On Some Branched Surfaces Which Admit Expanding Immersions

Eijirou HAYAKAWA

Kyoto University
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Abstract. We deal with the class of branched surfaces $K$ such that 1) the branch set $S$ of $K$ is an embedded circle, 2) all connected components of $K\setminus S$ are orientable and their number is two or three. We show that in this class only two topological types admit expanding immersions. In the proof of the result, the Euler class of the tangent bundle of $K$ plays an important role.

§ 0. Introduction.

R. Williams [1], [2], [3] introduced the concept of branched manifolds and expanding immersions in order to study the dynamics of expanding attractors. Using his own tools, he succeeded in classifying 1 dimensional expanding attractors. Our final aim is to study the topological conjugacy classes of 2 dimensional expanding attractors. As the first step toward it we propose the following problem:

Find some topological invariants of branched surfaces which admit expanding immersions.

As an approach to solve this problem, we consider the simplest class of them, i.e., the class of branched surfaces with branch sets a circle.

First of all let us give two examples of expanding immersions. First take a rectangle $[0, 1]\times[0, 2]$ in the coordinate plane, and take two disks $D_1$ and $D_2$ whose radii are $1/10$ and centers are $(4/5, 4/5)$ and $(4/5, 4/5+1)$ respectively. We define the equivalence relation among the points in the rectangle; $(s, t)\sim (s', t')\iff 1) (s, t)$ and $(s', t')$ do not belong to $D_1$ and $D_2$, and $(s-t)\equiv 0, (s'-t')\equiv 0$ mod 1. 2) $(s, t)=(s', t')\in D_1$ or $D_2$. We denote the quotient space by this equivalence relation by $T^*$. Then $T^*$ is a branched surface whose branch set is homeomorphic to a circle. Notice that there exists a canonical projection $p: T^*\to T^c$. The dilation by 2
yields a map $f: T^2 \to T^2$. Clearly $f$ lifts to a map $\tilde{f}: T^* \to T^*$ in a way that $\tilde{f}$ is surjective. Thus $T^*$ admits an expanding immersion.

The second example is as follows. We regard $T^2$ as a rectangle $[0,1] \times [0,1]$, and take two disks $D_i$ and $D_s$ in it whose radii are $1/10$ and centers are $(1/2, 1/4)$ and $(1/2, 3/4)$ respectively. We define the following equivalence relation in $T^2$: $(s, t) \sim (s', t') \iff (s, t) \in D_i$ and $(s', t') \in D_i$, or $(s, t) \in D_s$ and $(s', t') \in D_s$, and $2t \equiv 2t'$, $s \equiv s' \mod 1$. 2) $(s, t) \equiv (s', t')$. We consider the quotient space by this equivalence relation and denote it by $T_*$. $T_*$ is a branched surface whose branch set is homeomorphic to a circle, too. The dilation by 2, $f: T^2 \to T^2$, projects down to a map $\tilde{f}: T_* \to T_*$ via the natural projection $T^2 \to T_*$. This shows that $T_*$ admits an expanding immersions.

Suppose a branched surface $K$ has a branch set $S$ homeomorphic to a circle. Then a neighborhood of $S$ is homeomorphic to one of the following $N_0$ and $N_1$. Take two copies of a rectangle $I \times I$, where $I = [-1, 1]$, and identify the subsets $I \times [-1, 0]$ of them. (See Figure 1.)

![Figure 1](image)

$\tilde{N}$ denotes the quotient space. We take subsets $I_s$ and $I'_s$ in $\tilde{N}$ which are the images of $(-1) \times I$ and $(1) \times I$, contained in one of two copies, respectively, and let $I_s$ and $I'_s$ be the images of $(-1) \times I$ and $(1) \times I$, contained in the other of them, respectively. Then $N$ is obtained by connecting $I_s$ with $I'_s$ and $I_i$ with $I'_i$, or connecting $I_s$ with $I'_s$ and $I_i$ with $I'_i$. We denote the former by $N_0$ and the latter by $N_1$. We define subsets of $N_0$ and $N_1$ as follows. Let $J_i^+$ and $J_i^-$ be the images in $N_0$ of two copies of $I \times \{1\}$ in two copies of $I \times I$ respectively, and let $J^-$ be the image in $N_0$ of $I \times \{-1\}$. In $N_1$, let $J^+$ and $J^-$ be the images of $I \times \{1\}$ and $I \times \{-1\}$.

