TOKYO J. MATH.
VOL. 14, NO. 1, 1991

Generalized Spencer Cohomology Groups and Quasi-Regular Bases

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(Communicated by T. Nagano)

Dedicated to Professor Tadashi Nagano on his 60th birthday

Introduction. Let $\mathfrak{g}=\bigoplus_{p\geq -\mu}\mathfrak{g}_p$ be a transitive graded Lie algebra of depth $\mu>0$, and let $\mathfrak{g}_-=\bigoplus_{p<0}\mathfrak{g}_p$ be its negative part. Associated with the adjoint representation of $\mathfrak{g}_-$ on $\mathfrak{g}$, there is defined a cohomology group $H(\mathfrak{g}_-, \mathfrak{g})=\bigoplus_{p,r}H^p(\mathfrak{g}_-, \mathfrak{g})_r$ which we call the generalized Spencer cohomology group (see §6).

If $\mu=1$, $\mathfrak{g}_-$ is abelian and the cohomology group is well known as the Spencer cohomology groups (see e.g., [2], [6]). The generalized Spencer cohomology group was introduced by Tanaka [7] and has been used extensively in our studies of filtered Lie algebras [3], geometric structures [4] and differential equations [5], based on filtered manifolds, where it is this cohomology group that takes the rôle of the Spencer cohomology group.

We know that $H^p(\mathfrak{g}_-, \mathfrak{g})_r$ vanishes for large r by Noetherian property (see [3]), however in various concrete problems we need further to compute this cohomology group explicitly or to determine the range of (p, r) in which $H^p(\mathfrak{g}_-, \mathfrak{g})_r$ vanishes. In the case $\mu=1$, as is well known, there is a fundamental theorem conjectured by Guillemin and Sternberg and proved by Serre (see Appendix of [2]), which relates the vanishing of the cohomology group with the existence of a quasi-regular basis.

The main purpose of this paper is to extend the theorem of Serre to the generalized Spencer cohomology group. We shall give a criterion (explicit and in some extent calculable) in terms of quasi-regular bases for the vanishing of the generalized Spencer cohomology group, and also make clear the difference which lies between the special case $\mu=1$ and the general case $\mu\geq 1$.

The nature of our problem being better adapted to its dualized form, we shall mainly discuss homology groups of graded modules, and in the last section we translate the main results to the cohomology groups of graded Lie algebras.

Received June 6, 1990

NOTATION AND CONVENTIONS. All vector spaces are considered over a field F of characteristic zero. Graded vector spaces are always Z-graded. If we write $V=\bigoplus_{p=s}^{t} V_p$, it is understood that $V_p=0$ for $p<s$ or $p>t$. We say an element $x\in V$ is of degree p and write $\deg x=p$ if $x\in V_p$. For graded vector spaces $V=\bigoplus V_p$, $W=\bigoplus W_p$, gradations considered on various associated vector spaces are those ones which are defined in the standard way; for example, $(V\oplus W)_p=V_p\oplus W_p$, $(V\otimes W)_p=\bigoplus_{i+j=p}V_i\otimes W_j$, $(V/W)_p=V_p/W_p$ (if W is a graded subspace of V), $(V^*)_p=(V_{-p})^*$. A linear map $f: V\to W$ is therefore of degree r iff $f(V_p)\subset W_{p+r}$ for all p. For a graded vector space V, we define its filtration $\{\mathscr{F}^l V\}_{l\in Z}$ by setting

$$\mathscr{F}^l V=\bigoplus_{p\geq l} V_p\,.$$

A graded Lie algebra $\mathfrak{g}=\bigoplus_{p\in Z}\mathfrak{g}_p$ is a graded vector space endowed with a Lie algebra structure such that $[\mathfrak{g}_p, \mathfrak{g}_q]\subset\mathfrak{g}_{p+q}$ for all $p, q\in Z$.

