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On the Existence and Smoothness of Invariant Manifolds of Semilinear Evolution Equations

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§1. Introduction.

Let us consider semilinear evolution equations in a Hilbert space X

(E) $$du/dt = Lu + Nu\ , \qquad t>0\ .$$

Here L is the generator of an analytic semigroup and N is a nonlinear operator defined near 0. We suppose that the spectrum $\sigma(L)$ of L is divided into two parts $\sigma_1(L)$ and $\sigma_2(L)$ in such a way that

$$(\alpha_2 \equiv)\ \sup_{\sigma \in \sigma_2(L)} \mathrm{Re}\,\sigma < \inf_{\sigma \in \sigma_1(L)} \mathrm{Re}\,\sigma\ (\equiv \alpha_1)\ .$$

If N is identically zero, the eigenspace X_i, $i=1, 2$, corresponding to $\sigma_i(L)$ is invariant in the following sense: If an initial value x is contained in X_i then the solution $u(t, x)$ of (E) with the initial value x is also contained in X_i for $t>0$.

In this paper we are interested in the persistency of the invariance and smoothness of the manifolds X_i under small perturbation N. Let $N(x)$ be a C^k-mapping, $1 \leqq k < \infty$, with $N(0)=0$. We first ask if there exists an invariant manifold M_i "near X_i", provided that $\|D_x N\|$ is small enough. ($D_x N$ denotes the Fréchet derivative of $N(x)$ with respect to x.) If it does, we next ask if invariant manifolds are C^k.

The following facts have been known. See, e.g., [1-11, 14-17, 19-22].

(i) If $\inf_{\sigma \in \sigma_1(L)} \mathrm{Re}\,\sigma \geqq 0$, then an invariant C^k-manifold M_1 "near X_1" exists. It is called a center-unstable manifold. In particular, if $\inf_{\sigma \in \sigma_1(L)} \mathrm{Re}\,\sigma > 0$ (resp. $\mathrm{Re}\,\sigma = 0$ for $\sigma \in \sigma_1(L)$), then the manifold is called an unstable (resp. a center) manifold.

(ii) If $\sup_{\sigma \in \sigma_2(L)} \mathrm{Re}\,\sigma < 0$, then an invariant C^k-manifold "near X_2" exists.

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It is called a stable manifold.

In this paper we shall prove that an invariant C^1-manifold M_i "near X_i" exists if $\|D_xN\|$ is small enough. The problem of smoothness of M_i is more delicate. It depends on the structure of the spectrum of L. We state a result on the smoothness of M_1. A similar result holds on the smoothness of M_2.

THEOREM. *Let the above hypotheses hold. Suppose that there exists an integer r with $\alpha_2 < r\alpha_1$, $1 \leqq r \leqq k$. Then a C^r-invariant manifold exists if $\|D_xN\|$ is small enough.*

This result is optimal in the following sense. If $\alpha_2 = k\alpha_1$ (<0), then there is an example such that there does not exist any C^k-invariant manifold M_1 "near X_1". Such an example is given in section 2.

In section 2 we state our hypotheses and results. The proofs of Theorems 1, 2, and 3 are based on results of Hirsch, Pugh, and Shub [12]. In section 3 we state them in an adequate form to our use. In section 4 we prove our results by applying them to a time s-mapping $u(s, \cdot)$.

§ 2. Main results.

Throughout the present paper we postulate the following two hypotheses concerning L and N.

HYPOTHESIS 1. (i) L generates an analytic semigroup $\{e^{tL}\}_{t>0}$ in X. (ii) The spectrum $\sigma(L)$ of L is divided into two parts:

$$\sigma(L) = \sigma_1(L) \cup \sigma_2(L)\ , \quad (\alpha_2 \equiv) \sup_{\sigma \in \sigma_2(L)} \operatorname{Re} \sigma < \inf_{\sigma \in \sigma_1(L)} \operatorname{Re} \sigma\ (\equiv \alpha_1)\ .$$

By Hypothesis 1 (i) there exists a constant α with $\sup_{\sigma \in \sigma_1(L)} \operatorname{Re} \sigma < \alpha$. In the following we fix such a number α. Let β be such that $0 \leqq \beta < 1$. We denote by X_β the Banach space consisting of all elements in the domain of $(-L+\alpha)^\beta$. The norm of X_β is the graph norm of $(-L+\alpha)^\beta$, which we denote by $\|\ \ \|$.