Using $N_0$ and $N_1$, we define the types of $S$. $S$ is called untwisted (or twisted) if $S$ has a neighborhood homeomorphic to $N_0$ (or $N_1$).

The main result of this paper is as follows. We consider the class of branched surfaces $K$ such that 1) the branch set $S$ of $K$ is an em-
bedded circle, 2) all connected components of $K\setminus S$ are orientable and their number is two or three. In this class, only $T^*$ and $T_*$ admit expanding immersions.

In §1, after giving definitions of branched surfaces and expanding immersions, a precise statement of our result is described. §2 and §3 are devoted to its proof.

The author thanks the referee for suggesting the use of the Euler class of the tangent bundle of $K$. It makes the proof of the theorem clear and simple.

§1. Definitions and the statement of the result.

In order to define branched surfaces, three types of local neighborhoods are needed. Let us define:

1) $U_{(1)}=I\times I$, where $I$ is an open interval $(-1, 1)$.
2) $U_{(2)}=U_{(1)}^1 \sqcup U_{(1)}^2 \sim$, which means a quotient space of two copies of $U_{(1)}$, $U_{(1)}^1$ and $U_{(1)}^2$, by the equivalence relation generated by $(t, s)\sim(t', s')\iff (t, s)\in U_{(1)}^1$, $(t', s')\in U_{(1)}^2$ and $-1<t=t'\leq 0$, $s=s'$.
3) $U_{(3)}=U_{(2)} \sqcup U_{(1)}^3 / \sim$, which means a quotient space of the copy $U_{(1)}^3$ of $U_{(1)}$ and $U_{(2)}$ by the equivalence relation generated by $(t, s)\sim(t', s')\iff (t, s)\in U_{(1)}^3 \subset U_{(2)}$, $(t', s')\in U_{(1)}^1$ and $t=t'$, $-1<s=s'\leq 0$.

Here we have natural maps $\pi_z: U_{(2)} \to U_{(1)}$ and $\pi_z: U_{(3)} \to U_{(1)}$ such that $\pi_z|U_{(i)}^j$ is a natural identification of the copy $U_{(i)}^j$ with $U_{(1)}$ itself, where $i=2$ and $j=1$ or 2, or $i=3$ and $j=1, 2$ or 3.

**Definition 1** [3]. A compact Hausdorff space $K$ is called a $C^r$ branched surface if it has a finite family $\{(U_j, \varphi_j)\}$ satisfying

1) $K=\bigcup U_j$,
2) For each $j$ there exists a homeomorphism $g_j: U_j \to U_{(i)}$ ($i=1, 2$ or 3) such that $\varphi_j=\pi_{(i)} \circ g_j$,
3) For $j$ and $j'$ such that $U_j \cap U_{j'} \neq \emptyset$, there exists a $C^r$ map $\pi_{j',j}: \varphi_j(U_j \cap U_{j'}) \rightarrow \varphi_{j'}(U_{j'} \cap U_j)$ such that $\pi_{j',j} \circ \varphi_j = \varphi_{j'}$. We call $(U_j, \varphi_j)$ a coordinate neighborhood and $\{(U_j, \varphi_j)\}$ a coordinate neighborhood system of $K$.

$S = \{x \in K; x$ does not have a neighborhood homeomorphic to an open disk $D\}$ is called the branch set of $K$.

As in the case of ordinary manifolds, we define the tangent bundle $TK$ of $K$ as the quotient space of $\bigsqcup_j \varphi_j^*TU_{(1)}$ by the natural identification induced by the coordinate change, where $\varphi_j^*TU_{(1)}$ denotes the pull back of the tangent bundle $TU_{(1)}$ by $\varphi_j$. (For detail, see [3].) For $x \in K$, $p^{-1}(x)$ is called the tangent space at $x$ and is denoted by $T_xK$, where $p: TK \rightarrow K$ denotes the projection map, which is induced by $p_j: \varphi_j^*TU_{(1)} \rightarrow U_j$ naturally.

