TOKYO J. MATH.
VOL. 14, NO. 1, 1991

A Proof of Thurston's Uniformization Theorem of Geometric Orbifolds

Yukio MATSUMOTO and José María MONTESINOS-AMILIBIA

University of Tokyo and Universidad Complutense de Madrid
(Communicated by N. Iwahori)

Dedicated to Professor Tatsuo Homma in his fifth Jupiterian year

§1. The uniformization theorem.

A smooth m-dimensional orbifold (briefly, an m-orbifold) is a σ-compact Hausdorff space M which is locally modelled on a quotient space of a finite group action on a smooth m-dimensional manifold ([Sa], [Th]). More precisely, an *m-orbifold M* is covered by an *atlas* of *folding charts* $\{(\tilde{U}_i, G_i, f_i, U_i)\}_{i\in I}$, each chart consisting of a smooth connected m-manifold $\tilde{U}_i$, a finite group G_i acting on $\tilde{U}_i$ smoothly and effectively, an open set U_i of M and a *folding map* $f_i : \tilde{U}_i \to U_i$ which induces a natural homeomorphism $G_i \backslash \tilde{U}_i \to U_i$. These charts must satisfy a certain compatibility condition. In a simplified version due to Bonahon and Siebenmann [BS], the condition states the following: for every $x \in \tilde{U}_i$ and $y \in \tilde{U}_j$ such that $f_i(x) = f_j(y) \in U_i \cap U_j$, there exists a diffeomorphism $\psi : \tilde{V}_x \to \tilde{V}_y$ from an open neighborhood of x in $\tilde{U}_i$ to an open neighborhood of y in $\tilde{U}_j$ such that $\psi(x) = y$ and $f_j\psi = f_i$. (For an explanation of this compatibility condition, see Appendix A.)

Two atlases on an m-orbifold give the same orbifold structure iff their union is again a compatible atlas.

For instance, let Γ be a group acting on a manifold $\tilde{U}$ smoothly, effectively and properly discontinuously. Then the quotient space $\Gamma \backslash \tilde{U}$ has the structure of a smooth orbifold. This type of an orbifold is said to be *good*.

The notion of being good is defined more formally in terms of orbifold coverings as follows. Let $h : N \to M$ be a continuous map of a connected orbifold N onto another orbifold M. h is called an *orbifold covering* of M if M admits an atlas of folding charts $\{(\tilde{U}_i, G_i, f_i, U_i)\}_{i\in I}$ such that, for each component V of $h^{-1}(U_i)$, there exists a folding chart $k : \tilde{U}_i \to V$ in the maximal atlas of N so that $f_i = hk$.

DEFINITION ([Th]). A connected orbifold M is *good* if there exists an orbifold

Received June 6, 1990

covering $h: N \to M$ in which N is a manifold.

Now suppose that we are given a *geometry* (G, X) consisting of a smooth connected manifold X and a subgroup G of Diff(X). Here Diff(X) denotes the group of all diffeomorphisms of X onto itself. Note that in our definition X need not be a Riemannian manifold. Then assuming, for each folding chart $(\tilde{U}_i, G_i, f_i, U_i)$, that G_i is a finite subgroup of G and $\tilde{U}_i$ is a G_i-invariant open set of X, and also that each gluing map $\psi: \tilde{V}_x \to \tilde{V}_y$ (that appeared in the compatibility condition) is the restriction of a member of G, we obtain the notion of a *geometric orbifold modelled on* (G, X), or simply, a (G, X)-*orbifold*. Clearly a (G, X)-orbifold has the same dimension as X.

In §13 of his lecture notes [Th], Thurston stated a proposition (Proposition 13.3.2) that *if G is an analytic group of diffeomorphisms of a real analytic manifold X, then every (G, X)-orbifold is good.* Thurston did not give a detailed proof of this "uniformization theorem" of geometric orbifolds.

As far as we know, there is no published proof of this theorem in the literature, though it is tacitly assumed in some articles (cf. [Du]).

The purpose of this paper is to give a proof of this theorem of Thurston. In order to clarify the essential point, we will prove a slightly more general statement than the original one, in which we will impose the following condition $(*)$ to our geometry (G, X) instead of its analyticity:

$(*)$ If two diffeomorphisms belonging to G coincide on a non-empty open set of X, then they coincide identically on X.

Of course, a real analytic geometry (G, X) satisfies condition $(*)$. As another example, if G is a *finite group* of diffeomorphisms of X, (G, X) satisfies condition $(*)$, ([Nw], [Dr]).

The uniformization theorem to be proved in this paper is as follows:

THEOREM 1.1. *If (G, X) satisfies condition $(*)$, then every (G, X)-orbifold is good.*

This theorem will be proved in §5 after some preliminary observations made in §2–§4. In §6 we will give an application. Appendix A indicates the proof of the fact that Bonahon and Siebenmann's compatibility condition assures the well-definedness of the isotropy group attached to a point. Finally, Appendix B sketches "silvered covering" theory and the orientable double covering of an orbifold.

§2. Developing maps.

The proof of the uniformization theorem heavily depends on *developing maps* which we will briefly review in this section, (see [Th, §3]). Unless otherwise stated, the geometry (G, X) will be assumed to satisfy condition $(*)$ of §1.

A (G, X)-*manifold* means a (G, X)-orbifold for which every G_i is trivial, or equivalently, every folding chart $f_i : \tilde{U}_i \to U_i$ is a homeomorphism. In this case, we will simply call $f_i : \tilde{U}_i \to U_i$ a *chart*.

DEFINITION. Let M_1 and M_2 be (G, X)-manifolds. *A* map $D : M_1 \to M_2$ is called a *(G, X)-map*, if for every chart $f_i : \tilde{U}_i \to U_i$ of M_1 and for every chart $g_j : \tilde{V}_j \to V_j$ of M_2 such that $U_i \cap D^{-1}(V_j) \neq \varnothing$, the composition

$$g_j^{-1} D f_i : f_i^{-1}(U_i \cap D^{-1}(V_j)) \to \tilde{V}_j$$

extends to a diffeomorphism $X \to X$ which is a member of G. (Note that the sets $f_i^{-1}(U_i \cap D^{-1}(V_j))$ and $\tilde{V}_j$ are both open sets of X.) In the special case when $M_2 = X$, a (G, X)-map $D : M_1 \to X$ is called a *developing map*.

