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Hyperbolic 3-Manifolds with Totally Geodesic Boundary Which Are Decomposed into Hyperbolic Truncated Tetrahedra

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§0. Introduction.

In this paper we study compact oriented hyperbolic 3-manifolds each of which has a totally geodesic boundary. By a hyperbolic manifold, we mean a Riemannian manifold with constant sectional curvature -1. We construct such 3-manifolds by gluing the hexagonal faces of hyperbolic truncated tetrahedra by isometries, where a hyperbolic truncated tetrahedron is a polyhedron in the hyperbolic 3-space $\boldsymbol{H}^3$ bounded by four totally geodesic right-angled hexagons and four triangles (precise definition is given by Proposition 2.1). Such a construction was presented by W. P. Thurston in a lecture at the University of Warwick in July in 1984 (the author learned it from Professor Sadayoshi Kojima).

We deal with a detailed study of constructing hyperbolic 3-manifolds by gluing hyperbolic truncated tetrahedra in §1 and §2, and after these preparations, we will show the following result in §3.

THEOREM 3.1. *There are exactly eight mutually non-isometric compact oriented hyperbolic* 3*-manifolds with totally geodesic boundary such that they can be decomposed into two hyperbolic truncated tetrahedra and that their boundaries are closed surfaces of genus* 2.

We can easily obtain the presentations of fundamental groups of the above eight 3-manifolds. But it is quite difficult to distinguish them, even though we compute the first homology group of the n-fold covering of each of them by using the Reidemeister-Schreier method (for each $n\geqq 2$). So we shall distinguish the above eight 3-manifolds geometrically by using the shortest return path, where a return path is a geodesic arc which starts from and returns back to the boundary surface with the right angle.

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By the way, there exist infinitely many mutually non-isometric compact oriented hyperbolic 3-manifolds each of which has a totally geodesic boundary of genus 2 (Fujii [2]). Among such 3-manifolds, the above eight hyperbolic 3-manifolds have the following special property: i.e., these eight hyperbolic 3-manifolds have the same volume and this value is the minimal one among all compact oriented hyperbolic 3-manifolds with totally geodesic closed boundary (Kojima-Miyamoto [3]). This value is about 6.452. Then the minimal volume among all closed oriented hyperbolic 3-manifolds each of which has two sided totally geodesic closed surfaces is about 12.904. This value is extremely large in comparison with the minimal volume among all complete oriented hyperbolic 3-manifolds (by Thurston, there exists such a value and it is at most 0.98 which is the value of the volume of the manifold obtained by performing (5, 1) Dehn surgery on the figure-eight knot in the 3-sphere).

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§ 1. Preliminaries.

In this section we prepare some formulae in hyperbolic geometry and consider hyperbolic 3-manifolds obtained by gluing the faces of geodesic polyhedra in the hyperbolic 3-space $\boldsymbol{H}^3$.

PROPOSITION 1.1 (Beardon [1]). *For polygons in the hyperbolic plane* $\boldsymbol{H}^2$ *as illustrated in Fig.* 1.1, *the following equations hold*:

$$\cosh C = \frac{\cos\alpha\cos\beta+\cos\gamma}{\sin\alpha\sin\beta} \tag{a}$$

$$\cosh C = \frac{\cosh\alpha\cosh\beta+\cosh\gamma}{\sinh\alpha\sinh\beta} \tag{b}$$

$$\sinh A\sinh B=\cosh D \tag{c}$$

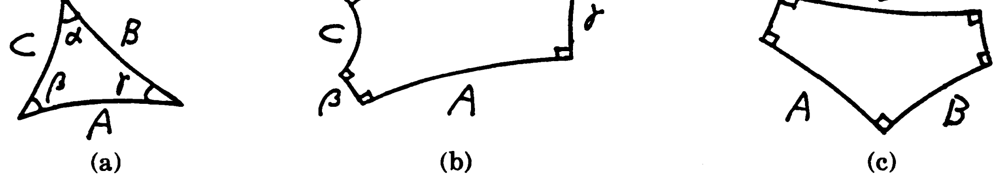


FIGURE 1.1. Various values indicated above give the lengths of the corresponding geodesics or the angles.

Now consider polyhedra in $\boldsymbol{H}^3$ each of which is bounded by totally geodesic surfaces. Suppose that we are given finitely many oriented polyhedra in $\boldsymbol{H}^3$ from which we are going to construct some hyperbolic 3-manifold. We assume that the following conditions are satisfied. First of all, the faces of these polyhedra are divided into two kinds: one is *the boundary face* and the other is *the interior face*. For any boundary face, we assume that it intersects perpendicularly with all edges of the corresponding polyhedron. For the interior faces, we assume that the following *gluing pattern* is given. Namely for each interior face another such face is paired so that they are mutually isometric and moreover there is given an orientation reversing isometry between them. We also assume that the dihedral angles add up to π around each edge of the boundary faces. Now choose some gluing pattern such that we identify pairs of 2-faces of these polyhedra with isometries, so we can only glue two congruent faces. Let M^3 be a resulting 3-manifold by using such a gluing pattern.

Polyhedra in $\boldsymbol{H}^3$ which consist M^3 determine a hyperbolic structure on $M^3 \setminus \{\text{interior edges}\}$, since we only glue congruent faces. Now consider the condition for the hyperbolic structure on $M^3 \setminus \{\text{interior edges}\}$ to give a hyperbolic structure on M^3 itself (not necessarily complete). By Thurston [4], it is the condition that its developing map, in a neighborhood of each interior edge, should come from a local homeomorphism of M^3 itself. Then the hyperbolic structure extends over each interior edge of M^3 if and only if the dihedral angles between the two faces which have this interior edge as their boundaries add up to 2π.

§2. Hyperbolic truncated tetrahedron.

In this section we deal with a hyperbolic truncated tetrahedron (definition is given by the following Proposition 2.1) which is a main constituent element of a hyperbolic 3-manifold with boundary.

PROPOSITION 2.1. *Let θ_i $(i=1, \cdots, 6)$ be six real numbers each of which satisfies the following*:

$$\begin{aligned}
&\theta_i \in (0, \pi) \qquad i=1, \cdots, 6\\
&\theta_1+\theta_2+\theta_3<\pi\ ,\\
&\theta_3+\theta_4+\theta_5<\pi\ ,\\
&\theta_2+\theta_4+\theta_6<\pi\ ,\\
&\theta_1+\theta_5+\theta_6<\pi\ .
\end{aligned}$$

Then there exists a unique (up to conformal transformation) polyhedron in $\boldsymbol{H}^3$ whose four boundary hexagons are all right-angled as shown in Fig. 2.1. We call such a polyhedron a hyperbolic truncated tetrahedron.

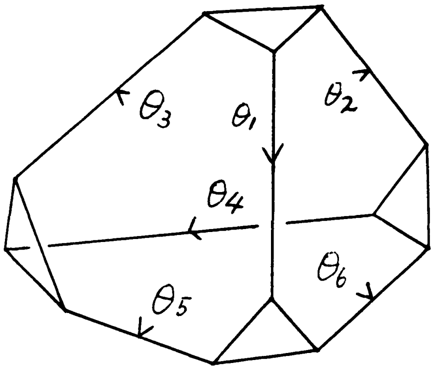


FIGURE 2.1. Parameters θ_i $(i=1, \cdots, 6)$ indicated above give the corresponding dihedral angles. The dihedral angles are $\pi/2$ except where indicated.

In the case where we glue some hyperbolic truncated tetrahedra to construct hyperbolic 3-manifolds, all its hexagonal faces are interior faces and all its triangle faces are boundary faces.

