{R}^1)}=\overline{\lim_{T\to\infty}}\left(\frac{1}{2T}\int_{-T}^{T}|f(x)|^pdx\right)^{1/p}<\infty\right\}, \tag{0.2}$$

which is called the Marcinkiewicz space, and

$$B^p(\boldsymbol{R}^1)=\left\{f\in L^p_{\mathrm{loc}}(\boldsymbol{R}^1)\ :\ \|f\|_{B^p(\boldsymbol{R}^1)}=\sup_{T\geq 1}\left(\frac{1}{2T}\int_{-T}^{T}|f(x)|^pdx\right)^{1/p}<\infty\right\}, \tag{0.3}$$

where $1<p<\infty$. Recently, K. Lau [8, 9] investigated the multiplier theory on $M^p(\boldsymbol{R}^1)$. Also, Y. Chen and K. Lau [5] developed the harmonic analysis on $B^p(\boldsymbol{R}^1)$ and the related spaces (e.g., the Hardy-Littlewood maximal function, the Hardy spaces, John-Nirenberg's *BMO*, the Carleson measure, the atomic decomposition, and Fefferman-Stein's duality).

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It is the purpose of this paper to study the complex interpolation between $B^p(\boldsymbol{R}^n)$ spaces, i.e. the $\boldsymbol{R}^n$ case of $B^p(\boldsymbol{R}^1)$ spaces, and also the related spaces, which corresponds to that between $L^p(\boldsymbol{R}^n)$ spaces.

§ 1. Preliminaries.

Let $1<p<\infty$, and let

$$(1.1)\qquad \begin{aligned} B^p &= B^p(\boldsymbol{R}^n) \\ &= \left\{ f \in L^p_{\mathrm{loc}}(\boldsymbol{R}^n) \ :\ \|f\|_{B^p} = \sup_{r \geq 1} \left(\frac{1}{|S_r|} \int_{S_r} |f(x)|^p dx \right)^{1/p} < \infty \right\}, \end{aligned}$$

where S_r is the open ball in $\boldsymbol{R}^n$, having center 0 and radius $r>0$, and

$$(1.2)\qquad B_0^p = B_0^p(\boldsymbol{R}^n) = \left\{ f \in B^p \ :\ \lim_{r\to\infty} \frac{1}{|S_r|} \int_{S_r} |f(x)|^p dx = 0 \right\}.$$

Also let

$$(1.3)\qquad \begin{aligned} A^p &= A^p(\boldsymbol{R}^n) \\ &= \left\{ f \ :\ \|f\|_{A^p} = \inf_{\omega \in \Omega} \|f\|_{L^p_{\omega^{-(p-1)}}} = \inf_{\omega \in \Omega} \left(\int_{\boldsymbol{R}^n} |f(x)|^p \omega(x)^{-(p-1)} dx \right)^{1/p} < \infty \right\}, \end{aligned}$$

where Ω is the class of functions ω on $\boldsymbol{R}^n$ such that ω is positive, radial, nonincreasing with respect to $|x|$, and

$$\omega(0) + \int_{\boldsymbol{R}^n} \omega(x) dx = 1 .$$

We call A^p the Beurling algebra. Then, it follows easily that B_0^p and A^p, $1<p<\infty$, are separable Banach spaces, and both spaces contain $C_c^\infty(\boldsymbol{R}^n)$, i.e. the class of infinitely differentiable functions with compact support, as dense subspace (see e.g., Y. Chen and K. Lau [4] and J. Garcia-Cuerva [6]).

Now, we will list two results on B^p, B_0^p and A^p which are relevant to our discussions.