REMARK. To check if $D : M_1 \to M_2$ is a (G, X)-map, it is only necessary to check for a set of charts covering M_1 and M_2 (not for all) due to the compatibility condition of charts in the definition of orbifold modelled on (G, X).

Given a (G, X)-manifold M, a developing map $D : M \to X$ does not always exists. However, we have the following:

LEMMA 2.1. *The universal covering $\tilde{M}$ of a (G, X)-manifold M admits a developing map $D : \tilde{M} \to X$.*

Take a point $p \in M$, and let $f_i : \tilde{U}_i \to U_i$ be a chart with U_i containing p. Then $f_i^{-1} : U_i \to \tilde{U}_i \subset X$ is a developing map of U_i. One can extend this map along paths in M to obtain the required developing map $D : \tilde{M} \to X$. Condition $(*)$ imposed to our geometry (G, X) assures the well-definedness of this construction. Compare [Th, §3].

LEMMA 2.2. *If $D : M \to X$ is a developing map and $h : N \to M$ is an unbranched covering, then the composition $Dh : N \to X$ is a developing map.*

Whenever we speak of an unbranched covering $h : N \to M$ of a (G, X)-manifold M, the (G, X)-manifold structure on N will be the one inherited from M. Thus h is always a (G, X)-map. Lemma 2.2 would be self-evident.

LEMMA 2.3. *If M is a connected (G, X)-manifold, two developing maps D, D' : $M \to X$ differ only by an element of G, namely, there exists a diffeomorphism $g : X \to X$ which belongs to G so that $D' = gD$.*

PROOF. Let $p \in M$ be a point, $f_i : \tilde{U}_i \to U_i$ a chart containing p. By the definition of a developing map, the maps Df_i, $D'f_i : \tilde{U}_i \to X$ extend to diffeomorphisms g_i, g'_i : $X \to X$ which belong to G. Define $g := g'_i g_i^{-1} \in G$. Then $D' = gD$ holds on $f_i(\tilde{U}_i)$. Let $f_j : \tilde{U}_j \to U_j$ be another chart with U_j having non-empty intersection with U_i. Condition $(*)$ assures that $D' = gD$ also holds on $f_j(\tilde{U}_j)$. Proceeding in this way, we can prove $D' = gD$ on X by the connectivity of M. □

Let $D : \tilde{M} \to X$ be a developing map of the universal covering $\tilde{M}$ of a (G, X)-manifold

M. Let $l \in \pi_1(M)$ be an element of the fundamental group of M, and $T_l : \tilde{M} \to \tilde{M}$ the corresponding covering translation. Then, by Lemma 2.2, $DT_l : \tilde{M} \to X$ is a developing map, and by Lemma 2.3, D and DT_l differ by an element of G:

$$DT_l = gD, \qquad g \in G.$$

The correspondence $l \mapsto g$ gives a *holonomy homomorphism*

$$H : \pi_1(M) \longrightarrow G.$$

If $D' : \tilde{M} \to X$ is a different developing map, then, by Lemma 2.3, there exists $h \in G$ such that

$$D' = hD.$$

Then

$$D'T_l = hDT_l = hgD = hgh^{-1}D'$$

and therefore H is uniquely defined up to conjugation in G.

LEMMA 2.4. *Let M_Γ be the covering of a (G, X)-manifold M corresponding to the subgroup $\Gamma \leq \pi_1(M)$. Then M_Γ admits a developing map $D : M_\Gamma \to X$ if and only if $\Gamma \leq \mathrm{Kernel}(H : \pi_1(M) \to G)$.*

PROOF. We will prove the "if" part. Let $h : \tilde{M} \to M_\Gamma$ be the covering between the universal covering $\tilde{M}$ of M and the covering M_Γ. Let $D : \tilde{M} \to X$ be a developing map given by Lemma 2.1. Then D preserves fibers of h. In fact, let $x, y \in \tilde{M}$ be points such that $h(x) = h(y)$. Then x, y correspond to two paths l_x, l_y based at $p \in M$ ending at $q \in M$ and such that $l_x l_y^{-1} \in \Gamma \leq \mathrm{Kernel}(H : \pi_1(M) \to G)$. This means that the developing $D : \tilde{M} \to X$ along l_x and l_y coincide at their ends. Therefore, $D(x) = D(y)$. This means that M_Γ admits a developing map $D : M_\Gamma \to X$.

The converse is proved also easily by the definition of a developing map. □

COROLLARY 2.4.1. *A (G, X)-manifold M admits a developing map $D : M \to X$ if and only if the holonomy $H : \pi_1(M) \to G$ is trivial.*

§3. Orbifold coverings as branched coverings.

In this section, we will investigate the relationship between orbifold coverings and Fox's branched coverings [F]. The results will be valid for general orbifolds which are not necessarily geometric.

Let M be an m-orbifold, p a point $\in M$. Let $(\tilde{U}_i, G_i, f_i, U_i)$ be a folding chart containing p. Then for a point $x \in \tilde{U}_i$ which is projected onto p by f_i, the isomorphism class of the isotropy group $(G_i)_x := \{g \in G_i \mid g(x) = x\}$ depends only on p (see Appendix A). We will call this isomorphism class the *isotropy group* of p and denote it by G_p.

The set $\Sigma M := \{p \in M \mid G_p \neq \{1\}\}$ is called the *singular set* of M. Its complement $M - \Sigma M$ is called the *non-singular part* of M and is denoted by M_*.

Obviously, M is a manifold if and only if $\Sigma M = \emptyset$. The singular set ΣM admits a natural stratification $\mathscr{S} = \{S_\alpha\}_{\alpha \in A}$ with the following properties:

(i) Each stratum S_α is a smooth connected manifold of dimension $\leq m-1$,

(ii) for any two points $p, q \in S_\alpha$, the isotropy groups G_p and G_q are the same, in other words, the isotropy group is "constant" along a stratum, and

(iii) $\bar{S}_\alpha - S_\alpha$ consists of strictly lower dimensional strata, where $\bar{S}_\alpha$ denotes the closure of S_α in M.

The stratification $\mathscr{S} = \{S_\alpha\}_{\alpha \in A}$ is easily constructed starting from the stratification of the singular set ΣU_i according to the orbit types of the action of G_i on $\tilde{U}_i$, and patching them together by the compatibility condition of charts, cf. [Ka].