PROOF OF PROPOSITION 2.1. For any $\theta_1, \theta_2, \theta_3 \in (0, \pi)$, where $\theta_1+\theta_2+\theta_3<\pi$, up to conformal transformation, we can write three circles on $\boldsymbol{C}\cup\{\infty\}$ with dihedral angles equal to $\theta_1, \theta_2, \theta_3$. By a conformal transformation, we can transfer one of the intersection points to ∞ on $\boldsymbol{C}\cup\{\infty\}$. (See Fig. 2.2. Let us name these three circles C_1, C_2, C_3 respectively and let a be an intersection point of two circles C_2 and C_3.)

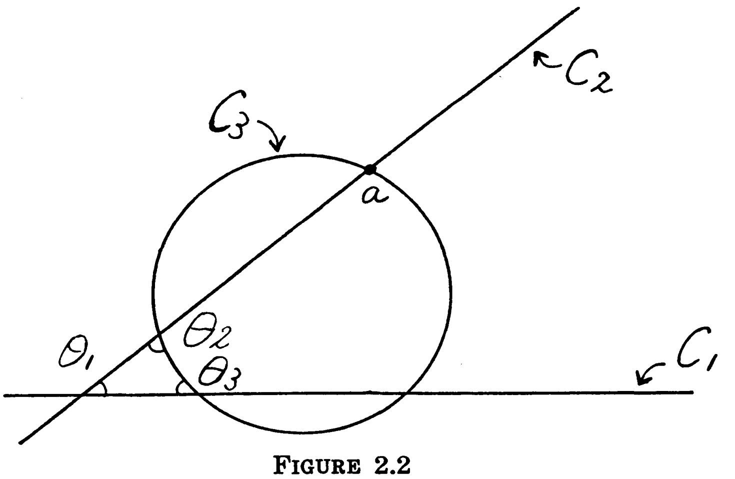


FIGURE 2.2

It is easy to see that the following assertion holds.

ASSERTION 2.2. *Let p be an arbitrary point on C_2. For any θ_5, $\theta_6 \in (0, \pi)$, where $\theta_1+\theta_5+\theta_6<\pi$, we can write the fourth circle C_4 such that the dihedral angles between the above two circles C_1 and C_2 are θ_5 and θ_6 respectively and that it passes through p* (see Fig. 2.3).

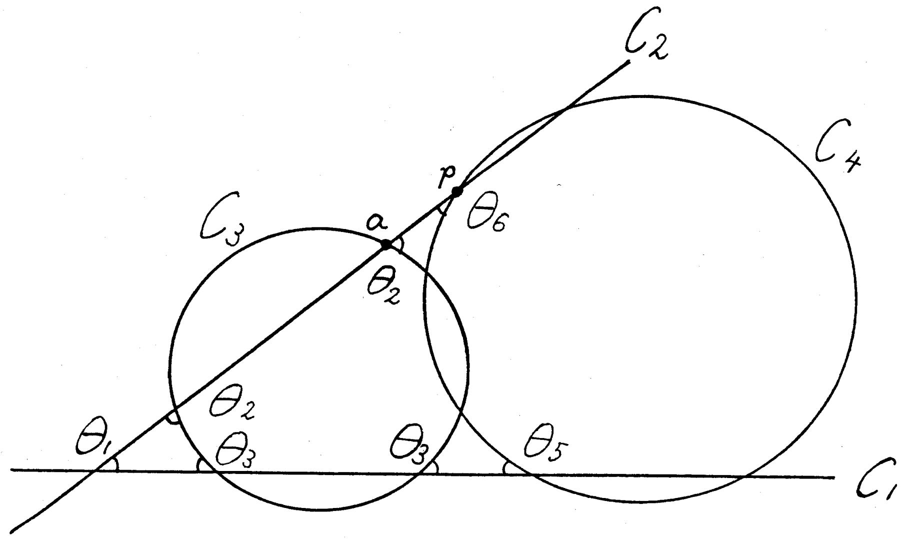


FIGURE 2.3

Consider the case as indicated in Fig. 2.4. This is the case when the dihedral angle between C_3 and C_4 is 0. Let b be an intersection of two circles C_2 and C_4 in this case.

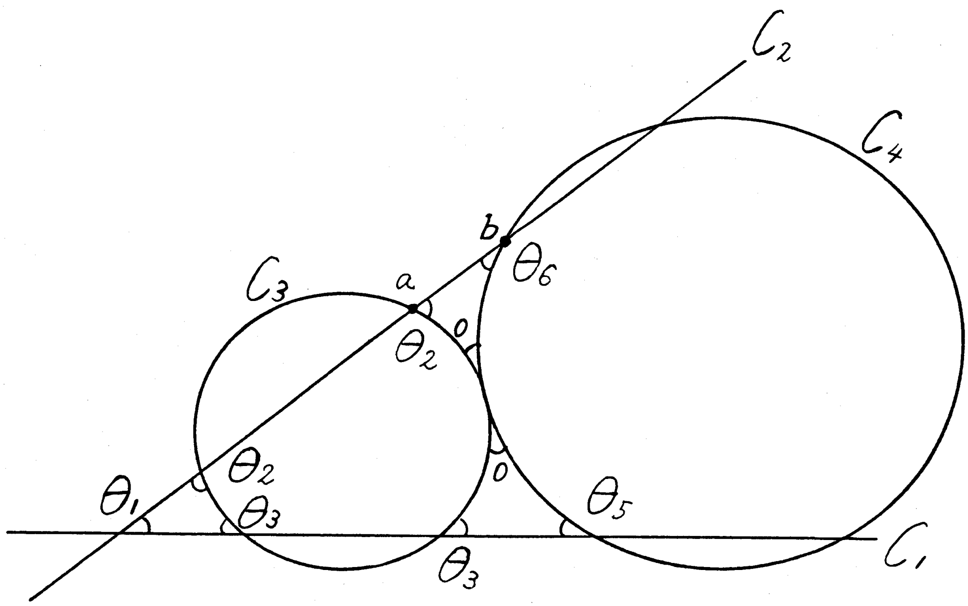


FIGURE 2.4

Now let us move p with the direction as illustrated in Fig. 2.5.

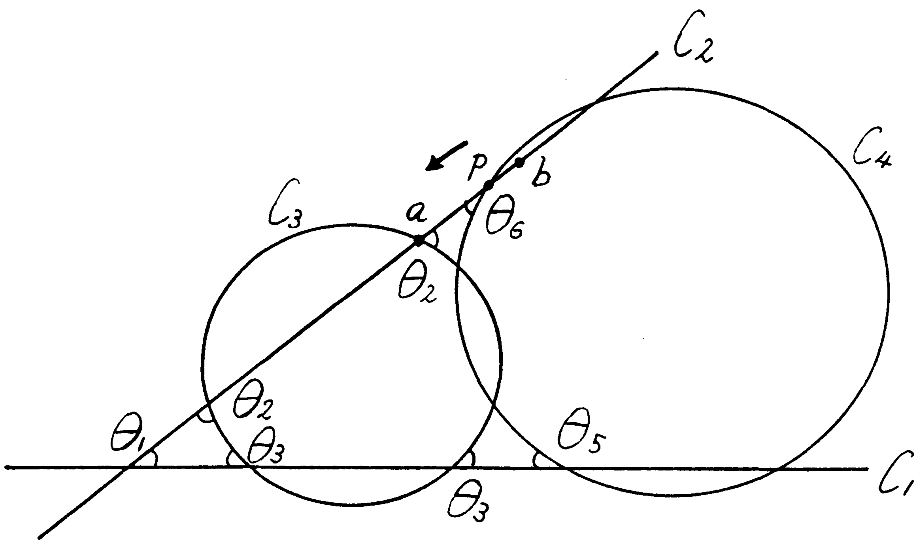


FIGURE 2.5

There is a point p between a and b such that the dihedral angle between C_3 and C_4 is equal to θ_4, because $\theta_4 \in (0, \pi)$ satisfies $\theta_2+\theta_4+\theta_6<\pi$ and $\theta_3+\theta_4+\theta_5<\pi$ (see Fig. 2.6).