HYPOTHESIS 2. The nonlinear operator N is a C^k-mapping of some neighborhood U of 0 in X_β into X such that $N(0)=0$.

Let P_i, $i=1, 2$, be the projection associated with $\sigma_i(L)$. The restriction $P_i|X_\beta$ of P_i to X_β is also the projection of X_β onto P_iX_β. Then X_β is decomposed into the direct sum: $X_\beta = P_1X_\beta \oplus P_2X_\beta$. For simplicity we write X_i for P_iX_β.

We give a definition of local invariance. Let $M \subset U$ ($\subset X_\beta$). We say that a set M is locally invariant if the following holds: Let $x \in M$. Then there exists a $t>0$ such that $u(s, x) \in M$, $0<s\leqq t$.

Our problem can be formulated as follows.

PROBLEM. Let $\sup_{x\in U}\|D_xN(x)\|$ be small enough. Does there exist a C^k-mapping w_i, $i=1, 2$, defined in a neighborhood V_i of 0 in X_i into X_{3-i} which satisfies
(i) $w_i(0)=0$, $\sup_{x\in V_i}\|D_xw_i(x)\|$ is small,
(ii) the graph of w_i is locally invariant.

Our main results are given by the following theorems.

THEOREM 1. *Assume that Hypotheses* 1 *and* 2 *are satisfied. Then there exist an open neighborhood* V_i, $i=1, 2$, *of* 0 *in* X_i *and a* C^1*-mapping* w_i *of* V_i *into* X_{3-i} *with the properties* (i) *and* (ii) *in Problem.*

THEOREM 2 (Smoothness of w_1). *Assume that Hypotheses* 1 *and* 2 *are satisfied. Let* r *be the largest integer which satisfies* $\alpha_2<r\alpha_1$, $1\leqq r\leqq k$. *Then if* $\|D_xN(0)\|$ *is small enough, then there exist an open neighborhood* V_1 *of* 0 *in* X_1 *and a* C^r*-mapping* w_1 *of* V_1 *into* X_2 *which satisfy* (i) *and* (ii) *in Problem.*

THEOREM 3 (Smoothness of w_2). *Assume that Hypotheses* 1 *and* 2 *are satisfied. Let* r *be the largest integer which satisfies* $r\alpha_2<\alpha_1$, $1\leqq r\leqq k$. *Then if* $\|D_xN(0)\|$ *is small enough, then there exist an open neighborhood* V_2 *of* 0 *in* X_2 *and a* C^r*-mapping* w_2 *of* V_2 *into* X_1 *which satisfy* (i) *and* (ii) *in Problem.*

REMARK. The smoothness of a mapping w_i, $i=1, 2$, is optimal in the following sense. Suppose that $\alpha_2=k\alpha_1$ (resp. $k\alpha_2=\alpha_1$). Then there is an example such that even if N is a C^∞-mapping, there does not exist any C^k-mapping of a neighborhood of 0 in X_1 into X_2 (resp. X_2 into X_1). In the rest of this section we give such an example.

Consider the system of equations

$$dx/dt=-x\ , \qquad dy/dt=-ky+x^k\ .$$

Straightforward computation shows that the solution (x, y) with initial value (x_0, y_0), $x_0\neq 0$, satisfies

$$y=(y_0/x_0{}^k)x^k-x^k\log(x/x_0)\ .$$

If we set

$$C = y_0/x_0^k + \log |x_0|$$

then (x, y) satisfies

$$y = -x^k \log |x| + Cx^k \ .$$

If $x_0=0$, then the solution satisfies $x=0$. Therefore we conclude that the invariant curve $x=0$ corresponds to the graph of w_2 and that the other invariant curves are not C^k at $x=0$.