PROPOSITION 1.1 (A. Beurling [3]). *Let $1<p$, $p'<\infty$ with $1/p+1/p'=1$. Then A^p is a Banach algebra contained in $L^1 \cap L^p(\boldsymbol{R}^n)$. The dual of A^p is $B^{p'}$, and the duality is given by*

$$(1.4)\qquad \langle f, g\rangle = \int_{\boldsymbol{R}^n} f(x)\overline{g(x)} dx \qquad (f \in B^{p'},\ g \in A^p) .$$

PROPOSITION 1.2. *For $1<p$, $p'<\infty$, $1/p+1/p'=1$, $(B_0^{p'})^*$ is isometrically isomorphic to A^p.*

PROOF. The $\boldsymbol{R}^1$ case was given by Y. Chen and K. Lau [4, Theorem

2.3]. By applying the same argument that was used in [4], the $\boldsymbol{R}^n$ case immediately follows (cf. J. Garcia-Cuerva [6]). □

§2. The complex interpolation method.

We will recall here the two definitions of complex interpolation spaces (see J. Bergh and J. Löfström [2], C. Sadosky [12] and H. Triebel [13] for details).

Let A_0 and A_1 be two Banach spaces. Then we shall say that A_0 and A_1 are compatible if there is a Hausdorff topological vector space V such that $A_0 \hookrightarrow V$ and $A_1 \hookrightarrow V$. Here, the symbol "$\hookrightarrow$" means that the left hand side is continuously embedded in the right hand side. Let us consider in the complex plane the closed strip

$$S=\{z \in \boldsymbol{C}: 0 \leqq \operatorname{Re} z \leqq 1\}$$

and the open strip

$$S_0=\{z \in \boldsymbol{C}: 0<\operatorname{Re} z<1\} .$$

For a couple of compatible Banach spaces A_0 and A_1, let $\mathscr{F}=\mathscr{F}(A_0, A_1)$ be the space of all functions f from S to A_0+A_1 which satisfy the following properties:

(i) f is continuous and bounded on S;
(ii) f is analytic on S_0;
(iii) the function $t \rightarrow f(j+it)$ is a continuous and bounded function from $\boldsymbol{R}$ into A_j $(j=0, 1)$.

We provide $\mathscr{F}$ with the norm

$$\|f\|_{\mathscr{F}}=\max\{\sup_{-\infty<t<\infty}\|f(it)\|_{A_0}, \sup_{-\infty<t<\infty}\|f(1+it)\|_{A_1}\} . \tag{2.1}$$

Then, for $0<\theta<1$, the complex interpolation space $(A_0, A_1)_{[\theta]}$ is defined by

$$(A_0, A_1)_{[\theta]}=\{f(\theta) \in A_0+A_1 : f \in \mathscr{F}\} \tag{2.2}$$

and the norm is given by

$$\|a\|_{(A_0,A_1)[\theta]}=\inf\{\|f\|_{\mathscr{F}} : f(\theta)=a\} . \tag{2.3}$$

We now proceed to another definition of complex interpolation spaces. For a couple of compatible Banach spaces B_0 and B_1, let $\mathscr{G}=\mathscr{G}(B_0, B_1)$ be the space of all functions g from S to B_0+B_1 which satisfy the following properties:

(i) g is continuous on S and

$$\|g(z)\|_{B_0+B_1} \leqq c(1+|z|) \qquad (z \in S) ;$$

(ii) g is analytic on S_0;

(iii) $\Delta_{t_1,t_2} g(j+it) = g(j+it_1) - g(j+it_2)$ has values in B_j for all real values of t_1, t_2 $(j=0, 1)$ and

$$\|g\|_{\mathscr{G}} = \max\left\{ \sup_{-\infty<t_1,t_2<\infty} \left\| \frac{\Delta_{t_1,t_2} f(it)}{t_1-t_2} \right\|_{B_0}, \sup_{-\infty<t_1,t_2<\infty} \left\| \frac{\Delta_{t_1,t_2} f(1+it)}{t_1-t_2} \right\|_{B_1} \right\} \tag{2.4}$$

is finite.