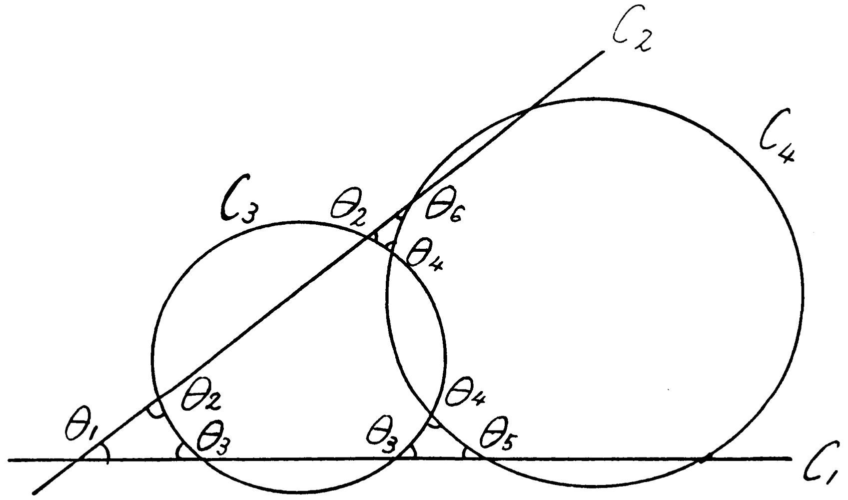


FIGURE 2.6

The above configuration of four circles is unique up to conformal transformation.

ASSERTION 2.3. *For any configuration of three circles such that there is a hyperbolic triangle between these three circles, there exists a unique circle which intersects the three circles perpendicularly* (see Fig. 2.7).

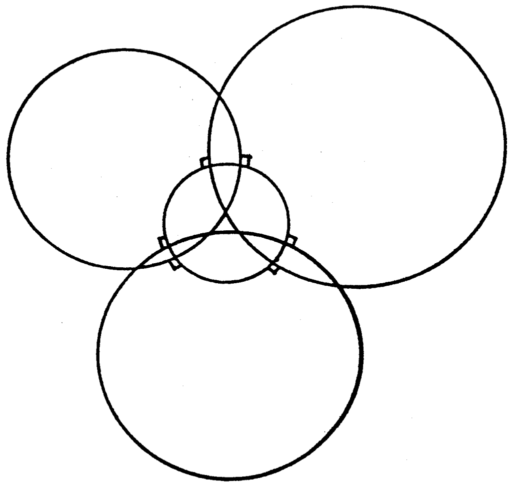


FIGURE 2.7

PROOF OF ASSERTION 2.3. By a conformal transformation, we can transfer one of the intersection points to ∞ on $C \cup \{\infty\}$ (see Fig. 2.8).

Then draw a circle centered at 0 which passes through the point of tangency of one of the three circles (see Fig. 2.9).

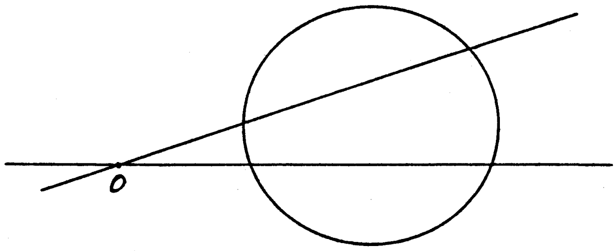


FIGURE 2.8

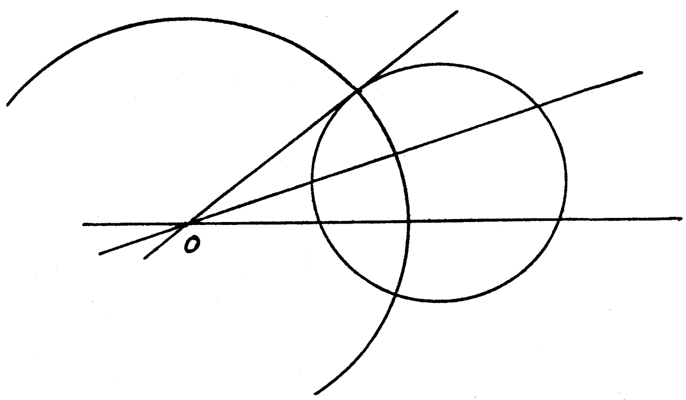


FIGURE 2.9

This circle perpendicularly intersects the remaining two circles. $\square$

Now consider the hyperbolic 3-space $\boldsymbol{H}^3$ which is bounded by $\boldsymbol{C}\cup\{\infty\}$. By applying this Assertion 2.3 to our circles, we can say that in $\boldsymbol{H}^3$ there are four geodesic hemispheres each of which intersects the three geodesic hemispheres perpendicularly. These geodesic hemispheres are hypersurfaces in $\boldsymbol{H}^3$ each of which has the above circle as its boundary on $\boldsymbol{C}\cup\{\infty\}$ (observe Fig. 2.10).

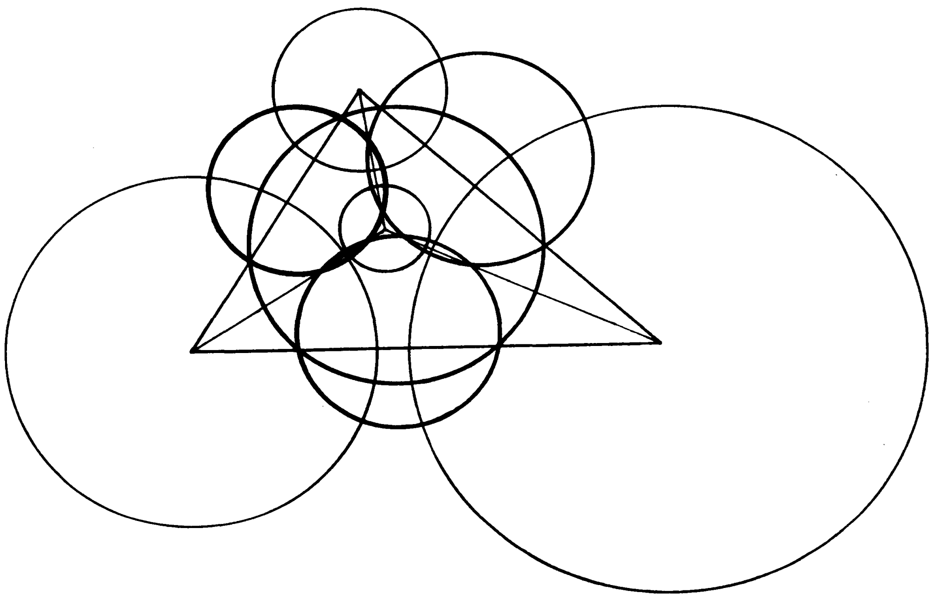


FIGURE 2.10

Then consider a polyhedron which is bounded by the above eight geodesic hemispheres. This is the required hyperbolic truncated tetrahedron. □

§ 3. Hyperbolic 3-manifolds with totally geodesic boundary which can be obtained by gluing two hyperbolic truncated tetrahedra.

In this section we deal with hyperbolic 3-manifolds constructed by identifying all hexagonal faces of two hyperbolic truncated tetrahedra with isometries. Actually we have:

THEOREM 3.1. *There are exactly eight mutually non-isometric compact oriented hyperbolic 3-manifolds with totally geodesic boundary such that they can be decomposed into two hyperbolic truncated tetrahedra and that their boundaries are closed surfaces of genus* 2.