A Riemannian metric on $K$ is defined as a positive definite symmetric bilinear form on $TK$.

Next, we define a $C^r$ map from a branched surface to a branched surface, a $C^r$ immersion and an expanding immersion.

**DEFINITION 2.** Let $K$ and $L$ be $C^r$ branched surfaces, and $\{(U_j, \varphi_j)\}$ and $\{(V_k, \psi_k)\}$ be their coordinate neighborhood systems respectively.  

1) A map $f: K \rightarrow L$ is called a $C^r$ map if for any $i$, $j$ and $k$ with $f^{-1}(V_k) \cap U_j \neq \emptyset$, the composite

$$U_{(1)} \xrightarrow{\varphi_j|U_{ij}^{-1}} f^{-1}(V_k) \cap U_j^i \xrightarrow{f} V_k \xrightarrow{\psi_k} U_{(1)}$$

is $C^r$, where $U_{ij} = g_j^{-1}(U_{ij}')$.

For a $C^r$ map $f: K \rightarrow L$, we can define the differential of $f$, $df: TK \rightarrow TL$, by using the above local representation of $f$ (See [3]). We denote $df|T_xK$ by $df_x$.

2) A map $f: K \rightarrow L$ is called a $C^r$ immersion if $f$ is a $C^r$ map and $df_x: T_xK \rightarrow T_{f(x)}L$ is injective for any $x \in K$.

3) A map $f: K \rightarrow K$ is called a $C^r$ expanding immersion if it satisfies

i) $f$ is a $C^r$ immersion,

ii) there exist numbers $\alpha > 0$ and $\nu > 1$ such that for any positive integer $n$ and $v \in T_xK$, $\|df^n_x(v)\| \geq \alpha \nu^n \|v\|$, where $\| \cdot \|$ means a Riemannian metric,

iii) there exists a positive integer $\tilde{n}$ such that for any $x \in K$ and some neighborhood $U$ of $x$, $f^{\tilde{n}}(U)$ is homeomorphic to an open disk,

iv) the nonwandering set $\Omega(f)$ of $f$ is equal to $K$.

Our branched surfaces are more restrictive than Williams'. His original definition admits more varied types of neighborhoods. But
Williams himself showed that ours are sufficiently general to study expanding immersions.

**Theorem.** Suppose $K$ is a $C^1$ branched surface such that

1) $K$ admits an expanding immersion,
2) The branch set $S$ of $K$ is homeomorphic to a circle,
3) All connected components of $K\setminus S$ are orientable and their number is 2 or 3.

Then $K$ is homeomorphic to $T^*$ or $T_*$.

§ 2. Proof of Theorem (1).

In this section we deal with the case when the number of connected components of $K\setminus S$ is equal to 3. We show that in this case only $T^*$ admits expanding immersions.

Assume that $K$ admits an expanding immersion $f$. Let $K_0$, $K_1$ and $K_2$ be connected components of $K\setminus S$ such that $K_0 \supset J^-$, $K_1 \supset J^+_1$ and $K_2 \supset J^+_2$. For $i=0, 1$ or 2, we attach $\partial K_i$ to $K_i$, and denote the obtained space by $K_i$. (Below, generally for an open subspace $X \subset Y$, we denote the one obtained by attaching the copies of boundary $\partial X$ to $X$ as $X^\sim$. For example, $K_i=K_i^\sim$.)

We construct manifolds $M_1$ and $M_2$ from $K_0$ and $K_1$, and $K_0$ and $K_2$ by identifying their boundaries respectively. $M_1$ and $M_2$ are embedded in $K$ by natural inclusions $\iota_i: M_i \to K$ and $\iota_i: M_i \to K$. By easy calculation, we know that $H_3(K; \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}$ and is generated by $m_i=(\iota_i)_*[M_i]$ and $m_i=(\iota_i)_*[M_i]$, where $[M_i]$ and $[M_i]$ are the fundamental homology classes of $M_i$ and $M_i$ such that they induce the same orientation on $K$.