Then, for $0<\theta<1$, the second complex interpolation space $(B_0, B_1)^{[\theta]}$ is defined by

$$(B_0, B_1)^{[\theta]} = \{g'(\theta) \in B_0+B_1 : g \in \mathscr{G}\} \tag{2.5}$$

and the norm is given by

$$\|b\|_{(B_0,B_1)^{[\theta]}} = \inf\{\|g\|_{\mathscr{G}} : g'(\theta)=b\} . \tag{2.6}$$

Concerning the relation between the two complex interpolation spaces, there are the following two well-known theorems.

THEOREM A (The complex equivalence theorem). *For any couple of compatible Banach spaces A_0 and A_1, we have*

$$(A_0, A_1)_{[\theta]} \hookrightarrow (A_0, A_1)^{[\theta]} \qquad (\|\cdot\|_{(A_0,A_1)_{[\theta]}} \geqq \|\cdot\|_{(A_0,A_1)^{[\theta]}}) . \tag{2.7}$$

THEOREM B (The duality theorem). *Assume that (A_0, A_1) is a couple of compatible Banach spaces, and that $A_0 \cap A_1$ is dense in both A_0 and A_1. Then*

$$(A_0, A_1)_{[\theta]}^* = (A_0^*, A_1^*)^{[\theta]} \qquad (\textit{equal norms}) . \tag{2.8}$$

We finish this section with a proposition which plays an important role in Section 3.

PROPOSITION 2.1. *Let $0<\theta<1$ and (A_0, A_1) be a couple of compatible Banach spaces such that $A_0 \cap A_1$ is dense in both A_0 and A_1. Assume that*

$$(A_0, A_1)_{[\theta]} = A_\theta \qquad (\textit{equal norms}) \tag{2.9}$$

and

$$(A_0^*, A_1^*)_{[\theta]} \hookleftarrow A_\theta^* \qquad (\|\cdot\|_{(A_0^*,A_1^*)_{[\theta]}} \leqq \|\cdot\|_{A_\theta^*}) . \tag{2.10}$$

Then

(2.11) $$(A_0^*, A_1^*)_{[\theta]}=(A_0^*, A_1^*)^{[\theta]}=A_\theta^* \qquad (\textit{equal norms}) .$$

PROOF. Using Theorems A and B, we obtain, by (2.9),

(2.12) $$(A_0^*, A_1^*)_{[\theta]}\hookrightarrow(A_0^*, A_1^*)^{[\theta]}=(A_0, A_1)_{[\theta]}^*=A_\theta^* .$$

Thus, combining this with (2.10), we easily have (2.11). □

§ 3. Interpolation theorems.

In this section, we shall characterize the complex interpolation spaces $(A^{p_0}, A^{p_1})_{[\theta]}$, $(A^{p_0}, A^{p_1})^{[\theta]}$ and $(B^{p_0}, B^{p_1})_{[\theta]}$, $(B^{p_0}, B^{p_1})^{[\theta]}$.

THEOREM 3.1. *Suppose* $1<p_0, p_1<\infty$ and $0<\theta<1$. *Then*

(3.1) $$(A^{p_0}, A^{p_1})_{[\theta]}=(A^{p_0}, A^{p_1})^{[\theta]}=A^p \qquad (\textit{equal norms}) ,$$

where $1/p=(1-\theta)/p_0+\theta/p_1$.

THEOREM 3.2. *Suppose* $1<p_0, p_1<\infty$ *and* $0<\theta<1$. *Then*

(3.2) $$(B^{p_0}, B^{p_1})_{[\theta]}=(B^{p_0}, B^{p_1})^{[\theta]}=B^p \qquad (\textit{equal norms}) ,$$

where $1/p=(1-\theta)/p_0+\theta/p_1$.

Before proving the theorems, we show the following lemma.

LEMMA 3.3. *Suppose* $1<p_0, p_1<\infty$ *and* $0<\theta<1$. *Then*

(3.3) $$(B_0^{p_0}, B_0^{p_1})_{[\theta]}=B_0^p \qquad (\textit{equal norms}) ,$$

where $1/p=(1-\theta)/p_0+\theta/p_1$.