The isotropy group corresponding to a stratum of codimension 1 or 2 is particularly simple:

(1) dim $S_\alpha = m-1$, iff the isotropy group G_p $(p \in S_\alpha)$ is a group of order 2 acting on $\tilde{U}_i$ as a reflection with respect to an $(m-1)$-submanifold (a "mirror"). The union of codimension 1 strata is called the *silvered boundary*.

(2) If dim $S_\alpha = m-2$, then the isotropy group G_p $(p \in S_\alpha)$ is a dihedral or a cyclic group. If M has no silvered boundary, it is necessarily a cyclic group of order $b_\alpha > 1$, say. The integer b_α is called the *order* of the codimension 2 stratum S_α.

In the rest of this section, we will assume that M is connected and without silvered boundary. Let μ_α be a "meridian" around a codimension 2 stratum S_α of order $b_\alpha > 1$. Define a subgroup $\langle \mu^b \rangle_M$ of $\pi_1(M_*)$ to be the subgroup which is normally generated by the totality of the elements of the form $(\mu_\alpha)^{b_\alpha}$, the b_α-th power of μ_α, where α runs over all the indices of the codimension 2 strata of ΣM. We will call $\langle \mu^b \rangle_M$ the *characteristic subgroup* of $\pi_1(M_*)$.

For an orbifold covering $h : N \to M$ (or more generally, for a branched covering along ΣM), we denote $N - h^{-1}(\Sigma M)$ by N_0. Thus N_0 is the part of N which spreads over the non-singular part M_*. Generally speaking, N_0 is contained in the non-singular part N_* $(= N - \Sigma N)$ of N, but may not coincide with N_*. These notations N_0 and N_* will sometimes be used without further comments.

The image of $\pi_1(N_0)$ under the induced homomorphism $(h|N_0)_* : \pi_1(N_0) \to \pi_1(M_*)$ is called the *subgroup of $\pi_1(M_*)$ associated with the orbifold covering* (or *the branched covering*) $h : N \to M$.

THEOREM 3.1. *Let M be a connected m-orbifold without silvered boundary. A branched covering $h : N \to M$ branched along ΣM is an orbifold covering if and only if the associated subgroup $(h|N_0)_*(\pi_1(N_0))$ contains the characteristic subgroup $\langle \mu^b \rangle_M$.*

REMARK. Conversely, an orbifold covering $h : N \to M$ is always a branched covering of M branched along ΣM.

Before proving Theorem 3.1, let us examine the local situation. Let M be as in Theorem 3.1. Fix any folding chart $(\tilde{U}_i, G_i, f_i, U_i)$ of M, and suppose that $\tilde{U}_i$ is 1-connected. Let $\langle \mu^b \rangle_{U_i}$ be the characteristic subgroup of $\pi_1(U_{i*})$, where $U_{i*} = U_i - \Sigma U_i$, the non-singular part of U_i.

LEMMA 3.2. *Let $h : V \to U_i$ be a branched covering branched along ΣU_i. Suppose that the associated subgroup $(h|V_0)_* \pi_1(V_0)$ $(\leq \pi_1(U_{i*}))$ contains the characteristic subgroup $\langle \mu^b \rangle_{U_i}$ for U_i, where $V_0 = V - h^{-1}(\Sigma U_i)$ as usual. Then there exists a folding map $k : \tilde{U}_i \to V$ so that the diagram*

$$\begin{array}{ccc} \tilde{U}_i & \xrightarrow{\;k\;} & V \\ & {\scriptstyle f_i}\searrow & \downarrow {\scriptstyle h} \\ & & U_i \end{array}$$

commutes.

PROOF. As an m-orbifold, U_i $(\cong G_i \backslash \tilde{U}_i)$ has a natural stratification $\mathscr{T} = \{T_\beta\}_{\beta \in B}$ of its singular set ΣU_i. Let T_β be any codimension 2 stratum, $\tilde{T}_\beta$ a lift of T_β to $\tilde{U}_i$. Then we can find an element of G_i which acts on $\tilde{U}_i$ as a rotation around $\tilde{T}_\beta$ through angle $2\pi/b_\beta$, b_β being the order of T_β. Thus for a meridian loop μ_β of T_β, its b_β-th power $(\mu_\beta)^{b_\beta}$ lifts to a meridian loop $\tilde{\mu}_\beta$ of $\tilde{T}_\beta$. This together with the 1-connectivity of $\tilde{U}_i$ implies that the subgroup $(f_i|\tilde{U}_{i0})_* \pi_1(\tilde{U}_{i0})$ of $\pi_1(U_{i*})$ coincides with the characteristic subgroup for U_i, $\langle \mu^b \rangle_{U_i}$, where $\tilde{U}_{i0} := \tilde{U}_i - f_i^{-1}(\Sigma U_i)$.

By the general theory of covering spaces, the group of covering translations of the (regular) unbranched covering $f_i|\tilde{U}_{i0} : \tilde{U}_{i0} \to U_{i*}$ is isomorphic to $\pi_1(U_{i*})/(f_i|\tilde{U}_{i0})_* \pi_1(\tilde{U}_{i0})$, but by the observation above, the latter group is isomorphic to $\pi_1(U_{i*})/\langle \mu^b \rangle_{U_i}$. On the other hand, the same group of covering translations of $f_i|\tilde{U}_{i0} : \tilde{U}_{i0} \to U_{i*}$ is isomorphic to G_i, because G_i acts on $\tilde{U}_{i0}$ freely and the orbit space is U_{i*}.

Comparing these isomorphisms, one obtains $G_i \cong \pi_1(U_{i*})/\langle \mu^b \rangle_{U_i}$.