§1. Let $\mathfrak{n}=\bigoplus_{k=1}^{\mu}\mathfrak{n}_k$ be a finite dimensional graded Lie algebra and let $U(\mathfrak{n})$ denote its universal enveloping algebra. Let $E=\bigoplus_{p\in Z}E_p$ be a graded right $U(\mathfrak{n})$-module satisfying:

$$E_p\cdot\mathfrak{n}_k\subset E_{p+k}\,.$$

Then we have the homology group $H_p(\mathfrak{n}, E)$ associated with the complex $(E\otimes\bigwedge\mathfrak{n}, \partial)$, where the boundary operator

$$\partial: E\otimes\bigwedge^p\mathfrak{n}\longrightarrow E\otimes\bigwedge^{p-1}\mathfrak{n}$$

is defined by

$$\partial(\alpha\otimes X_1\wedge\cdots\wedge X_p)=\sum_{i=1}^{p}(-1)^{i-1}\alpha\cdot X_i\otimes X_1\wedge\cdots\wedge\hat{X}_i\wedge\cdots\wedge X_p$$
$$+\sum_{i<j}(-1)^{i+j}\alpha\otimes[X_i, X_j]\wedge X_1\wedge\cdots\wedge\hat{X}_i\wedge\cdots\wedge\hat{X}_j\wedge\cdots\wedge X_p$$

for $\alpha\in E$, $X_1, \cdots, X_p\in\mathfrak{n}$.

Since both E and $\mathfrak{n}$ are graded, we have the natural gradation: $E\otimes\bigwedge\mathfrak{n}=\bigoplus(E\otimes\bigwedge\mathfrak{n})_r$, where

$$(E\oplus\bigwedge\mathfrak{n})_r=\bigoplus_{p+a+b+\cdots+c=r}E_p\otimes\mathfrak{n}_a\wedge\mathfrak{n}_b\wedge\cdots\wedge\mathfrak{n}_c\,.$$

Clearly $(E\otimes\bigwedge\mathfrak{n})_r$ forms a subcomplex of $(E\otimes\bigwedge\mathfrak{n}, \partial)$, of which homology group (resp. p-th homology group) will be denoted by $H(\mathfrak{n}, E)_r$ (resp. $H_p(\mathfrak{n}, E)_r$). Then we have:

$$H(\mathfrak{n}, E)=\bigoplus H_p(\mathfrak{n}, E)=\bigoplus H(\mathfrak{n}, E)_r=\bigoplus H_p(\mathfrak{n}, E)_r\,.$$

The following theorem is fundamental (for a proof see Morimoto [3]):

THEOREM 1.1. *If E is finitely generated as $U(\mathfrak{n})$-module, then $H(\mathfrak{n}, E)_r=0$ for r large enough.*

In the following sections we will study explicit criterions in order for $H_p(\mathfrak{n}, E)_r$ to vanish.

§2. Let $\mathfrak{n}$ and E be as in the preceding section. An admissible sequence of $\mathfrak{n}$ of length r is a linearly independent sequence $\{u_1, \cdots, u_r\}$ in $\mathfrak{n}$ such that $u_i \in \mathfrak{n}_k$ if $\dim \bigoplus_{p\geq k}\mathfrak{n}_p \geq i > \dim \bigoplus_{p>k}\mathfrak{n}_p$. An admissible basis of $\mathfrak{n}$ is an admissible sequence of length n, where $n=\dim \mathfrak{n}$. Take an admissible basis $\{u_1, \cdots, u_n\}$ of $\mathfrak{n}$, and set:

$$\chi(p)\ (=\chi(\mathfrak{n}, p))=\sum_{i=1}^{p} d(u_i)\,, \qquad \text{where} \quad d(u_i)=\deg u_i\,.$$

Obviously $\chi(p)$ does not depend on the choice of admissible basis.

DEFINITION 2.1. An admissible sequence $\{u_1, \cdots, u_r\}$ of $\mathfrak{n}$ is called "injective to E_m" if the following maps (multiplications by u_i) are all injective:

$$\begin{cases} E_{m-d(u_1)} \xrightarrow{u_1} E_m \\ (E/E(u_1))_{m-d(u_2)} \xrightarrow{u_2} (E/E(u_1))_m \\ \cdots\cdots\cdots \\ (E/E(u_1, \cdots, u_{r-1}))_{m-d(u_r)} \xrightarrow{u_r} (E/E(u_1, \cdots, u_{r-1}))_m\,, \end{cases}$$

where $(u_1, \cdots, u_i)$ denotes the ideal of $U(\mathfrak{n})$ generated by $u_1, \cdots, u_i$. We say furthermore that $\{u_1, \cdots, u_r\}$ is a quasi-regular sequence of $\mathfrak{n}$ relative to $\mathscr{F}^l E$ if it is injective to E_m for all $m\geq l+\chi(1)$.