We prepare the following two lemmas to prove Theorem 3.1.

LEMMA 3.2. *A boundary surface of a hyperbolic 3-manifold which can be decomposed into two hyperbolic truncated tetrahedra is a totally* *geodesic closed surface of genus* 2.

LEMMA 3.3. *Let M^3 be a complete hyperbolic 3-manifold which is decomposed into two hyperbolic truncated tetrahedra and has a totally geodesic boundary ∂M^3. Then there is a unique minimizing geodesic which intersects ∂M^3 perpendicularly at the both ends of it, and it is the edge which is constructed by identifying all edges of the two hyperbolic truncated tetrahedra.*

PROOF OF LEMMA 3.2. First let us consider the hyperbolic truncated tetrahedron as a purely topological one and let us consider the gluing diagram as a purely combinatorial one.

Let us remove neighborhoods of all edges of truncated tetrahedra as in Fig. 3.1.

See the stunted hexagon face of these removed truncated tetrahedra (look at Fig. 3.2).

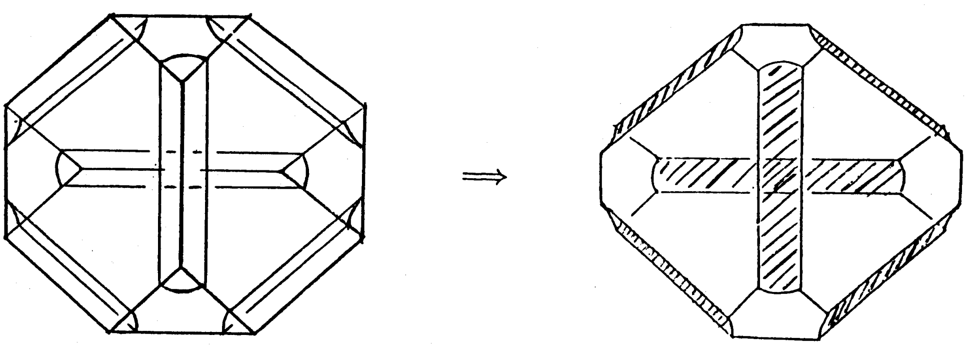


FIGURE 3.1

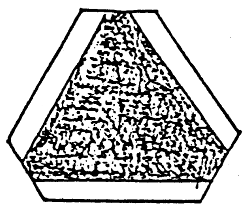


FIGURE 3.2

Then glue these stunted hexagons of truncated tetrahedra according to some gluing diagram such that the resulting object is connected. Topologically we obtain a 3-dimensional handlebody of genus 3, and on this boundary there exist some annuli which are decomposed into the shaded portion in Fig. 3.1. (They do not have any intersections.) Then attach back the removed neighborhoods of edges along these annuli according to the gluing diagram. This means that we attach some $D^2\times[0, 1]$'s along these annuli. Let us retract these annuli to simple closed curves each of which is a centerline of the annulus. Then *topologically* we can regard this resulting 3-manifold as a 3-manifold which is constructed by attaching $D^2\times[0, 1]$'s to a handlebody of genus 3 along some simple closed curves. If the number of these closed curves are greater than or equal to 2, then the boundary of the resulting 3-manifold is a torus or S^2. Now if we try to give a hyperbolic metric to this resulting 3-manifold, then these boundary surfaces cannot be totally geodesic. Thus, the number of these closed curves must be 1. In this case, all edges of hexagons each of which is a boundary edge of the adjacent two hexagons in one truncated tetrahedron are identified by the gluing diagram. Now there is one simple closed curve on the boundary of the handlebody of genus 3. This curve intersects three times with at least one non trivial homology representative of the boundary of the above handlebody. (Consider one of the identified stunted hexagons. The boundary of this hexagon becomes a representative of the boundary of the above handlebody. Observe Fig. 3.3.)

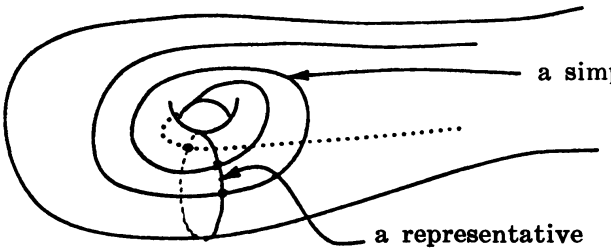


FIGURE 3.3

Then this simple closed curve is not homologous to zero. Therefore the boundary of the resulting 3-manifold in this case is a closed surface of genus 2. □

PROOF OF LEMMA 3.3. In the situation indicated in the lemma, all edges of hyperbolic truncated tetrahedra are identified so that all dihedral angles are $\pi/6$ (see Fig. 3.4).

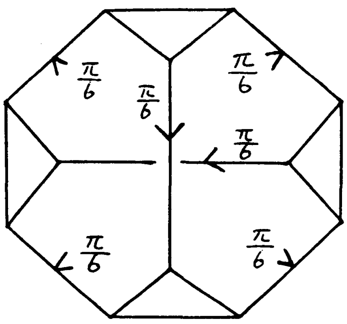
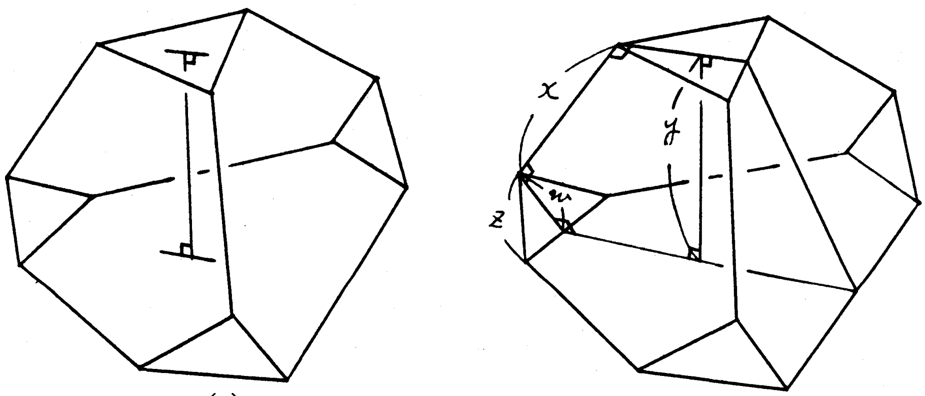


FIGURE 3.4. The dihedral angles are $\pi/2$ except where indicated.

In a hyperbolic truncated tetrahedron, there is a geodesic which leaves a boundary triangle and reaches another one and intersects them perpendicularly at the both ends. It is exactly an edge of adjacent two right-angled hexagons of a hyperbolic truncated tetrahedron. Also there exists the minimizing geodesic which leaves a boundary triangle of a hyperbolic truncated tetrahedron and does not reach another boundary triangle. It is exactly the geodesic which leaves a boundary triangle perpendicularly and reaches the opposite side right-angled hexagon and intersects them perpendicularly at the both ends (look at Fig. 3.5 (a)).

Let us take x, y, z, w as indicated in Fig. 3.5 (b).



(a) (b)

FIGURE 3.5. Values x, y, z, w give the lengths of the corresponding geodesic arcs.

Now we claim

$$y>x\,. \tag{$*$}$$

If we can show this, then the proof of Lemma 3.3 is complete.