§ 3. Results of Hirsch, Pugh, and Shub.

In this section we recall results of Hirsch, Pugh, and shub [12, Theorem 5.1]. We state them in a modified form which are adequate to our use.

Let E be a Banach space divided into the direct sum: $E=E_1 \oplus E_2$. Let T be a bounded linear operator on E. We denote by P_i, $i=1, 2$, the projection of E onto E_i, $i=1, 2$. We suppose that the following conditions hold.

HYPOTHESIS T. (i) $TE_i \subset E_i$, $i=1, 2$.
(ii) The restriction T_1 of P_1T to E_1 has a bounded inverse.
(iii) The following inequality holds:

$$\|T_1^{-1}\| \, \|T_2\| < 1 \ , \tag{3.1}$$

where T_2 denotes the restriction of P_2T to E_2.

The results of Hirsch, Pugh, and Shub are stated in the following theorems.

THEOREM 4. *Assume that Hypothesis* T *holds. Then there exist* $\varepsilon_1>0$ *and two constants* ρ_i, $i=1, 2$, *with* $\rho_2<\rho_1$ *such that the following statements hold. Suppose that a* C^1*-mapping* $f: E \to E$ *which satisfies* $f(0)=0$, $\|D_x f - T\| \ (=\varepsilon) < \varepsilon_1$, *then there exist two maps* $\xi_f : E_1 \to E_2$ *and* $\xi_{f^{-1}} : E_2 \to E_1$ *with the following properties*:
(i) $\sup_{x \in E_1} \|D_x \xi_f(x)\|$ *and* $\sup_{x \in E_2} \|D_x \xi_{f^{-1}}(x)\|$ *tends to* 0 *as* $\varepsilon \to 0$.
(ii) *If* $x \in W_1 \equiv \text{graph}\, \xi_f = \{x=(y, z) : z=\xi_f(y), y \in E_1\}$, *then there exists a unique sequence* $\{x_{-n}\}$ *in* W_1, $n \in N$, *which satisfies* $f^n(x_{-n})=x$, *and*

$$\|x_{-n}\| \leqq \rho_1^{-n} \|x\| \ . \tag{3.4}$$

Conversely if there exists a sequence $\{x_{-n}\}$ *which satisfies* $f^n(x_{-n})=x$, *and* $\|x_{-n}\| \rho_1^n$ *is bounded, then* $x \in W_1$.
(iii) *If* $x \in W_2 \equiv \text{graph}\, \xi_{f^{-1}} = \{x=(y, z) : y=\xi_{f^{-1}}(z), z \in E_2\}$, *then for* $n \geqq 0$

(3.5) $$\|f^n(x)\| \leqq \rho_2^n \|x\| .$$

Conversely if $\|f^n(x)\|/\rho_2^n$ *is bounded, then* $x \in W_2$.

On further smoothness of ξ_f and $\xi_{f^{-1}}$ the following theorems hold.

THEOREM 5 (Smoothness of ξ_f). *Under the hypotheses of Theorem 4, we further assume that*

(3.6) $$\|T_1^{-1}\|^k \|T_2\| < 1 .$$

Then there exists $0 < \varepsilon_k \leqq \varepsilon_1$ *such that for any* C^k*-mapping* f *which satisfies* $f(0)=0$ *and* $\|D_x f - T\| < \varepsilon_k$, *the mapping* ξ_f *obtained in Theorem 4 is* C^k.