Now let $h : V \to U_i$ be a branched covering satisfying the assumption of Lemma 3.2, namely, it is a branched covering branched along ΣU_i such that the associated subgroup $(h|V_0)_* \pi_1(V_0)$ $(\leq \pi_1(U_{i*}))$ contains $\langle \mu^b \rangle_{U_i}$. Recall that $\langle \mu^b \rangle_{U_i} = (f_i|\tilde{U}_{i0})_* \pi_1(\tilde{U}_{i0})$. Then $(h|V_0)_* \pi(V_0)$ turns out to contain $(f_i|\tilde{U}_{i0})_* \pi_1(\tilde{U}_{i0})$. This implies, by the usual covering space theory, that there exists an (unbranched) covering map $k_0 : \tilde{U}_{i0} \to V_0$ so that the diagram

$$\begin{array}{ccc} \tilde{U}_{i0} & \xrightarrow{\;k_0\;} & V_0 \\ & {\scriptstyle f_i|\tilde{U}_{i0}}\searrow & \downarrow {\scriptstyle h|V_0} \\ & & U_{i*} \end{array}$$

commutes. Moreover, $k_0 : \tilde{U}_{i0} \to V_0$ is a regular covering whose covering translation group K is isomorphic to $(h|V_0)_{\#}\pi_1(V_0)/\langle\mu^b\rangle_{U_i}$. Note that K is identified with a subgroup of G_i, because

$$K \cong (h|V_0)_{\#}\pi_1(V_0)/\langle\mu^b\rangle_{U_i} \leq \pi_1(U_{i*})/\langle\mu^b\rangle_{U_i} \cong G_i\,.$$

What we have proved so far is the fact that the unbranched covering $h|V_0 : V_0 \to U_{i*}$ is isomorphic to the natural quotient $K\backslash\tilde{U}_{i0} \to G_i\backslash\tilde{U}_{i0}$. Now by Fox [F], $h : V \to U_i$ is the completion of the *spread* $V_0 \to U_{i*} \subset U_i$, while $K\backslash\tilde{U}_i \to U_i$ is the completion of the spread $K\backslash\tilde{U}_{i0} \to G_i\backslash\tilde{U}_{i0} \subset U_i$ which was seen to be isomorphic to the spread $V_0 \to U_{i*} \subset U_i$. By the uniqueness of a completion of a spread ([Fox; see Hunt, p. 149]), $h : V \to U_i$ and $K\backslash\tilde{U}_i \to U_i$ are isomorphic branched coverings. In particular, $V \cong K\backslash\tilde{U}_i$, and we have a required folding map $k : \tilde{U}_i \to V$. $\square$

Proof of Theorem 3.1. Let M be as in Theorem 3.1. Let $h : N \to M$ be a branched covering branched along ΣM whose associated subgroup $(h|N_0)_{\#}\pi_1(N_0)$ contains the characteristic subgroup $\langle\mu^b\rangle_M$. We will show that N is an m-orbifold and that $h : N \to M$ is an orbifold covering.

To make the argument precise, let us take base points $p_0 \in M_*$ and $q_0 \in N_0$ for the fundamental groups $\pi_1(M_*, p_0)$ and $\pi_1(N_0, q_0)$, respectively, so that $p_0 = h(q_0)$.

Let p be any point in M, $(\tilde{U}_i, G_i, f_i, U_i)$ a small folding chart with U_i containing p. We may assume that $\tilde{U}_i$ is 1-connected. Let V $(\subset N)$ be any component of $h^{-1}(U_i)$. We will first prove that *the image of* $(h|V_0)_{\#} : \pi_1(V_0) \to \pi_1(U_{i*})$ *contains the characteristic subgroup* $\langle\mu^b\rangle_{U_i}$ *for* U_i.

Take a path $\tilde{c}$ in N_0 joining q_0 and a point q_1 in V_0 $(= V \cap N_0)$. Let $c : [0, 1] \to M_*$ be the image of $\tilde{c}$ projected down to M_* by h. Then the starting point $c(0)$ of c is p_0, and its terminal point p_1 $(= c(1) = h(q_1))$ belongs to U_{i*} $(= U_i \cap M_*)$. We regard the points p_1 and q_1 as the base points of $\pi_1(U_{i*}, p_1)$ and $\pi_1(V_0, q_1)$.

Let l be a loop representing an element of $\langle\mu^b\rangle_{U_i}$ $(\leq \pi_1(U_{i*}, p_1))$. The loop clc^{-1} belongs to $\langle\mu^b\rangle_M$, which is contained in $(h|N_0)_{\#}(\pi_1(N_0, q_0))$ by the hypothesis. Thus clc^{-1} can be lifted to a loop l' in N_0 based at $q_0 = \tilde{c}(0)$. Since the path c does lift to $\tilde{c}$ and V_0 is a connected component of $h^{-1}(U_{i*})$, l' must be of the form $\tilde{c}\tilde{l}\tilde{c}^{-1}$, where $\tilde{l}$ is a loop in V_0 based at $q_1 = \tilde{c}(1)$. This means that l is lifted to $\tilde{l}$, that is, l belongs to the image of $(h|V_0)_{\#} : \pi_1(V_0) \to \pi_1(U_{i*})$.

The loop l was arbitrarily chosen in $\langle\mu^b\rangle_{U_i}$. Hence the image $(h|V_0)_{\#}\pi_1(V_0)$ contains the characteristic subgroup $\langle\mu^b\rangle_{U_i}$ for U_i as asserted.

Now by Lemma 3.2, there exists a folding map $k : \tilde{U}_i \to V$ so that the diagram

$$\begin{array}{ccc} \tilde{U}_i & \xrightarrow{\;k\;} & V \\ & {\scriptstyle f_i}\searrow & \downarrow{\scriptstyle h} \\ & & U_i \end{array}$$

commutes.

This means that N is an orbifold covering of M with the projection $h: N \to M$, because $(\tilde{U}_i, G_i, f_i, U_i)$ is arbitrarily chosen and V is any component of $h^{-1}(U_i)$. This completes the "if" part of Theorem 3.1.

The "only if" part is proved similarly. The details will be left to the reader. □

Let M be a connected orbifold without silvered boundary. Let $h: \tilde{M} \to M$ be an orbifold covering such that $(h|\tilde{M}_0)_* \pi_1(\tilde{M}_0) = \langle \mu^b \rangle_M$. By Theorem 3.1 such an orbifold covering certainly exists, and by Fox [F], its isomorphism class as a branched covering is unique. In this paper, we will adopt the following definition for convenience.

DEFINITION. The orbifold covering $h: \tilde{M} \to M$ satisfying $(h|\tilde{M}_0)_* \pi_1(\tilde{M}_0) = \langle \mu^b \rangle_M$ is called the *universal orbifold covering* of M.

This is equivalent to Thurston's definition [Th, §13] of the universal covering of an orbifold, at least for an orbifold without silvered boundary. See [Ka, Theorem 1] and also Appendix B of the present paper.