THEOREM 2.1. *Let $\mathfrak{n}$ and E be as above. If there exists a quasi-regular sequence of $\mathfrak{n}$ of length r relative to $\mathscr{F}^l E$, then*

$$\mathscr{F}^{l+\chi(p)}H_p(\mathfrak{n}, E)=0$$

for $p=\dim \mathfrak{n}-r+1, \cdots, \dim \mathfrak{n}$.

PROOF. Let us prove the theorem by induction on r and $\dim \mathfrak{n}$. Suppose that the theorem holds if $\dim \mathfrak{n}<n$ or if $\dim \mathfrak{n}=n$ and the length of the quasi-regular sequence is less than r. Now, given a quasi-regular sequence $\{u_1, \cdots, u_r\}$ of length r, it suffices to prove $\mathscr{F}^{l+\chi(p)}H_p(\mathfrak{n}, E)=0$ for $p=n-r+1$.

First recall that $\mathfrak{n}$ acts on $E\otimes\bigwedge\mathfrak{n}$ through Lie derivative, that is, for $Y\in\mathfrak{n}$, $\alpha\otimes X_1\wedge\cdots\wedge X_q\in E\otimes\bigwedge^q\mathfrak{n}$, we set

$$(\alpha\otimes X_1\wedge\cdots\wedge X_q)\theta(Y)=\alpha Y\otimes X_1\wedge\cdots\wedge X_q+\sum_{i=1}^{q}\alpha\otimes X_1\wedge\cdots\wedge[X_i, Y]\wedge\cdots\wedge X_q\,.$$

Then we have the following formulae:

$$\theta(Y)\cdot\partial=\partial\cdot\theta(Y)\,, \tag{2.1}$$

$$\theta([X, Y])=-\theta(X)\theta(Y)+\theta(Y)\theta(X)\,, \tag{2.2}$$

$$\partial\cdot\lambda(Y)+\lambda(Y)\cdot\partial=\theta(Y)\,, \tag{2.3}$$

where $\lambda(Y): E\otimes\bigwedge\mathfrak{n}\to E\otimes\bigwedge\mathfrak{n}$ is given by:

$$(\alpha\otimes X_1\wedge\cdots\wedge X_q)\lambda(Y)=\alpha\otimes Y\wedge X_1\wedge\cdots\wedge X_q\,.$$

We then have the following exact sequence of chain complexes:

$$E\otimes\bigwedge\mathfrak{n}\xrightarrow{\theta(u_1)}E\otimes\bigwedge\mathfrak{n}\xrightarrow{\pi}E^{(1)}\otimes\bigwedge\mathfrak{n}\,, \tag{2.4}$$

where $E^{(1)}=E/E(u_1)$, and π is the natural projection. Note that since $[u_1, \mathfrak{n}]=0$, $\theta(u_1)$ is just the multiplication by u_1. From our hypothesis the map:

$$\mathscr{F}^l E\xrightarrow{u_1}\mathscr{F}^{l+d(u_1)}E$$

is injective, therefore we have the following commutative diagram whose rows are all exact:

$$\begin{array}{ccccccc}
 & \mathscr{F}^{l'}(E\otimes\bigwedge^{p+2}\mathfrak{n}) & \xrightarrow{\theta(u_1)} & \mathscr{F}^{l'+d(u_1)}(E\otimes\bigwedge^{p+2}\mathfrak{n}) & \xrightarrow{\pi} & \mathscr{F}^{l'+d(u_1)}(E^{(1)}\otimes\bigwedge^{p+2}\mathfrak{n}) & \longrightarrow 0\\
 & \downarrow & & \downarrow & & \downarrow & \\
 & \mathscr{F}^{l'}(E\otimes\bigwedge^{p+1}\mathfrak{n}) & \xrightarrow{\theta(u_1)} & \mathscr{F}^{l'+d(u_1)}(E\otimes\bigwedge^{p+1}\mathfrak{n}) & \xrightarrow{\pi} & \mathscr{F}^{l'+d(u_1)}(E^{(1)}\otimes\bigwedge^{p+1}\mathfrak{n}) & \longrightarrow 0\\
 & \downarrow & & \downarrow & & \downarrow & \\
0\longrightarrow & \mathscr{F}^{l'}(E\otimes\bigwedge^{p}\mathfrak{n}) & \xrightarrow{\theta(u_1)} & \mathscr{F}^{l'+d(u_1)}(E\otimes\bigwedge^{p}\mathfrak{n}) & \xrightarrow{\pi} & \mathscr{F}^{l'+d(u_1)}(E^{(1)}\otimes\bigwedge^{p}\mathfrak{n}) & \longrightarrow 0\\
 & \downarrow & & \downarrow & & & \\
0\longrightarrow & \mathscr{F}^{l'}(E\otimes\bigwedge^{p-1}\mathfrak{n}) & \xrightarrow{\theta(u_1)} & \mathscr{F}^{l'+d(u_1)}(E\otimes\bigwedge^{p-1}\mathfrak{n}) & \longrightarrow\cdots\,, & &
\end{array}$$

where we have put $l'=l+\chi(p)$. We then have the following long exact sequence:

$$\mathscr{F}^{l'}H_{p+1}(\mathfrak{n}, E)\xrightarrow{\theta(u_1)_*}\mathscr{F}^{l'+d(u_1)}H_{p+1}(\mathfrak{n}, E)\xrightarrow{\pi_*}\mathscr{F}^{l'+d(u_1)}H_{p+1}(\mathfrak{n}, E^{(1)})$$

$$\xrightarrow{\delta}\mathscr{F}^{l'}H_p(\mathfrak{n}, E)\xrightarrow{\theta(u_1)_*}\mathscr{F}^{l'+d(u_1)}H_p(\mathfrak{n}, E)\longrightarrow\cdots\,.$$

Note that we have $\mathscr{F}^{l'+d(u_1)}H_{p+1}(\mathfrak{n}, E)=0$ by using the induction assumption on account of the inequality $l+\chi(p)+d(u_1)\geq l+\chi(p+1)$. Note also that $\theta(u_1)_*=0$ by the Stokes' formula (2.3). Hence we have:

$$0 \longrightarrow \mathscr{F}^{l'+d(u_1)}H_{p+1}(\mathfrak{n}, E^{(1)}) \underset{\cong}{\overset{\delta}{\longrightarrow}} \mathscr{F}^{l'}H_p(\mathfrak{n}, E) \longrightarrow 0\,.$$

On the other hand, we have the following exact sequence of complexes:

$$\begin{array}{l} 0 \longrightarrow E^{(1)} \otimes \bigwedge \mathfrak{n} \wedge u_1 \longrightarrow E^{(1)} \otimes \bigwedge \mathfrak{n} \longrightarrow E^{(1)} \otimes \bigwedge \mathfrak{n}^{(1)} \longrightarrow 0\,, \\ \qquad\qquad \Vert \\ \quad E^{(1)} \otimes \bigwedge \mathfrak{n}^{(1)} \otimes u_1 \end{array} \tag{2.5}$$

where $\mathfrak{n}^{(1)} = \mathfrak{n}/\langle u_1 \rangle$. This yields the following exact sequence:

$$\overset{\delta}{\longrightarrow} \mathscr{F}^{l'}H_p(\mathfrak{n}^{(1)}, E^{(1)}) \longrightarrow \mathscr{F}^{l'+d(u_1)}H_{p+1}(\mathfrak{n}, E^{(1)}) \longrightarrow \mathscr{F}^{l'+d(u_1)}H_{p+1}(\mathfrak{n}^{(1)}, E^{(1)}) \longrightarrow \cdots\,.$$