Proof of ($*$). We calculate the geodesic arc lengths x, y, z, w by using the formulae in Proposition 1.1. Fig. 3.6 (a) is a picture of one of the boundary triangles, so

$$\begin{aligned}\cosh z &= \frac{\cos \pi/6 \cos \pi/6 + \cos \pi/6}{\sin \pi/6 \sin \pi/6} \\ &= 3+2\sqrt{3}\,.\end{aligned}$$

See Fig. 3.6 (b). This is a picture of one of the boundary right-angled hexagons, so

$$\begin{aligned}\cosh x &= \frac{\cosh z \cosh z + \cosh z}{\sinh z \sinh z} \\ &= \frac{3+\sqrt{3}}{4}\,.\end{aligned}$$

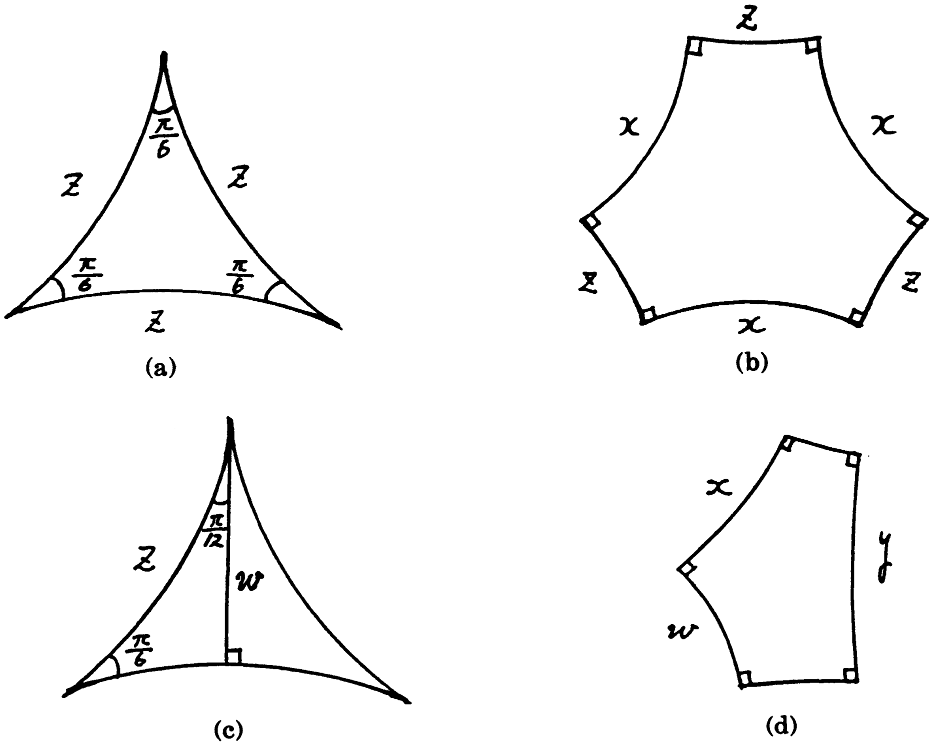


FIGURE 3.6

The geodesic arc corresponding to w in Fig. 3.6 (c) is a middle line of this triangle, so

$$\begin{aligned}\cosh w &= \frac{\cos \pi/6}{\cos \pi/12} \\ &= \sqrt{6+3\sqrt{3}}\ .\end{aligned}$$

Then by the formula for a right-angled pentagon, we get

$$\begin{aligned}\cosh y &= \sinh x \sinh w \\ &= \sqrt{\frac{17+9\sqrt{3}}{8}} \quad \text{(look at Fig. 3.6 (d))}\ .\end{aligned}$$

Thus

$$\cosh y > \cosh x\ .$$

Therefore

$$y > x\ . \qquad \square$$

Now we give the proof of Theorem 3.1. The proof is divided into nine steps (1), $\cdots$, (9).

(1) We mark all faces and all edges of hyperbolic truncated tetrahedra as indicated in Fig. 3.7. From now on, an edge of hyperbolic

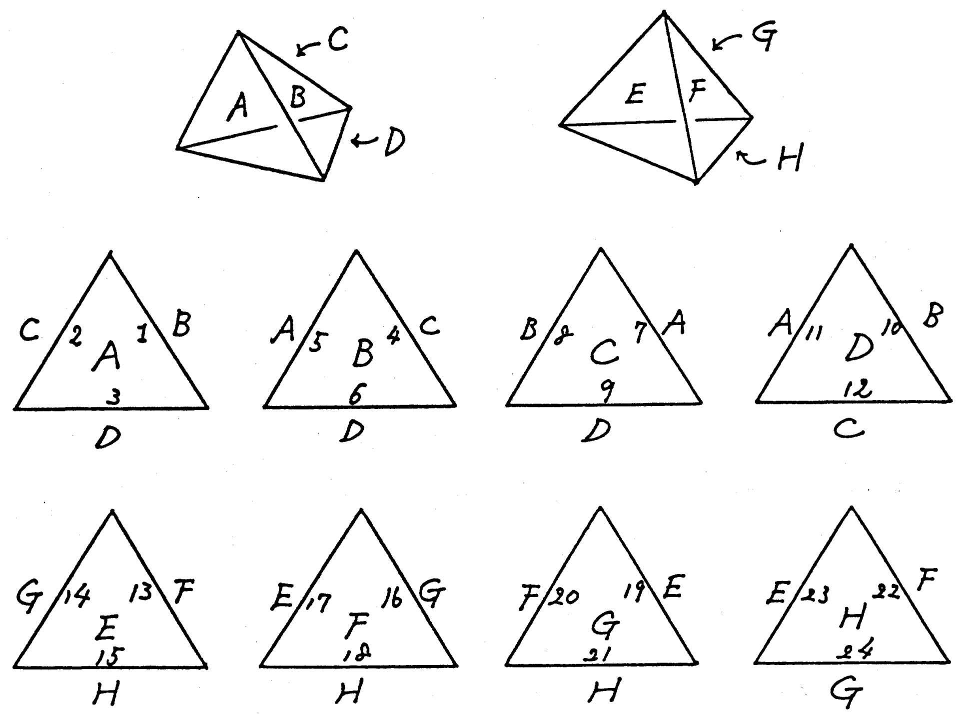


FIGURE 3.7. Numbers 1, $\cdots$, 24 are the labels of the corresponding edges of the boundary triangles of the tetrahedra.

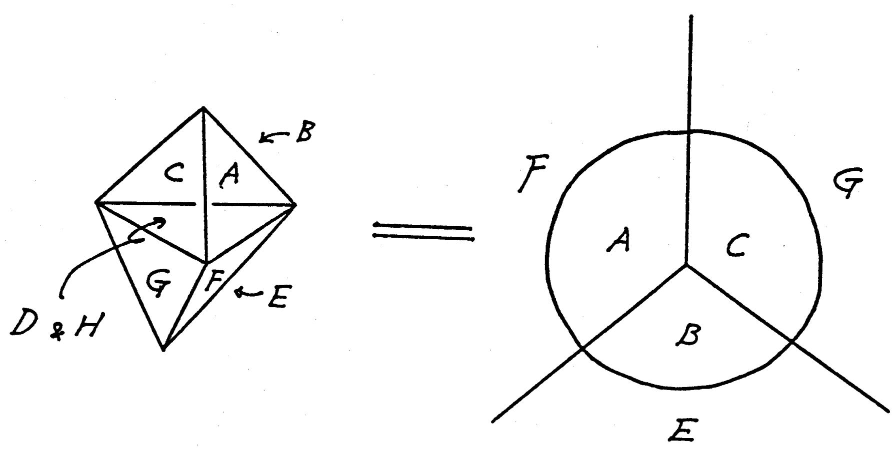


FIGURE 3.8. In the right picture, one vertex has been removed so that the polyhedron can be flattened out in the plane.

truncated tetrahedron is assumed not to be an edge of a boundary triangle unless otherwise indicated and we indicate hyperbolic truncated tetrahedra as purely Euclidean tetrahedra including all vertices.