REMARK. If M has no silvered boundary, then the orbifold fundamental group $\pi_1^{\text{orb}}(M)$ in Thurston's sense is isomorphic to $\pi_1(M_*)/\langle \mu^b \rangle_M$. See [Ka, Theorem 1].

§ 4. Lemmas on geometric orbifolds.

Let (G, X) be a geometry satisfying condition $(*)$ of §1.

LEMMA 4.1. *Let $f: \tilde{U} \to U$ be a folding map modelled on (G, X), $j: \tilde{U} \to X$ the inclusion. Let $h: V' \to U_*$ $(= U - \Sigma U)$ be an unbranched covering, where V' is a connected (G, X)-manifold. Suppose that there exists a developing map $D: V' \to X$. Then the covering projection $h: V' \to U_*$ factors through $f|\tilde{U}_0$:*

$$\begin{array}{ccc} V' & \xrightarrow{\exists l'} & \tilde{U}_0 \\ & \searrow^{h} & \downarrow{\scriptstyle f|\tilde{U}_0} \\ & & U_* \end{array}$$

PROOF. Let $\tilde{f}: Z \to \tilde{U}_0$ be the universal covering of $\tilde{U}_0$ $(:= \tilde{U} - f^{-1}(\Sigma U))$. By Lemma 2.2, $\tilde{j} := (j|\tilde{U}_0)\tilde{f}$ is a developing map making the following diagram commutative:

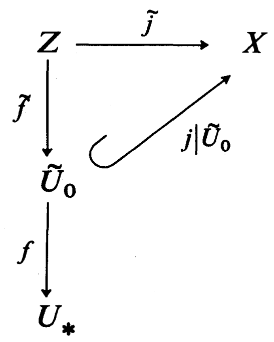


DIAGRAM (A)

Since $f\tilde{f} : Z \to U_*$ is the universal covering of U_*, there exists a covering map $\tilde{h} : Z \to V'$ so that the diagram commutes:

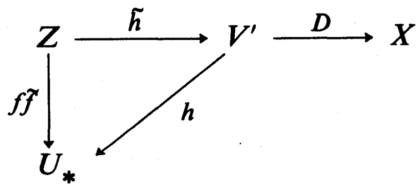


DIAGRAM (B)

We have two developing maps $\tilde{j}$ and $D\tilde{h}$ of Z into X; see diagrams (A) and (B). By Lemma 2.3, we can find a diffeomorphism $g : X \to X$ belonging to G such that

$$\tilde{j} = gD\tilde{h} .$$

Put $D' = gD$. Then $\tilde{j} = D'\tilde{h}$ and we have the following commutative diagram:

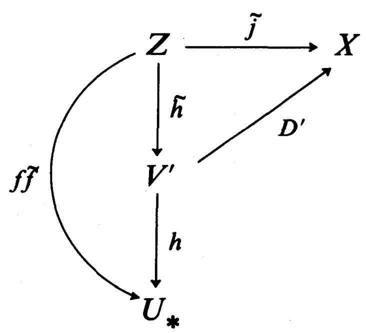


DIAGRAM (C)

From diagrams (A) and (C), we obtain

$$h\tilde{h} \overset{(C)}{=} f\tilde{f} \overset{(A)}{=} f(j|\tilde{U}_0)^{-1}\tilde{j} \overset{(C)}{=} f(j|\tilde{U}_0)^{-1}D'\tilde{h}.$$

Here $\tilde{h}: Z \to V'$ is surjective. Thus

$$h = f(j|\tilde{U}_0)^{-1}D'.$$

Putting $l' := (j|\tilde{U}_0)^{-1}D' : V' \to \tilde{U}_0$, we have the desired splitting $h = fl'$.

This completes the proof of Lemma 4.1. □

LEMMA 4.2. *Let M be a connected geometric orbifold without silvered boundary, modelled on (G, X). Let $h: \tilde{M} \to M$ be the universal orbifold covering. Then there exists a developing map $\tilde{D}: \tilde{M}_0 \to X$, where $\tilde{M}_0 = \tilde{M} - h^{-1}(\Sigma M)$.*

PROOF. We will show that the holonomy $H: \pi_1(\tilde{M}_0) \to G$ is trivial. This will imply the existence of a developing map $\tilde{D}: \tilde{M}_0 \to X$, (see Corollary 2.4.1).

Let $\tilde{l}$ be a loop in $\tilde{M}_0$ based at a point $\tilde{p}_0 \in \tilde{M}_0$. Since $h: \tilde{M} \to M$ is a universal orbifold covering, the associated subgroup $(h|\tilde{M}_0)_\sharp \pi_1(\tilde{M}_0)$ coincides with the characteristic subgroup $\langle \mu^b \rangle$ for M. Thus the projected loop $h_\sharp(\tilde{l})$ can be expressed as

$$(**) \qquad h_\sharp(\tilde{l}) = \prod_\alpha c_\alpha (\mu_\alpha)^{b_\alpha} c_\alpha^{-1} \in \pi_1(M_*, p_0),$$

where $p_0 = h(\tilde{p}_0)$ and c_α is a path in M_* joining p_0 and a point near the codimension 2 stratum S_α. If the meridian μ_α of S_α is small enough, we can find a folding chart $(\tilde{U}_i, G_i, f_i, U_i)$ containing μ_α. By the definition of the order b_α, the b_α-th power $(\mu_\alpha)^{b_\alpha}$ can be lifted to a *loop* $\tilde{l}_\alpha$ in $\tilde{U}_i$. Since $\tilde{U}_i$ is an open set of X, the holonomy along the loop $\tilde{l}_\alpha$ ($\subset \tilde{U}_i$) is the identity of X. Hence the holonomy along $(\mu_\alpha)^{b_\alpha}$ is the identity. By the expression $(**)$ above, the holonomy of M_* along $h_\sharp(\tilde{l})$ is again the identity of X. Since the (G, X)-structure on $\tilde{M}_0$ is inherited from M_*, the holonomy along the loop $\tilde{l}$ is trivial. This completes the proof. □

COROLLARY 4.2.1. *Let M be as in Lemma 4.2. Then the holonomy $H: \pi_1^{\text{orb}}(M) \to G$ is well-defined up to conjugation in G.*

PROOF. By Lemma 4.2, the holonomy $H: \pi_1(M_*) \to G$ is trivial on $\langle \mu^b \rangle_M$. Thus it factors through $\pi_1^{\text{orb}}(M) = \pi_1(M_*)/\langle \mu^b \rangle_M$. (See Remark at the end of §3.) □

§5. Proof of Theorem 1.1.