Now note that

$$p = \dim \mathfrak{n} - r + 1 = \dim \mathfrak{n}^{(1)} - (r-1) + 1$$

and

$$\chi(\mathfrak{n}, p) \geq \chi(\mathfrak{n}^{(1)}, p)\,, \qquad \chi(\mathfrak{n}, p) + d(u_1) \geq \chi(\mathfrak{n}^{(1)}, p+1)\,.$$

Note also that $\mathfrak{n}^{(1)}$ has a quasi-regular sequence of length $r-1$ relative to $\mathscr{F}^l E^{(1)}$. Then by induction assumption we obtain

$$\mathscr{F}^{l'}H_p(\mathfrak{n}^{(1)}, E^{(1)}) = \mathscr{F}^{l'+d(u_1)}H_{p+1}(\mathfrak{n}^{(1)}, E^{(1)}) = 0\,.$$

Hence

$$\mathscr{F}^{l'+d(u_1)}H_{p+1}(\mathfrak{n}, E^{(1)}) = 0\,.$$

Therefore

$$\mathscr{F}^{l'}H_p(\mathfrak{n}, E) = 0\,,$$

which completes the proof.

§3. In this section we examine in more detail the conditions for quasi-regular basis. For that, we define a module $M(E^{(k)})$ for the graded $U(\mathfrak{n})$-module E; the dual notion of prolongation.

We denote by $E^{(k)}$ the quotient module $E/\mathscr{F}^{k+1}E$, and by *Trun* the truncation map : $E \rightarrow E^{(k)}$. Let R be the $U(\mathfrak{n})$ submodule of $E^{(k)} \otimes U(\mathfrak{n})$ generated by

$$\{a \otimes \xi - a\xi \otimes 1 \mid a \in E^{(k)},\ \xi \in U(\mathfrak{n}) \text{ with } \deg a + \deg \xi \leq k\}\,,$$

and set $M(E^{(k)}) = E^{(k)} \otimes U(\mathfrak{n})/R$. We simply write $M(E^{(k)}) = M$ if there is no fear of confusion. Then, as easily seen, M possesses the following properties:

(P1) M is a graded $U(\mathfrak{n})$-module and is generated by $\bigoplus_{p\le k} M_p$.

(P2) There is a canonical isomorphism $q^{(k)}: M^{(k)} \to E^{(k)}$ of graded $U(\mathfrak{n})$-modules, where $M^{(k)} = M/\mathscr{F}^{k+1}M$.

(P3) Given a graded $U(\mathfrak{n})$-module F, and $U(\mathfrak{n})$-homomorphism of degree 0; $f^{(k)}: E^{(k)} \to F^{(k)}$, then there exists a unique $U(\mathfrak{n})$ homomorphism $f: M \to F$ of degree 0, such that the following diagram commutes:

$$\begin{array}{ccccc} M & = & M & \xrightarrow{f} & F \\ {\scriptstyle Trun}\downarrow & & {\scriptstyle Trun}\downarrow & & {\scriptstyle Trun}\downarrow \\ M^{(k)} & \xrightarrow{q^{(k)}} & E^{(k)} & \xrightarrow{f^{(k)}} & F^{(k)} . \end{array}$$

Moreover if F is generated by $\bigoplus_{p\le k} F_p$ and $f^{(k)}$ is surjective then f is also surjective.

In particular there is a canonical $U(\mathfrak{n})$-homomorphism $q: M \to E$ which makes the following diagram commutative:

$$\begin{array}{ccc} M & \xrightarrow{q} & E \\ {\scriptstyle Trun}\downarrow & & \downarrow{\scriptstyle Trun} \\ M^{(k)} & \underset{\cong}{\xrightarrow{q^{(k)}}} & E^{(k)} . \end{array}$$

We say that the $U(\mathfrak{n})$-module E is *k-determined* if $q: M(E^{(k)}) \to E$ is an isomorphism. We then have:

PROPOSITION 3.1. *Assume that E is generated by* $\bigoplus_{p\le k} E_p$. *For an integer* $l > k$ *the following conditions are equivalent*:

(1) *The surjection* $M(E^{(k)}) \to E$ *induces an isomorphism* $M(E^{(k)})^{(l)} \to E^{(l)}$.