THEOREM 6 (Smoothness of $\xi_{f^{-1}}$). *Under the hypotheses of Theorem 4, we further assume that*

(3.7) $$\|T_1^{-1}\| \, \|T_2\|^k < 1 .$$

Then there exists $0 < \varepsilon_k \leqq \varepsilon_1$ *such that for any* C^k*-mapping* f *which satisfies* $f(0)=0$ *and* $\|D_x f - T\| < \varepsilon_k$, *the mapping* $\xi_{f^{-1}}$ *obtained in Theorem 4 is* C^k.

§ 4. Proofs of Theorems 1, 2, and 3.

Instead of the evolution equation (E), we consider a modified equation

(E_ε) $$du/dt = Lu + \chi(u/\varepsilon)N(u) ,$$

where $\chi \in C^k(X_\beta, \boldsymbol{R})$ with $\chi(x)=1$ ($\|x\|<1$), $=0$ ($\|x\|>2$). We state the following lemma, which is elementary, but plays a fundamental role in the proofs.

LEMMA. *For any* $x \in X_\beta$ *a solution* $u_\varepsilon(t, x)$ *of* (E_ε) *with initial value* x *exists on* $[0, \infty)$. *For each* $t>0$ *a mapping* $x \to u_\varepsilon(t, x)$ *is a* C^k*-mapping and satisfies*

(4.1) $$\|D_x u_\varepsilon(t, x) - e^{tL}\| \leqq K(e^{\delta(\varepsilon)Kt} - 1)e^{\alpha t} , \qquad t \geqq 1$$

where K *and* $\delta(\varepsilon)$ *are constants independent of* t *and* x *such that* $\delta(\varepsilon)$ *tends to* 0 *as* $\varepsilon \to 0$.

The proof is standard. See, e.g., [11].

PROOF OF THEOREM 1. First we give an outline of the proof of Theorem 1. We set $T(s)=e^{sL}$. We write $f_\varepsilon(t)x$ for the solution $u_\varepsilon(t, x)$. Then, by Lemma, $f_\varepsilon(t)$ is a C^k-mapping of X_β into itself. We first choose

$s>0$ so large that the condition (3.1) in Hypothesis T holds with T replaced by $T(s)$. We next choose ε so small that the inequality

$$(4.2) \qquad \|D_x f_\varepsilon(s) - T(s)\| < \varepsilon_1$$

holds. Then we apply Theorem 4 with f replaced by f_ε. Thus we shall obtain a C^1-mapping w_i, the graph of which satisfies (i), (ii), and (iii) of Theorem 4. We set $W_{i,\varepsilon} = \text{graph}\, w_i$. For the proof of Theorem 1 we have only to establish that $W_{i,\varepsilon}$ is invariant under the semiflow $f_\varepsilon(t)$.

Now we determine s as follows. Choose real numbers β_1 and β_2 such that $\alpha_2 < \beta_2 < \beta_1 < \alpha_1$ $(<\alpha)$. Then the following inequalities hold.

$$(4.3) \qquad \|e^{tL}\| \leqq Ke^{\alpha t}, \qquad t \geqq 0,$$

$$(4.4) \qquad \|e^{-tL}|X_1\| \leqq Ke^{-\beta_1 t}, \qquad t \geqq 0,$$

$$(4.5) \qquad \|e^{tL}|X_2\| \leqq Ke^{\beta_2 t}, \qquad t \geqq 0,$$

where K is a constant independent of t, and $e^{-tL}|X_1$ is the inverse of $e^{tL}|X_1 : X_1 \to X_1$. Since $\beta_2 < \beta_1$, we can choose s so large that $K^2 e^{(\beta_2 - \beta_1)s} < 1$. Then, by (4.4) and (4.5), we get

$$\|e^{sL}|X_2\| \, \|e^{-sL}|X_1\| \leqq K^2 e^{(\beta_2 - \beta_1)s} < 1$$

and so the inequality (3.1) holds. By (4.1) we can choose $\varepsilon > 0$ so small that the inequality (4.2) holds. Thus we can apply Theorem 4 to a time s-mapping $T(s)$. Hence we obtain a mapping w_1 (resp. w_2) of X_1 into X_2 (resp. X_2 into X_1) which satisfies (i), (ii), and (iii) of Theorem 4.