Let M be a connected geometric orbifold modelled on (G, X) satisfying condition $(*)$ of §1. We assume that M has no silvered boundary, otherwise we take its double along the silvered boundary, [Th]. Let $h: \tilde{M} \to M$ be the universal orbifold covering. We will show that $\tilde{M}$ is a manifold.

By Lemma 4.2, there is a developing map $\tilde{D}: \tilde{M}_0 \to X$.

Now let p be any point of M, $(\tilde{U}_i, G_i, f_i, U_i)$ a folding chart containing p. We may assume as before that $\tilde{U}_i$ is 1-connected. Let $V\ (\subset \tilde{M})$ be any component of $h^{-1}(U_i)$. Then $V_0\ (= V \cap \tilde{M}_0)$ can be developed on X via $\tilde{D}|V_0$.

By Lemma 4.1, $h|V_0 : V_0 \to U_{i*}$ factors through $f_i|\tilde{U}_{i0}$:

$$\begin{array}{ccc} V_0 & \xrightarrow{\exists l'} & \tilde{U}_{i0} \\ & {\scriptstyle h|V_0}\searrow & \downarrow {\scriptstyle f_i|\tilde{U}_{i0}} \\ & & U_{i*} \end{array}$$

On the other hand, by the definition of an orbifold covering, there exists a folding map $k : \tilde{U}_i \to V$ which gives a factorization in the direction opposite to l' in the above diagram:

$$\begin{array}{ccc} \tilde{U}_i & \xrightarrow{k} & V \\ & {\scriptstyle f_i}\searrow & \downarrow {\scriptstyle h|V} \\ & & U_i \end{array}$$

These two diagrams show that the unbranched coverings associated to the branched coverings $\tilde{U}_i \to U_i$ and $V \to U_i$ are isomorphic. Thus the folding map $k : \tilde{U}_i \to V$ is in fact a (G, X)-diffeomorphism and V has no singular point. Since V is any component of $h^{-1}(U_i)$, $\tilde{M}$ is a manifold.

The proof of Theorem 1.1 is complete. □

§ 6. An application.

In this section, X and G will be *a connected real analytic Riemannian manifold which is complete* and *a group of analytic isometries of* X. Clearly (G, X) satisfies condition $(*)$.

Let M be a connected geometric orbifold modelled on (G, X). Let C be a *splitting complex* of M, namely a subcomplex of M satisfying the three conditions below ([Nu], [Mo]):

(i) C has Lebesgue measure zero;

(ii) the restriction of $h : \tilde{M} \to M$ to $h^{-1}(M - C)$ is the trivial covering, i.e., a disjoint union of copies of $M - C$, where $h : \tilde{M} \to M$ is the universal orbifold covering;

(iii) $M - C$ is connected.

Let K be the adherence of any component of $h^{-1}(M - C)$.

PROPOSITION 6.1. *Suppose that X is 1-connected and M is compact. Then the development of K in X is a fundamental domain for the action of $H(\pi_1)$ on X. Here*

$H: \pi_1 = \pi_1^{orb}(M) \to G$ is the holonomy and $H(\pi_1)$ is the image.

PROOF. Clearly K is a fundamental domain for the action of π_1 on $\tilde{M}$, but by Theorem 1.1 $\tilde{M}$ coincides with X. Cf. [Th. §13]. □

REMARK. This proposition gives a practical procedure to develop M in X if $\pi_1(X) = \{1\}$, which is particularly useful when M is defined as a polyhedron $K \subset X$ with identified faces by elements of G. These identifications define the holonomy $H(\pi_1)$. When $H(\pi_1)$ acts upon K, we fill X without overlappings (Poincaré's Theorem [A], [Ma], [Se]). The hypotheses of the Poincaré Theorem assure that $K \subset X$ self-identified by elements of G is a geometric orbifold modelled on (G, X).

Appendix A. Well-definedness of isotropy groups.

Let M be a smooth m-orbifold, $(\tilde{U}_i, G_i, f_i, U_i)$ a folding chart. Take a point $x \in \tilde{U}_i$ and consider its isotropy group $(G_i)_x := \{g \in G_i \mid g(x) = x\}$. Appendix A aims to prove the basic fact that the isomorphism type of $(G_i)_x$ depends only on the point $p := f_i(x) \in M$, but not on the choice of $(\tilde{U}_i, G_i, f_i, U_i)$ nor $x \in f_i^{-1}(p)$.

LEMMA A.1. *Let $\tilde{V}_x$ be an open neighborhood of x in $\tilde{U}_i$. Then there exists a connected open neighborhood $\tilde{W}_x$ of x such that*

(i) $\tilde{W}_x \subset \tilde{V}_x$;

(ii) *$g(\tilde{W}_x) \cap \tilde{W}_x \neq \emptyset$ iff $g \in (G_i)_x$, and $\tilde{W}_x$ is $(G_i)_x$-invariant;*

(iii) *$f_i(\tilde{W}_x)$ $(\subset U_i)$ is homeomorphic to the quotient $(G_i)_x \backslash \tilde{W}_x$.*

PROOF. Let us fix a metric on $\tilde{U}_i$ for convenience, and choose a sequence of open neighborhoods of x in $\tilde{V}_x$, $\{\tilde{V}_n\}_{n=1}^{\infty}$, with diameter$(\tilde{V}_n) \leq 1/n$ for $n = 1, 2, 3, \cdots$. Using the fact that $G_i - (G_i)_x$ is a finite set, we can find a $\tilde{V}_n$ such that $g(\tilde{V}_n) \cap \tilde{V}_n = \emptyset$ for $g \notin (G_i)_x$. Since $(G_i)_x$ is also a finite set, the intersection

$$\tilde{W}'_x = \bigcap_{g \in (G_i)_x} g(\tilde{V}_n)$$

is a $(G_i)_x$-invariant open neighborhood of x. Let $\tilde{W}_x$ be the connected component of $\tilde{W}'_x$ which contains x. Then $\tilde{W}_x$ is a required open neighborhood. □

We will call $f_i | \tilde{W}_x : \tilde{W}_x \to f_i(\tilde{W}_x)$ a *sub-folding chart* of $f_i : \tilde{U}_i \to U_i$ *with center at* x.