Now we are considering connected 3-manifolds, hence there is at least one glued pair of faces between the two tetrahedra. So, we may consider that face D and face H are always glued (see Fig. 3.8).

(2) There are two cases of glued face pairs of tetrahedra:

case (α) $(A\text{-}E,\ B\text{-}G,\ C\text{-}F)$,
case (β) $(A\text{-}E,\ B\text{-}C,\ F\text{-}G)$.

(i.e., the resulting 3-manifolds by other identifications are isometric to some resulting 3-manifolds in the above two cases.)

Proof of (2). There are

$$\frac{{}_6C_2 \times {}_4C_2 \times {}_2C_2}{{}_3P_3} = 15$$

cases of glued face pairings of tetrahedra.

They are indicated below:

$(A\text{-}E,\ B\text{-}G,\ C\text{-}F)$, $(A\text{-}E,\ B\text{-}F,\ C\text{-}G)$,
$(A\text{-}E,\ B\text{-}C,\ F\text{-}G)$,
$(A\text{-}G,\ B\text{-}F,\ C\text{-}E)$, $(A\text{-}G,\ B\text{-}E,\ C\text{-}F)$,
$(A\text{-}G,\ B\text{-}C,\ E\text{-}F)$,
$(A\text{-}F,\ B\text{-}E,\ C\text{-}G)$, $(A\text{-}F,\ B\text{-}G,\ C\text{-}E)$,
$(A\text{-}F,\ B\text{-}C,\ E\text{-}G)$,
$(A\text{-}B,\ C\text{-}E,\ F\text{-}G)$, $(A\text{-}B,\ C\text{-}F,\ E\text{-}G)$,
$(A\text{-}B,\ C\text{-}G,\ E\text{-}F)$,
$(A\text{-}C,\ B\text{-}E,\ F\text{-}G)$, $(A\text{-}C,\ B\text{-}F,\ E\text{-}G)$,
$(A\text{-}C,\ B\text{-}G,\ E\text{-}F)$.

Apparently we can reduce them to five cases below:

$(A\text{-}E,\ B\text{-}G,\ C\text{-}F)$,
$(A\text{-}E,\ B\text{-}F,\ C\text{-}G)$,
$(A\text{-}E,\ B\text{-}C,\ F\text{-}G)$,
$(A\text{-}F,\ B\text{-}E,\ C\text{-}G)$,
$(A\text{-}F,\ B\text{-}C,\ E\text{-}G)$.

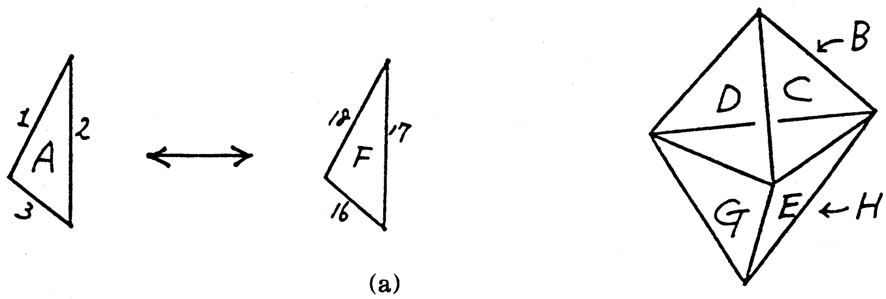
(i) Case (A-E, B-F, C-G). In this case, we cannot construct a gluing diagram such that all edges are identified. Therefore we take off this case.

(ii) Case (A-F, B-E, C-G).

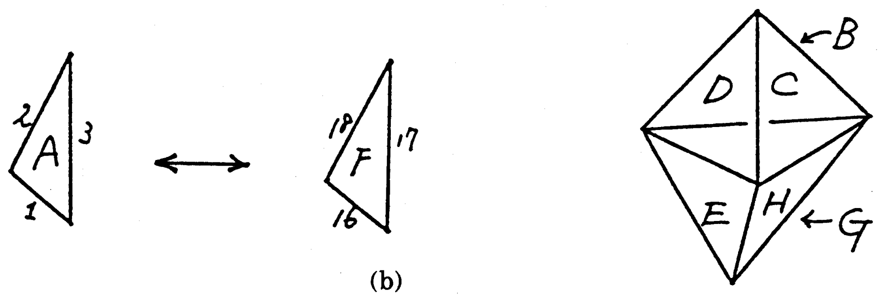
(ii_1) Case (1-18, 2-17, 3-16). See Fig. 3.9 (a). In this case, this 3-manifold is a mirror image to some 3-manifold which is realized in the case (A-E, B-G, C-F).

(ii_2) Case (1-17, 2-16, 3-18). We cannot construct a gluing diagram such that all edges are identified.

(ii_3) Case (1-16, 2-18, 3-17). See Fig. 3.9 (b). In this case, this 3-manifold is the same as some 3-manifold which is realized in the case (A-E, B-G, C-F).



(a)



(b)

FIGURE 3.9

By (ii_1), (ii_2), (ii_3), the case (ii) are all reduced to the case (A-E, B-G, C-F).

(iii) Case (A-F, B-C, E-G).

In the same way as the case (ii), the case (iii) are all reduced to the case (A-E, B-C, F-G).

Then, by (i), (ii), (iii), all cases are reduced to the two cases (α) and (β). □

(3) In the case (α), there are sixteen gluing diagrams such that all edges are identified. Though, up to isometry, there are three cases at the most and they are (a), (b), (j). (See Table 3.10. For example, let (a)

TABLE 3.10

1	15	15	15	15	15	15	14	14	14	14	13	13	13	13	13	13
2	14	14	14	14	14	14	13	13	13	13	15	15	15	15	15	15
3	13	13	13	13	13	13	15	15	15	15	14	14	14	14	14	14
4	21	21	20	20	19	19	21	21	19	19	21	21	20	20	19	19
5	20	20	19	19	21	21	20	20	21	21	20	20	19	19	21	21
6	19	19	21	21	20	20	19	19	20	20	19	19	21	21	20	20
7	18	17	18	16	17	16	18	16	18	16	17	16	18	16	18	17
8	17	16	17	18	16	18	17	18	17	18	16	18	17	18	17	16
9	16	18	16	17	18	17	16	17	16	17	18	17	16	17	16	18
	(*a*)	(*b*)	(*c*)	(*d*)	(*e*)	(*f*)	(*g*)	(*h*)	(*i*)	(*j*)	(*k*)	(*l*)	(*m*)	(*n*)	(*o*)	(*p*)