**Lemma 1.** Let

$$(f^{2n})_* m_1 = \alpha_n m_1 + \beta_n m_2, \quad (f^{2n})_* m_2 = \gamma_n m_1 + \delta_n m_2.$$ 

Then $\alpha_n$, $\beta_n$, $\gamma_n$ and $\delta_n \geq 0$, and both $\alpha_n + \beta_n$ and $\gamma_n + \delta_n$ become large as $n$ becomes large.

**Proof.** Since $f^{2n}$ is orientation preserving, $\alpha_n$, $\beta_n$, $\gamma_n$ and $\delta_n \geq 0$.

Let $\omega$ be the volume form on $K$ whose local representation is $\sqrt{\det(g_{ij})} dx_i \wedge dx_j$ when the local representation of the Riemannian metric is $\sum_{0 \leq i, j \leq 2} g_{ij} dx_i \otimes dx_j$. Let us denote the areas of $M_i$, $M_2$ and $K$ by $a(M_i)$, $a(M_2)$ and $a(K)$ respectively.

We calculate the Kronecker product of $(f^{2n})_* m_i$ and $\omega$.
\[ \langle \omega, (f^{2n})_* m_i \rangle = \alpha_n \langle \omega, m_i \rangle + \beta_n \langle \omega, m_2 \rangle = \alpha_n \int_{r_1} (\ell_1)^* \omega + \beta_n \int_{r_2} (\ell_2)^* \omega = \alpha_n \cdot a(M_1) + \beta_n \cdot a(M_2). \]

On the other hand, we have
\[ \langle \omega, (f^{2n})_* m_i \rangle = \langle (f^{2n})^* \omega, m_i \rangle = \int_{M_1} (\det Df^{2n}) \cdot (\ell_1)^* \omega \geq \min_{p \in M_1} \det(Df^{2n})_p \cdot a(M_1), \]
where \( \det(Df^{2n})_p \) denotes the determinant of \( (Df^{2n})_p \) for the orthonormal bases of \( T_p K \) and \( T_{f^{2n}(p)} K \). Hence we obtain the following inequality:
\[ \alpha_n \cdot a(M_1) + \beta_n \cdot a(M_2) \geq \min_{p \in M_1} \det(Df^{2n})_p \cdot a(M_1). \]

By Definition 2, 3), ii), the right-hand side of the above inequality becomes large as \( n \) becomes large. Hence we have the desired result for \( \alpha_n + \beta_n \).

For \( \gamma_n + \delta_n \), we can show the lemma in the same way as for \( \alpha_n + \beta_n \). \( \square \)

Let \( e(K) \) be the Euler class of the tangent bundle of \( K \). We calculate the Kronecker product of \( e(K) \) and \( m_i \):
\[ \langle e(K), m_i \rangle = \langle e(K), (\ell_1)_*[M_1] \rangle = \langle (\ell_1)^* e(K), [M_1] \rangle = \langle e(M_1), [M_1] \rangle = \chi(M_1). \]

On the other hand, since \( (f^{2n})^* e(K) = e(K) \),
\[ \langle e(K), m_i \rangle = \langle (f^{2n})^* e(K), m_i \rangle = \langle e(K), (f^{2n})_* m_i \rangle = \alpha_n \chi(M_1) + \beta_n \chi(M_2). \]

Hence we obtain for any \( n \):
\[ \chi(M_1) = \alpha_n \chi(M_1) + \beta_n \chi(M_2). \quad (1) \]

Calculating \( \langle e(K), m_2 \rangle \), we also have:
\[ \chi(M_2) = \gamma_n \chi(M_1) + \delta_n \chi(M_2). \quad (2) \]

By Lemma 1, for sufficiently large \( n \), \( \alpha_n + \beta_n \) and \( \gamma_n + \delta_n \) are large. Then from the equalities (1) and (2), we have only the following two cases:
\( 1^\circ \) \( \chi(M_1) = 0 \) or \( \chi(M_2) = 0 \), \( 2^\circ \) \( \chi(M_1) > 0 \) and \( \chi(M_2) < 0 \).