PROOF. We can assume, without loss of generality, that $1<p_0<p_1<\infty$. It is sufficient to prove that for all $a\in C_c^\infty(\boldsymbol{R}^n)$,

(3.4) $$\|a\|_{(B_0^{p_0},B_0^{p_1})_{[\theta]}}=\|a\|_{B^p} .$$

First, we assume that $\|a\|_{B^p}=1$. Now, let us put for any $\varepsilon>0$,

(3.5) $$f(z)=e^{\varepsilon(z^2-\theta^2)}|a(x)|^{p/p(z)}\frac{a(x)}{|a(x)|} \qquad (z\in S) ,$$

where $1/p(z)=(1-z)/p_0+z/p_1$. Then, clearly, f is a function from S to $B_0^{p_0}+B_0^{p_1}$, which is bounded and continuous on S, and analytic on S_0 and moreover, the function $t\to f(j+it)$ is a continuous and bounded function from $\boldsymbol{R}$ into $B_0^{p_j}$ $(j=0, 1)$, that is, $f\in\mathscr{F}(B_0^{p_0}, B_0^{p_1})$. Also, (3.6) $$\sup_{-\infty<t<\infty}\|f(j+it)\|_{B^{p_j}}\leqq e^{j\varepsilon}\|a\|_{B^p}^{p/p_j}=e^{j\varepsilon} \qquad (j=0,1)\,.$$

Hence, it follows from $f(\theta)=a$ that $a\in(B_0^{p_0}, B_0^{p_1})_{[\theta]}$, and

$$\|a\|_{(B_0^{p_0},\,B_0^{p_1})[\theta]}\leqq\|f\|_{\mathscr{F}}=e^{\varepsilon}\,.$$

Therefore, because $\varepsilon>0$ is arbitrarily close to 0, we get

$$\|a\|_{(B_0^{p_0},B_0^{p_1})[\theta]}\leqq\|a\|_{B^p}\,,$$

which implies half of (3.4).

Next, we assume that $\|a\|_{(B_0^{p_0},B_0^{p_1})[\theta]}=1$. In order to prove the remaining half of (3.4), we note that by Proposition 1.2,

(3.7) $$\|a\|_{B^p}=\sup\{|\langle a, b\rangle| : \|b\|_{A^{p'}}=1,\ b\in C_c^\infty(\boldsymbol{R}^n)\}\,.$$

Because of the definition $\|\cdot\|_{(B_0^{p_0},B_0^{p_1})[\theta]}$, for any $\varepsilon>0$, we can choose a function $f\in\mathscr{F}(B_0^{p_0}, B_0^{p_1})$ such that

(3.8) $$\|f(j+it)\|_{B^{p_j}}<1+\varepsilon \qquad (j=0,1)\,.$$

Now, letting $b\in C_c^\infty(\boldsymbol{R}^n)$ such that $\|b\|_{A^{p'}}=1$, for any $\varepsilon'>0$, there exists an $\omega_0\in\Omega$ such that

(3.9) $$\left(\int_{\boldsymbol{R}^n}|b(x)|^{p'}\omega_0(x)^{-(p'-1)}dx\right)^{1/p'}<1+\varepsilon'\,.$$

Let us put for any $\varepsilon''>0$,

$$g_{\omega_0}(z)=e^{\varepsilon''(z^2-\theta^2)}|b(x)|^{p'/p'(z)}\frac{b(x)}{|b(x)|}\omega_0(x)^{-(p'/p'(z)-1)} \qquad (z=s+it\in S)\,,$$

where $1/p'(z)=(1-z)/p_0'+z/p_1'$. Then, since $g_{\omega_0}(z)\in L^{p'(s)}_{\omega_0^{-(p'(s)-1)}}$ and $A^{p_0'}\subset A^{p_1'}$, it is readily seen that g_{ω_0} is a function from S to $A^{p_0'}+A^{p_1'}$, which is bounded and continuous on S, and analytic on S_0. Consequently, writing