Now recall Bonahon and Siebenmann's compatibility condition [BS]. Let $(\tilde{U}_i, G_i, f_i, U_i)$, $(\tilde{U}_j, G_j, f_j, U_j)$ be two folding charts of M with $U_i \cap U_j \neq \emptyset$. Take points $x \in \tilde{U}_i$ and $y \in \tilde{U}_j$ such that $p := f_i(x) = f_j(y) \in U_i \cap U_j$. Then the compatibility condition states that there exists diffeomorphism $\psi : \tilde{V}_x \to \tilde{V}_y$ of an open neighborhood of x in $\tilde{U}_i$ to an open neighborhood of y in $\tilde{U}_j$ such that $\psi(x) = y$ and $f_j \psi = f_i$.

LEMMA A.2. *In the above situation, there exist sub-folding charts $f_i | \tilde{Z}_x : \tilde{Z}_x \to f_i(\tilde{Z}_x)$*

and $f_j|\tilde{Z}_y : \tilde{Z}_y \to f_j(\tilde{Z}_y)$ *with center at x and y, respectively, such that*
(i) $\tilde{Z}_x \subset \tilde{V}_x$, $\tilde{Z}_y \subset \tilde{V}_y$; *and*
(ii) $\psi(\tilde{Z}_x) = \tilde{Z}_y$.
(We will say that these sub-folding charts are *compatible*.)

PROOF. Let $f_i|\tilde{W}_x : \tilde{W}_x \to f_i(\tilde{W}_x)$ be a sub-folding chart of $f_i : \tilde{U}_i \to U_i$ with center at x which is contained in $\tilde{V}_x$ (Lemma A.1).

Since $\psi(\tilde{W}_x)$ is an open neighborhood of y in $\tilde{U}_j$, there exists, again by Lemma A.1, a sub-folding chart $f_j|\tilde{Z}_y : \tilde{Z}_y \to f_j(\tilde{Z}_y)$ with center at y such that $\tilde{Z}_y \subset \psi(\tilde{W}_x)$. Since $f_j\psi = f_i$, we have

$$f_j(\tilde{Z}_y) \subset f_j\psi(\tilde{W}_x) = f_i(\tilde{W}_x) \ (\approx (G_i)_x \backslash \tilde{W}_x)\,.$$

Define $\tilde{Z}_x := f_i^{-1}(f_j(\tilde{Z}_y)) \cap \tilde{W}_x$. Then Z_x is (G_i)-invariant, and $f_i|\tilde{Z}_x : \tilde{Z}_x \to f_i(\tilde{Z}_x)$ is a desired sub-folding chart with $\psi(\tilde{Z}_x) = \tilde{Z}_y$. □

Let $f_i|\tilde{Z}_x : \tilde{Z}_x \to f_i(\tilde{Z}_x)$ and $f_j|\tilde{Z}_y : \tilde{Z}_y \to f_j(\tilde{Z}_y)$ be compatible sub-folding charts with center at x and y respectively. We will prove that $(G_i)_x \cong (G_j)_y$.

A point $z \in \tilde{Z}_x$ is said to be *admissible* if $(G_i)_z = \{1\}$ or D_2 (dihedral group of order 2). Here we remark that if D_2 is used instead of C_2 (cyclic group of order $2 \cong D_2$), it is to be understood that D_2 acts as reflection with respect to an $(m-1)$-submanifold, and in this case z is also called a *mirror point*.

With the definition of admissibility as above, it is easy to see that $z \in \tilde{Z}_x$ is admissible iff $\psi(z) \in \tilde{Z}_y$ is admissible. Denote the set of admissible points in $\tilde{Z}_x$ by $\tilde{Z}_x^{\mathrm{ad}}$. Then $\tilde{Z}_x^{\mathrm{ad}}$ is *connected* and $\psi(\tilde{Z}_y^{\mathrm{ad}}) = \tilde{Z}_y^{\mathrm{ad}}$. Moreover, denote the open set $f_i(\tilde{Z}_x^{\mathrm{ad}}) = f_j(\tilde{Z}_y^{\mathrm{ad}})$ of M by Z^{ad}. We see that

$$f_i|\tilde{Z}_x^{\mathrm{ad}} : \tilde{Z}_x^{\mathrm{ad}} \to Z^{\mathrm{ad}} \quad \text{and} \quad f_j|\tilde{Z}_y^{\mathrm{ad}} : \tilde{Z}_y^{\mathrm{ad}} \to Z^{\mathrm{ad}}$$

are *regular silvered coverings* (in the sense of Appendix B) whose group of covering translations are $(G_i)_x$ and $(G_j)_y$, respectively. But these silvered coverings are mutually isomorphic via $\psi : \tilde{Z}_x^{\mathrm{ad}} \to \tilde{Z}_y^{\mathrm{ad}}$, so that $(G_i)_x \cong (G_j)_y$ as asserted.

In the case when no mirror points appear, we need only ordinary covering spaces.

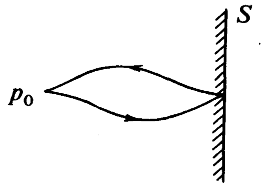
Appendix B. Silvered coverings.

A "silvered covering" is a generalization of an ordinary covering and a special case of an orbifold covering. However, it would be worthwhile to develop the theory of silvered coverings independently, because it is used in the proof of well-definedness of isotropy groups. (Appendix A)

DEFINITION. A silvered m-manifold M^s is a pair (M, S) of a smooth m-manifold M and an $(m-1)$-submanifold S of the boundary ∂M, such that S is a closed subset of ∂M. S is called the silvered boundary.

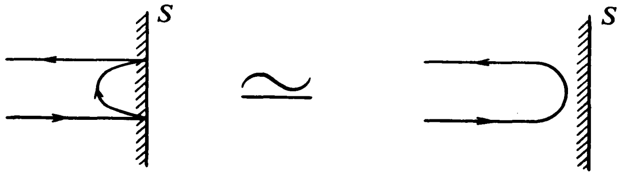
Given a connected silvered manifold $M^s=(M, S)$, fix a base point p_0 in Int(M). We will define the fundamental group $\pi_1(M^s)$ as follows:

A *loop* l in M^s means a loop in M based at p_0 which, whenever it meets S, meets it *transversely* like this:



Two loops l_1 and l_2 in M^s are *homotopic* ($l_1 \simeq l_2$ in "silvered" sense), if l_1 is transformed into l_2 by a finite sequence of operations 1) and/or 2) below:

1) ordinary homotopy in M which keeps the transversality of loops at S;
2) cancelation of successive two intersections with S like this:



DEFINITION. $\pi_1(M^s) := \{\text{loops in } M^s\}/\simeq$.