We show that the case \( 2^\circ \) cannot occur. In the case \( 2^\circ \), \( M_1 \) is a sphere \( S^2 \) and \( M_2 \) is the Riemann surface \( \Sigma_g \) of genus \( g \geq 2 \). Assume the case \( 2^\circ \) occurs. First we show that \( f(M_1) \) is not equal to \( M_2 \). If \( f(M_1) = M_2 \), then \( f|_{M_1} \) is a covering map from \( S^2 \) to \( \Sigma_g \). But it is im-
possible. So $f(M_i) \supset M_i$, and it is easy to show $(f|M_i)^{-1}(M_i) = M_i$. Hence $f(M_i) = M_i$. But, since $M_i = S^3$, the degree of the covering map $f|M_i$ is equal to 1. This contradicts Definition 2, 3, ii).

In the case 1°, first, we consider the case (a): $\chi(M_1) = 0$ and $\chi(M_2) = 0$. Next we deal with the case (b): $\chi(M_1) \neq 0$ and $\chi(M_2) = 0$.

In the case (a), we can consider two cases: i) $K_0 \approx D^2$ and $K_1 \approx K_2 \approx T^* - D^2$. ii) $K_0 \approx T^* - D^2$ and $K_1 \approx K_2 \approx D^2$. We show that the case i) cannot occur. Assume the case i) occurs. As $(f|M_i)^{-1}(K_0)$, for $i = 1$ or 2, are mutually disjoint disks embedded in $M_i$, $M_i \setminus (f|M_i)^{-1}(K_0)$ is connected. So we have that $f^2(M_i) = M_i$ and $f^2(M_i) = M_i$, because $f$ is surjective. Hence $f^2(K_0) \subset K_0$. This contradicts Definition 2, 3, ii). In the case ii) $K$ becomes $T^*$.

In the case (b), set $f_* m_z = \gamma m_z + \delta m_z$. Then, by the same calculation as above, we have $\chi(M_2) = \gamma \chi(M_2) + \delta \chi(M_2)$. As $\chi(M_1) \neq 0$ and $\chi(M_2) = 0$, we know $\gamma = 0$. Hence $f_* m_z = \delta m_z$, and this means that $f(M) = M$. If, for $x \in \hat{K}_1$, $f(x) \in M_z$, then $x$ is not a nonwandering point, because $f^n(f(x)) \in M_z$ for any integer $n \geq 1$. Hence $f(\hat{K}_1) \subset \hat{K}_1$. Moreover, by the equality (1), we know $\alpha_1 = 1$. So $f^n|\hat{K}_1$ is injective. This contradicts Definition 2, 3), ii), and so, in the case (b), we have no branched surface which admits expanding immersions. This completes the proof.

§ 3. Proof of Theorem (2).

In this section, we consider the case when the number of connected components of $K \setminus S$ is two. In this case, there are three types of branched surfaces, two of which have untwisted branch sets and one of which has a twisted branch set.

First we consider branched surfaces $K$ which have untwisted branch sets. Let $\hat{K}_1$ and $\hat{K}_2$ be connected components of $K \setminus S$. Two types of them are as follows: 1) $\hat{K}_1 \supset \hat{J}_1^+$ and $\hat{J}_1^+$, and $\hat{K}_2 \supset \hat{J}_2^+$, and $\hat{K}_2 \supset \hat{J}_2^+$.

In the case 1), we show that only $T_*$ admits expanding immersions. Set $K_1 = \hat{K}_1$ and $K_2 = \hat{K}_2$. We connect $K_1$ with two copies of $K_2$ by identifying their boundaries naturally, and denote the obtained space by $M$. Then $M$ has a differentiable structure such that the natural projection $\pi: M \to K$ becomes an immersion. We construct a lift $\hat{f}: M \to M$ of $f: K \to K$ as follows. For $x \in M - \pi^{-1} \circ f^{-1}(K_0)$, set $\hat{f}(x) = \pi^{-1} \circ f \circ \pi(x)$. For each connected component $\hat{K}$ of $\pi^{-1} \circ f^{-1}(K_0)$, we take a sufficiently small neighborhood $\hat{L}$ of $\hat{K}$. Then $f \circ \pi(\hat{L})$ is uniquely lifted to $M$ so as to be
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It is clear that \( \tilde{f} \) is an immersion, and then \( \tilde{f}: M \rightarrow M \) is a covering map whose degree is greater than 2. Hence we conclude that \( M \) is a torus, and \( K \approx D^2 \) and \( K \approx T^2 - (D^2 \sqcup D^2) \). By Definition 2, 3), iii), two copies of \( K \) in \( M \) have the same image for \( \tilde{f} \). Then \( K \) is obtained from \( M \) by identifying two copies of \( K \) by an orientation preserving \( C^1 \) diffeomorphism. It follows that \( K \approx T^* \).