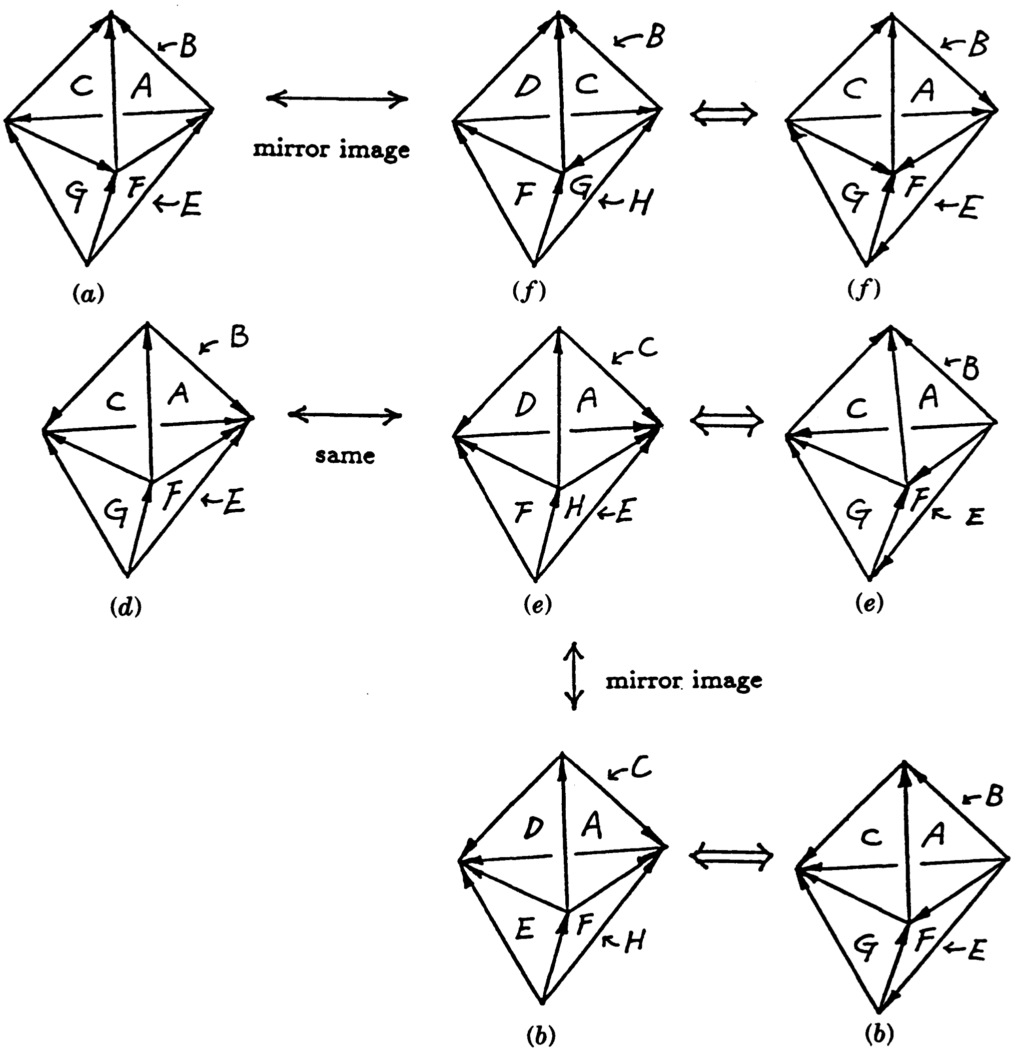


FIGURE 3.11

be the resulting 3-manifold by gluing the faces of two truncated tetrahedra according to the gluing diagram of type (a).)

Proof of (3). Immediately we can see that

$$
\begin{aligned}
&(b)\cong(c)\cong(g)\ ,\\
&(d)\cong(i)\cong(h)\ ,\\
&(e)\cong(h)\cong(m)\ ,\\
&(f)\cong(l)\cong(o)\ ,
\end{aligned}
$$

where $\cong$ means that both sides are isometric. So, up to isometry, there are at most six cases (i.e., (a), (b), (d), (e), (f), (j)).

Next, we shall say that

$$
\begin{aligned}
&(a)\cong(f)\ ,\\
&(d)\cong(e)\cong(h)\ ,
\end{aligned}
$$

but these are apparent by Fig. 3.11.

Thus, we have reached the conclusion. $\square$

(4) In the case (β), there are eight gluing diagrams such that all edges are identified. But, up to isometry, there are five cases at the most and they are (q), (r), (u), (v), (x). (See Table 3.12.)

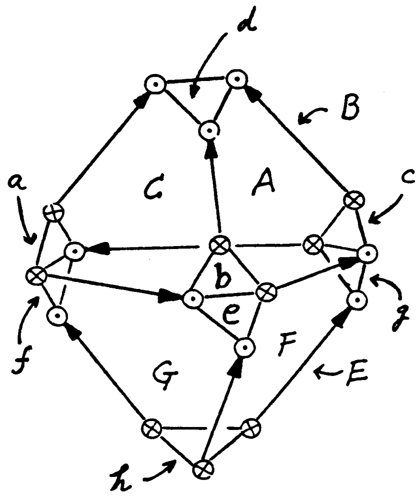
TABLE 3.12

1	15	15	15	15	14	14	14	14
2	14	14	14	14	13	13	13	13
3	13	13	13	13	15	15	15	15
4	9	9	7	7	9	9	7	7
5	8	8	9	9	8	8	9	9
6	7	7	8	8	7	7	8	8
16	21	19	21	19	21	19	21	19
17	20	21	20	21	20	21	20	21
18	19	20	19	20	19	20	19	20
	(q)	(r)	(s)	(t)	(u)	(v)	(w)	(x)

Proof of (4). In the same way as Step (3), it is shown that $(r)\cong(s)$, $(v)\cong(m)$ and $(q)\cong(t)$. Thus, we have reached the conclusion. $\square$

(5) By Lemma 3.3, if there are any isometries between (a), (b), (j), (q), (r), (u), (v), (x), then there are correspondences between the edges of the two truncated tetrahedra which are isometric to each other.

(6) Mark the both ends of an edge $\uparrow$ with $\odot$, $\otimes$. (See Fig. 3.13.) Now, we distinguish (a), (b), (j), (q), (r), (u), (v), (x) combinatorially.



example case (a)

FIGURE 3.13

If there is an isometric pair of 3-manifolds, then there is a correspondence

$$\begin{cases}\odot \longleftrightarrow \odot \\ \otimes \longleftrightarrow \otimes\end{cases} \quad \text{or} \quad \begin{cases}\odot \longleftrightarrow \otimes \\ \otimes \longleftrightarrow \odot\,.\end{cases}$$

Observe Fig. 3.14. On their boundaries, let us count the number of the edges of boundary triangles which connect $\odot$ and $\otimes$. In the case (a) and (j), there are 6. In the case (b), there are 4. In the case (q), (r), (u), (v), and (x), there are 8.

Then we have to distinguish

$$(a) \quad \text{and} \quad (j)\,,$$

and

$$(q),\ (r),\ (u),\ (v) \quad \text{and} \quad (x)\,.$$

(7) Watching the ordering of $\odot$ and $\otimes$ with $\odot$ as the center, we can distinguish (q), (r), (u), (v), (x) without the differences among (q), (r), (u).

(8) We can conclude that $(a)\not\cong(j)$.

Proof of (8). Assume $(a)\cong(j)$. This isometry maps $\uparrow$ of (a) to (or $\downarrow$) of (j). Then by the symmetry of these two truncated tetrahedra, we