DEFINITION. Let $M^s=(M, S_M)$, $N^s=(N, S_N)$ be connected silvered manifolds. Then $h: N^s \to M^s$ is called a silvered covering if

(i) h is continuous and surjective;

(ii) $h(S_N) \subset S_M$;

(iii) $\forall x \in M^s$ has an open neighborhood U such that $h^{-1}(U)$ is a disjoint union of (open) connected components U_i, and $h|U_i$ is either a homomorphism or quotient by an involution with codimension one fixed point set. In the latter case, the fixed point set in U_i has to be projected onto an open set of S_M under h.

A silvered covering $\tilde{h}$: $\tilde{M}^s \to M^s$ is a *universal silvered covering* if for any silvered covering $h: N^s \to M^s$, there exists a silvered covering $k: \tilde{M}^s \to N^s$ so that $\tilde{h}=hk$. A universal silvered covering $\tilde{M}^s$ is constructed by using paths in M^s as follows:

Fix a base point $p_0 \in \text{Int}(M)$. Two paths c_1, c_2, based at p_0 and ending in a point, correspond to the same point of $\tilde{M}^s$ iff the composition $c_1 c_2^{-1}$ is a loop which is homotopic (in the silvered sense) to a constant loop.

Let N be a smooth connected manifold, Γ a group acting on N smoothly and properly discontinuously. We assume that for $\forall x \in N$, the isotropy group Γ_x is either $\{1\}$ or D_2 (dihedral group of order 2) and in the latter case, that D_2 acts as involution with codimension one fixed point set. Then $\Gamma \backslash N$ is a silvered manifold and the projection

$N \to \Gamma \backslash N$ is a silvered covering. Such a silvered covering is said to be *regular*, and Γ the *group of covering translations*.

PROPOSITION B.1. *Let $h: N \to M^s$ be a regular silvered covering with the group of covering translations Γ. Then $h_\sharp \pi_1(N)$ is a normal subgroup of $\pi_1(M^s)$ and $\Gamma \cong \pi_1(M^s)/h_\sharp \pi_1(N)$.*

Silvered coverings over M^s are in one to one correspondence with the conjugacy class of subgroups of $\pi_1(M^s)$.

A silvered covering $h: N^s \to M^s$ is a *spread* in Fox's sense [F]. Thus one can define a *branched silvered covering* just as an "ordinary" branched covering [F].

Finally, we will construct the *orientable double covering M_O* for a given orbifold M. Let M be a connected orbifold. A point $p \in M$ is called *admissible* if there exists a folding chart $(\tilde{U}_i, G_i, f_i, U_i)$ containing p so that $f_i: \tilde{U}_i \to U_i$ is a silvered covering. Let M^{ad} denote the set of admissible points in M. Then M^{ad} is a silvered manifold. Take the subgroup $O \lhd \pi_1(M^{ad})$ of index 2 composed of orientation preserving loops. (Note that each time a loop intersects the silvered boundary, the orientation is understood to be reversed.) Construct the silvered covering $M_O^{ad} \to M^{ad}$ associated to the subgroup O. Let M_O be the completion of the spread $M_O^{ad} \to M^{ad} \subset M$. Then $M_O \to M$ is $2:1$ and M_O is not silvered. Also the universal covering of M_O coincides with the universal covering of M (in the sense of orbifolds).

In conclusion, the topological concept underlying an orbifold covering is a *branched silvered covering*.

References

[A] A. D. ALEKSANDROV, On completion of a space by polyhedra, Vestnik Leningrad Univ. Ser. Math. Fiz. Khim., **9:2** (1954), 33–43, MR 20 #1301.

[BS] F. BONAHON and L. SIEBENMANN, The classification of Seifert fibered 3-orbifolds, *Low Dimensional Topology*, ed. by R. Fenn, London Math. Soc. Lecture Note Series, **85** (1985), 19–85.

[Dr] A. DRESS, Newman's theorems on transformation groups, Topology, **8** (1969), 203–207.

[Du] W. D. DUNBAR, Geometric orbifolds, Revista Mat. Univ. Compl. Madrid, **1** (1988), 67–99.

[F] R. H. FOX, Covering spaces with singularities, *Algebraic Geometry and Topology*, Princeton Univ. Press (1957), 243–257.

[H] J. H. V. HUNT, Branched coverings as uniform completions of unbranched coverings (Résumé), Contemporary Math., **12** (1982), 141–155.

[K] M. KATO, On uniformization of orbifolds, Adv. Stud. Pure Math., **9** (1986), 149–172.

[Ma] B. MASKIT, On Poincaré's theorem for fundamental polygons, Adv. in Math., **7** (1971), 219–230.

[Mo] J. M. MONTESINOS, Sobre la conjetura de Poincaré y los recubridores ramificados sobre un nudo, Ph. D. Theses, Univ. Compl. Madrid (1971).

[Nu] L. P. NEUWIRTH, *Knot Groups*, Ann. Math. Stud., **56** (1965), Princeton Univ. Press.

[Nw] M. H. A. NEWMAN, A theorem on periodic transformations of spaces, Quart. J. Math., **2** (1931), 1–8.

[Sa] I. SATAKE, On a generalization of the notion of manifolds, Proc. Nat. Acad. Sci. USA, **42** (1956), 359–363.

[Se] H. SEIFERT, Komplexe mit Seitenzuordenung, MR 52 #9100, Nachr. Akad. Wiss. Göttingen Math. Phys. Kl. II, **6** (1975), 49–80.

[Th] W. THURSTON, *The Geometry and Topology of 3-Manifolds*, preprint, Princeton, 1976–79.

Present Address:

Y. MATSUMOTO
DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE, UNIVERSITY OF TOKYO
HONGO, TOKYO 113, JAPAN

J. M. MONTESINOS-AMILIBIA
UNIVERSIDAD COMPLUTENSE DE MADRID, FACULTAD DE CIENCIAS MATHEMÁTICAS
28040 MADRID, SPAIN