Next we show that in the case 2) there exists no branched surface which admits expanding immersions. Assume \( K \) admits an expanding immersion \( f \), and we will deduce a contradiction. Set \( M = K \setminus \hat{K} \). Then \( M \) is a manifold. Remark that \( \hat{K} \) is orientable, but \( M \) is not necessarily orientable.

\textbf{Lemmas 2.} \( f(M) \) is equal to \( M \).

\textbf{Proof.} First in the case when \( M \) is orientable, we show the lemma. We know easily that \( H_2(K; \mathbb{Z}) \equiv \mathbb{Z} \) and it is generated by \( m = \iota_* [M] \), where \( \iota_* \) is the induced homomorphism of the inclusion \( \iota: M \rightarrow K \), and \([M] \) is the fundamental homology class of \( M \). Here we assume that \( f(M) \neq M \). Then \( f(M) = K \). Take \( x \in \hat{K} \), and consider the following commutative diagram:

\[
\begin{array}{ccc}
H_2(M; \mathbb{Z}) & \xrightarrow{f_*} & H_2(K; \mathbb{Z}) \\
\downarrow p & & \downarrow q \\
H_2(M, M \setminus f^{-1}(x); \mathbb{Z}) & \xrightarrow{\overline{f}_*} & H_2(K, K - \{x\}; \mathbb{Z})
\end{array}
\]

First we have \( q \circ f_* (m) = 0 \). Remark that we can define an orientation on \( \hat{K} \) compatible with the orientation of \( M \), and that \( f \) is orientation preserving or reversing. Then \( \overline{f}_* \circ p(m) = \pm (\# f^{-1}(x) \cap M) \cdot O_s \), where \( O_s \) is a generator of \( H_2(K, K - \{x\}; \mathbb{Z}) \). By the assumption, \( \# f^{-1}(x) \cap M \neq 0 \). This is a contradiction. Hence \( f(M) = M \).

Next we assume \( M \) is nonorientable. We take the orientation covering of \( K \), \( \pi: \hat{K} \rightarrow K \). We can construct it in the same way as for ordinary manifolds. We take a lift \( \tilde{f}: \hat{K} \rightarrow \hat{K} \) of \( f \). Notice that \( \hat{K} \) is a branched surface whose tangent bundle is orientable and \( \tilde{f} \) can be taken as an orientation preserving immersion satisfying \( \sigma \circ \tilde{f} \circ \sigma = \tilde{f} \), where \( \sigma \) is the nontrivial covering transformation of \( \pi: \hat{K} \rightarrow K \). Let \( \tilde{M} = \pi^{-1}(M) \). Then \( \tilde{M} \) is an orientable manifold.