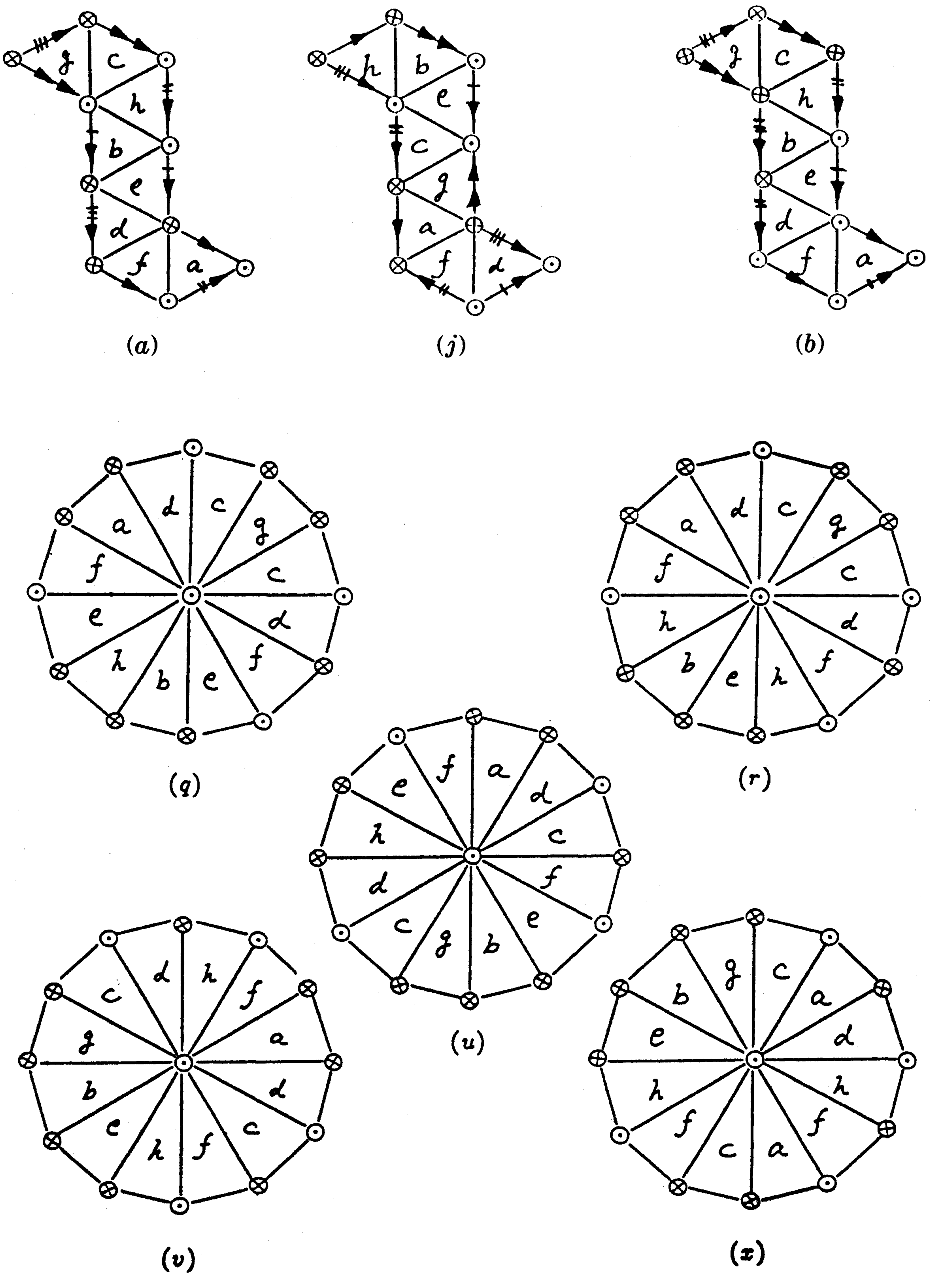
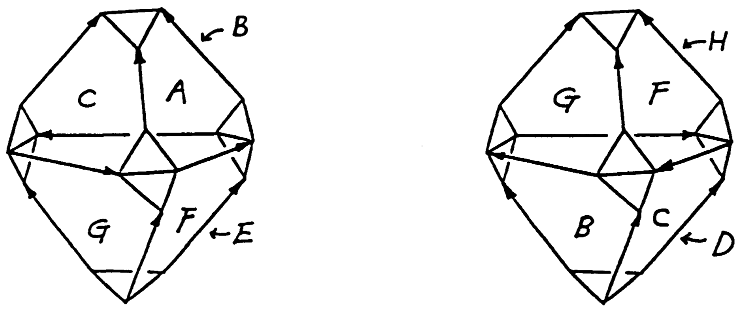


FIGURE 3.14

may assume that this isometry maps the glued face pair C–F of (a) to G–B of (j). (See Fig. 3.15.)

In this case, this isometry maps

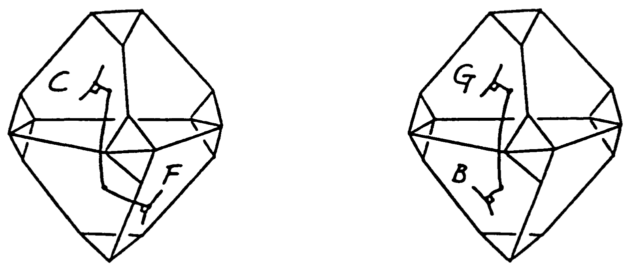
$$B\text{–}G \text{ of } (a) \longrightarrow F\text{–}C \text{ of } (j)\,,$$
$$A\text{–}E \text{ of } (a) \longrightarrow H\text{–}D \text{ of } (j)\,.$$



(a) (j)

FIGURE 3.15

Now note that the center of (a) corresponds to the center of (j). Draw perpendicular lines from the centers of (a) and (j), to C–F and G–B respectively. (See Fig. 3.16.)



(a) (j)

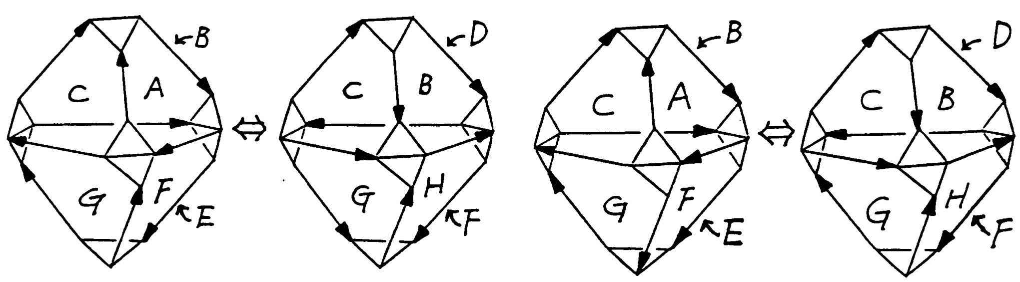
FIGURE 3.16

See the broken lines in (a) and (j). By our isometry, they must be correspondent to each other, which is a contradiction. □

(9) We can conclude that

$$(q)\not\cong(r)\ ,\quad (r)\not\cong(u)\ ,\quad (u)\not\cong(q)\ .$$

Proof of (9). Assume that $(q)\cong(r)$. Look at Fig. 3.17.



(q) (r)

FIGURE 3.17

By the above isometry, $\uparrow$ of (q) corresponds to $\uparrow$ (or $\downarrow$) of (r). Say $\uparrow \leftrightarrow \uparrow$. Combinatorially considering, tetrahedron $ABCD$ of (q) must correspond to tetrahedron $ABCD$ of (r), and there are two cases of correspondences between the faces of tetrahedra:

$$\begin{cases} A \text{ of } (q) \longleftrightarrow A \text{ of } (r) \\ B \text{ of } (q) \longleftrightarrow B \text{ of } (r) \\ C \text{ of } (q) \longleftrightarrow C \text{ of } (r) \\ D \text{ of } (q) \longleftrightarrow D \text{ of } (r), \end{cases} \quad \text{or} \quad \begin{cases} A \text{ of } (q) \longleftrightarrow D \text{ of } (r) \\ B \text{ of } (q) \longleftrightarrow C \text{ of } (r) \\ C \text{ of } (q) \longleftrightarrow B \text{ of } (r) \\ D \text{ of } (q) \longleftrightarrow A \text{ of } (r). \end{cases}$$

In this case, tetrahedron $EFGH$ of (q) must correspond to tetrahedron $EFGH$ of (r), which is impossible and is a contradiction.

In the case when $\uparrow$ of (q) corresponds to $\downarrow$ of (r), by combinatorial reason, we also have a contradiction.

$(q) \not\cong (u)$ is shown similarly to the case $(a) \not\cong (j)$.

$(r) \not\cong (u)$ is shown similarly to the case $(q) \not\cong (r)$. $\square$

By (1), $\cdots$, (9), Theorem 3.1 has been proved completely. $\square$

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