We first claim that $W_{2,\varepsilon}$ is invariant under $f_\varepsilon(t)$.

Proof of the claim. Let $t>0$ and let $k=[t/s]$. Then, by the semigroup property of $f_\varepsilon(t)$, we get

$$f_\varepsilon^n(s) u_\varepsilon(t, x) = u_\varepsilon(ns+t, x) = u_\varepsilon(t-ks, f_\varepsilon^{n+k}(s)x) .$$

On the other hand, since $u_\varepsilon(t, 0) = 0$, we get by Lemma and (4.3)

$$\|u_\varepsilon(t, x)\| \leqq Ke^{(\alpha + \delta(\varepsilon)K)t} \|x\| , \qquad t \geqq 0 .$$

Hence we obtain

$$\begin{aligned} \|f_\varepsilon^n(s) u_\varepsilon(t, x)\| &\leqq Ke^{(\alpha+\delta(\varepsilon)K)(t-ks)} \|f_\varepsilon^{n+k}(s)x\| \\ &\leqq Ke^{(\alpha+\delta(\varepsilon)K)s} \rho_2^{n+k} \|x\| . \end{aligned}$$

Therefore $\|f_\varepsilon^n(s) u_\varepsilon(t, x)\| / \rho_2^n$ is bounded for $n \geqq 0$. Thus, by Theorem 4 (iii) we conclude that $u_\varepsilon(t, x) \in W_{2,\varepsilon}$, $t>0$.

We next claim that $W_{1,\varepsilon}$ is invariant under the semiflow $f_\varepsilon(t)$.

Proof of the claim. Let $x\in W_{1,\varepsilon}$. Then by Theorem 4 (ii) there exists a sequence $\{x_{-n}\}_{n\geqq 0}$ which satisfies $f_\varepsilon^n(s)x_{-n}=x$ and

$$\|x_{-n}\|\leqq \rho_1^{-n}\|x\| . \tag{4.6}$$

Since $f_\varepsilon^n(s)f_\varepsilon(t)x_{-n}=f_\varepsilon(t)x$, for the proof of invariance of $W_{1,\varepsilon}$ under $f_\varepsilon(t)$, it suffices to show that $\|f_\varepsilon(t)x_{-n}\|\rho_1^n$ is bounded. Let $k=[t/s]$. Then, by Lemma and (4.6) we have

$$\begin{aligned}\|f_\varepsilon(t)x_{-n}\|\rho_1^n&=\|f_\varepsilon(t-ks)x_{k-n}\|\rho_1^n\\&\leqq Ke^{\alpha s}\|x_{k-n}\|\rho_1^n\leqq Ke^{\alpha s}\rho_1^{k-n}\rho_1^n\|x\|=Ke^{\alpha s}\rho_1^k\|x\| .\end{aligned}$$

Hence it follows that $\|f_\varepsilon(t)x_{-n}\|\rho_1^n$ is bounded for $n\geqq 0$. Thus we conclude that $W_{1,\varepsilon}$ is invariant under the semiflow $f_\varepsilon(t)$. Q.E.D.

PROOFS OF THEOREMS 2 AND 3. We have only to show that (3.6) and (3.7) hold, respectively, with T replaced by e^{sL}. We show that (3.6) holds. Suppose that $\alpha_2<r\alpha_1$. Then there exists β_1 and β_2 with $\beta_2<r\beta_1$ such that (4.4) and (4.5) hold. Choose $s>0$ so large that $K^2e^{(-\beta_1 r+\beta_2)s}<1$. Then we obtain

$$\|e^{-sL}|X_1\|^r\|e^{sL}|X_2\|\leqq K^2e^{(-\beta_1 r+\beta_2)s}<1 .$$

The proof of (3.7) is similar. Thus the proofs of Theorems 2 and 3 are complete.

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