We know \( H_2(\hat{K}; \mathbb{Z}) \equiv \mathbb{Z} \oplus \mathbb{Z} \), and we take a pair of generators as follows. We take submanifolds \( K^{(1)} \) and \( K^{(2)} \) in \( \hat{K} \) such that \( \pi(K^{(1)}) = \)
\[ \pi(K^{(1)}) = M, \ K^{(1)} \cup K^{(2)} = \tilde{M} \text{ and } K^{(1)} \cap K^{(2)} = \pi^{-1}(S), \] and take submanifolds \( K^{(1)} \) and \( K^{(2)} \) such that \( \pi(K^{(1)}) = \pi(K^{(2)}) = K_i \). Set \( L_i = K^{(1)} \cup K^{(2)} \) and \( L_i = K^{(1)} \cup K^{(2)} \). We choose a pair of generators \( l_i \) and \( l_2 \) of \( H_{i}(L_i; Z) \) and \( H_{2}(L_2; Z) \) such that \( \tilde{l}_1 + \tilde{l}_2 = \tilde{m} \), where \( \tilde{l}_1 = (\ell_i)_{*}l_i \), \( \tilde{l}_2 = (\ell_2)_{*}l_2 \) and \( \tilde{m} = \epsilon_x[\tilde{M}] \), and \( \ell_i : L_i \to \tilde{K}, \ell_2 : L_2 \to \tilde{K} \) and \( \epsilon : \tilde{M} \to \tilde{K} \) are inclusions. Then \( \tilde{l}_1 \) and \( \tilde{l}_2 \) are generators of \( H_{i}(\tilde{K}; Z) \). Let \( \tilde{f}_* l_i = \alpha \cdot \tilde{l}_1 + \beta \cdot \tilde{l}_2 \) and \( \tilde{f}_* l_2 = \gamma \cdot \tilde{l}_1 + \delta \cdot \tilde{l}_2 \). Since \( \sigma \circ \tilde{f} \circ \sigma = \tilde{f} \), we have \( \alpha = \delta \) and \( \beta = \gamma \). Then \( \tilde{f}_* \tilde{m} = \tilde{f}_* \tilde{l}_1 + \tilde{f}_* \tilde{l}_2 = (\alpha + \beta) \cdot (\tilde{l}_1 + \tilde{l}_2) = (\alpha + \beta) \cdot \tilde{m} \). Hence in the same way as the above case, we obtain that \( \tilde{f}(\tilde{M}) = \tilde{M} \) and \( f(M) = M \).

By Definition 2, 3), iii), for some positive integer \( n \) and \( x \in \hat{K}_2 \) sufficiently near \( S \), there exists \( y \in M \) such that \( f^{-n}(x) = f^{-n}(y) \). Since \( f(M) \subset M \) by Lemma 2, for any positive integer \( m \), \( f^{-n+m}(x) = f^{-n+m}(y) \in M \). This contradicts Definition 2, 3), iv).

Finally we consider the last type, each of which has a twisted branch set. We also assume that \( K \) admits an expanding immersion \( f \). Let \( \hat{K}_1 \) and \( \hat{K}_2 \) be connected components of \( K \setminus S \) such that \( \hat{K}_1 \supset J^+ \) and \( \hat{K}_2 \supset J^- \), and let \( K_i^N \) be connected components of \( K \setminus N \) such that \( K_i^N \subset \hat{K}_1 \) and \( K_i^N \subset \hat{K}_2 \), where \( N \) is a neighborhood of \( S \) homeomorphic to \( N_1 \). Easily we have \( H_{i}(\tilde{K }; Z) \cong Z \), and denote a generator by \( [K] \).

**Lemma 3.** Set \( f^{2n}[K] = \alpha_* [K] \). Then as \( n \) becomes large, \( \alpha_* \) becomes large.

**Proof.** Consider the following commutative diagram:

\[
0 \longrightarrow H(S; Z) \mathrel{\overset{p}{\longrightarrow}} H(S; Z) \mathrel{\overset{\partial}{\longrightarrow}} H(\partial S; Z) \longrightarrow H(S; Z)
\]

Similarly, we get:

\[
0 \longrightarrow H_{i}(K; Z) \mathrel{\overset{p}{\longrightarrow}} H_{i}(K; S; Z) \mathrel{\overset{\partial}{\longrightarrow}} H_{i}(S; Z) \longrightarrow H_{i}(K; Z)
\]

\[
H_{i}(K; \partial K; Z) \oplus H_{i}(K; \partial K; Z) \mathrel{\overset{\partial}{\longrightarrow}} H_{i}(\partial K; Z) \oplus H_{i}(\partial K; Z)
\]

Take fundamental homology classes \([K_i^N, \partial K_i^N]\) and \([K_i^N, \partial K_i^N]\) of \( K_i^N \) and \( K_i^N \) such that they induce the same orientation on \( K \) induced by \( [K] \). Moreover let \([S] \) be a generator of \( H_{i}(S; Z) \) such that \([S] = r_1 \circ \ell_1 \circ \partial_2 [K_i^N, \partial K_i^N] \). Since \( r_1 \circ \ell_1 \circ \partial_2 [K_i^N, \partial K_i^N] + 2[K_i^N, \partial K_i^N] = -2[S] + 2[S] = 0 \), we have \( \partial \circ r_1 \circ \ell_1 [K_i^N, \partial K_i^N] + 2[K_i^N, \partial K_i^N] = 0 \). Hence, \( p[K] = \alpha_* [K_i^N, \partial K_i^N] + 2[K_i^N, \partial K_i^N] \).