(3.10) $$F(z)=\langle f(z), g_{\omega_0}(z)\rangle=\int_{\boldsymbol{R}^n}f(z)g_{\omega_0}(z)dx \qquad (z\in S)\,,$$

F is a function from S to $\boldsymbol{R}$, which is bounded and continuous on S, and analytic on S_0. Further,

(3.11) $$\begin{aligned}|F(j+it)|&\leqq\|f(j+it)\|_{B^{p_j}}\cdot e^{j\varepsilon''}\left(\int_{\boldsymbol{R}^n}|b(x)|^{p'}\omega_0(x)^{-(p'-1)}dx\right)^{1/p_j'}\\&<(1+\varepsilon)(1+\varepsilon')^{p'/p_j'}e^{j\varepsilon''} \qquad (j=0,1)\,.\end{aligned}$$

Therefore, by three line theorem, we obtain

$$|\langle a, b\rangle| = |F(\theta)| < (1+\varepsilon)(1+\varepsilon')^{p'/p_1'} e^{\varepsilon''} ,$$

which implies that

$$\|a\|_{B^p} \leqq \|a\|_{(B_0^{p_0},\, B_0^{p_1})_{[\theta]}} .$$

Thus, (3.4) holds. This completes the proof of Lemma 3.3. □

PROOF OF THEOREM 3.1. $B_0^{p_0} \cap B_0^{p_1}$ is dense in both $B_0^{p_0}$ and $B_0^{p_1}$. Hence, by Proposition 2.1 and Lemma 3.3, it is clearly sufficient to prove that

$$(A^{p_0}, A^{p_1})_{[\theta]} \hookleftarrow A^p . \tag{3.12}$$

In order to do this, we apply the same argument as in the second half of the proof of Lemma 3.3. Letting $a \in A^p$, for any $\varepsilon>0$, there exists an $\omega_0 \in \Omega$ such that

$$\left(\int_{R^n} |a(x)|^p \omega_0(x)^{-(p-1)} dx\right)^{1/p} < \|a\|_{A^p} + \varepsilon . \tag{3.13}$$

Therefore, putting for any $\varepsilon'>0$,

$$f_{\omega_0}(z) = e^{\varepsilon'(z^2-\theta^2)} |a(x)|^{p/p(z)} \frac{a(x)}{|a(x)|} \omega_0(x)^{-(p/p(z)-1)} \qquad (z \in S) , \tag{3.14}$$

where $1/p(z) = (1-z)/p_0 + z/p_1$, it follows from $f_{\omega_0} \in \mathscr{F}(A^{p_0}, A^{p_1})$ and $f_{\omega_0}(\theta) = a$ that $a \in (A^{p_0}, A^{p_1})_{[\theta]}$. Also,

$$\|a\|_{(A^{p_0}, A^{p_1})_{[\theta]}} \leqq \|a\|_{A^p} .$$

Thus, we have (3.12). This concludes the proof of Theorem 3.1. □

PROOF OF THEOREM 3.2. Let us put for any $\varepsilon>0$,

$$f(z) = e^{\varepsilon(z^2-\theta^2)} |a(x)|^{p/p(z)} \frac{a(x)}{|a(x)|} \qquad (z \in S) , \tag{3.15}$$

where $a \in B^p$. Then, arguing as in the first half of the proof of Lemma 3.3, we obtain

$$(B^{p_0}, B^{p_1})_{[\theta]} \hookleftarrow B^p . \tag{3.16}$$

Thus, since $A^{p_0} \cap A^{p_1}$ is dense in both A^{p_0} and A^{p_1}, the desired conclusion follows from Proposition 2.1 and Theorem 3.1. □

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