For \( x \in \hat{K}_i \), such that \( f^{-2n}(x) \cap S = \emptyset \), set \( \{y_i\}^{k(1)} = f^{-2n}(x) \cap \hat{K}_1 \) and \( \{y_j\}^{k(2)} = f^{-2n}(x) \cap \hat{K}_2 \). We consider the commutative diagram:
\[ H_{2}(K; Z) \xrightarrow{(f^{2n})_{*}} H_{2}(K; Z) \]

\[ \begin{array}{c}
\oplus_{i=1}^{k(1)} H_{2}(K_{1}, K_{1}-\{x_{i}^{1}\}; Z) + \oplus_{j=1}^{k(2)} H_{2}(K_{2}, K_{2}-\{x_{j}^{2}\}; Z) \\
\xrightarrow{(f^{2n})_{*}} H_{2}(K, K-\{x\}; Z)
\end{array} \]

Then

\[ (f^{2n})_{*} \circ p_{1}[K] = (f^{2n})_{*} \left( \sum_{i=1}^{k(1)} O_{i}^{1} + 2 \sum_{j=1}^{k(2)} O_{j}^{2} \right) = (k(1) + 2 \cdot k(2)) \cdot O_{x}, \]

since \( p[K] = r_{2} \circ \ell_{2}(K_{1}, \partial K_{1}^{N} + 2[K_{2}, \partial K_{2}^{N}]) \), where \( O_{i}^{1} \) and \( O_{j}^{2} \) denote generators of \( H_{2}(K_{1}, K_{1}-\{x_{i}^{1}\}; Z) \) and \( H_{2}(K_{2}, K_{2}-\{x_{j}^{2}\}; Z) \) respectively, and \( O_{x} \) denotes a generator of \( H_{2}(K, K-\{x\}; Z) \).

On the other hand, \( p_{2} \circ (f^{2n})_{*}[K] = \alpha_{n} \cdot O_{x} \), hence we have \( \alpha_{n} = k(1) + 2k(2) \geq \# f^{-2n}(x) \).

We calculate the Kronecker product of \([K]\) and \(e(K)\). First, since \((f^{2n})^{*}e(K) = e(K)\), \(\langle e(K), [K] \rangle = \langle (f^{2n})^{*}e(K), [K] \rangle = \langle e(K), (f^{2n})_{*}[K] \rangle = \alpha_{n} \langle e(K), [K] \rangle\). By Lemma 3, we have \(\langle e(K), [K] \rangle = 0\). On the other hand, in the same way as the proof of the index theorem \(\langle e[M], [M] \rangle = \chi(M)\) for an ordinary manifold \(M\), we calculate \(\langle e(K), [K] \rangle\) by using a vector field \(X\) with finite singularities such that the indices of \(X|K_{1}\) and \(X|K_{2}\) are equal to \(\chi(K_{1})\) and \(\chi(K_{2})\). As \(p[K] = r_{2} \circ c_{4}(K_{1}, \partial K_{1}) + 2[K_{2}, \partial K_{2}]\), we have \(\langle e(K), [K] \rangle = \chi(K_{1}) + 2\chi(K_{2})\). Hence \(\chi(K_{1}) + 2\chi(K_{2})\) must be zero, but \(\chi(K_{1})\) is odd since \(K_{1}\) has one boundary circle. It follows that in this case we have no branched surface which admits expanding immersions.

References


Present Address:
DEPARTMENT OF MATHEMATICS, KYOTO UNIVERSITY
Oiwake-cho, Kitashirakawa, Sakyō-ku, Kyoto 606, JAPAN