(2) $H_1(\mathfrak{n}, E)_j = 0$ *for* $k+1 \le j \le l$.

Therefore E is k-determined if and only if $\mathscr{F}^{k+1} H_1(\mathfrak{n}, E) = 0$.

PROOF. By the properties (P1) (P2) (P3), we have:

$$M(M(E^{(k)})^{(l)}) = M(E^{(k)}) \qquad \text{for} \quad l \ge k .$$

From this we see that in order to prove the proposition it suffices to show the assertion for $l = k+1$.

Let us first show that $H_1(\mathfrak{n}, M(E^{(k)}))_{k+1} = 0$. As before we write $M(E^{(k)}) = M$. Let $\omega \in (M \otimes \mathfrak{n})_{k+1}$ and suppose $\partial\omega = 0$. Consider the following commutative diagram:

$$\begin{array}{ccccc} M \otimes \bigwedge^2 \mathfrak{n} & \xrightarrow{\partial} & M \otimes \mathfrak{n} & \xrightarrow{\partial} & M \\ & & {\scriptstyle \pi' = \pi \otimes id_{\mathfrak{n}}}\uparrow & & \uparrow{\scriptstyle \pi} \\ & & E^{(k)} \otimes U(\mathfrak{n}) \otimes \mathfrak{n} & \xrightarrow{\gamma} & E^{(k)} \otimes U(\mathfrak{n}) , \end{array}$$

where γ is the multiplication tensored by $id_{E^{(k)}}$. Take $\Omega\in(E^{(k)}\otimes U(\mathfrak{n})\otimes\mathfrak{n})_{k+1}$ such that $\pi'(\Omega)=\omega$. Since $\pi\circ\gamma(\Omega)=0$, we can write

$$\gamma(\Omega)=\sum(a\otimes\xi-a\xi\otimes 1)\cdot\eta\,,$$

where the summation is taken over some number of $a\in E^{(k)}$, ξ, $\eta\in U(\mathfrak{n})$ with $\deg(a\xi)\leq k$. Since $\deg\gamma(\Omega)=k+1$, we can rewrite:

$$\gamma(\Omega)=\sum(a\otimes\xi\cdot\xi'-a\xi\otimes\xi')y\,,$$

where $y\in\mathfrak{n}$ and $\deg(a\otimes\xi\cdot\xi')\leq k$. Moreover we may assume without loss of generality that

$$\xi\cdot\xi'=\xi''\cdot x\qquad\text{with}\quad\xi''\in U(\mathfrak{n})\,,\quad x\in\mathfrak{n}\,.$$

Then

$$\begin{aligned}\gamma(\Omega)&=\sum a\otimes\xi''xy-a\xi\otimes\xi'y\\&=\sum a\otimes\xi''yx+a\otimes\xi''[x,y]-a\xi\otimes\xi'y\,.\end{aligned}$$

Therefore by setting

$$\Omega'=\sum a\otimes\xi''y\otimes x+a\otimes\xi''\otimes[x,y]-a\xi\otimes\xi'\otimes y\,,$$

we see that $\gamma(\Omega)=\gamma(\Omega')$ and $\pi'(\Omega')$ is a boundary. In fact

$$\begin{aligned}\pi'(\Omega')&=\sum\pi(a\otimes\xi''y)\otimes x+\pi(a\otimes\xi'')\otimes[x,y]-\pi(a\xi\otimes\xi')\otimes y\\&=\sum\pi(a\otimes\xi'')y\otimes x+\pi(a\otimes\xi'')\otimes[x,y]-\pi(a\otimes\xi'')x\otimes y\\&=\partial(\sum\pi(a\otimes\xi'')\otimes y\wedge x)\,.\end{aligned}$$

On the other hand, since $\gamma(\Omega-\Omega')=0$ we can write $\Omega=\Omega'+\Omega''$ with

$$\Omega''=\sum b\otimes\eta(u\otimes v-v\otimes u-1\otimes[u,v])\,,$$

where $b\in E^{(k)}$, $\eta\in U(\mathfrak{n})$, u, $v\in\mathfrak{n}$. Obviously $\pi'(\Omega'')$ is a boundary. Therefore $\pi'(\Omega)$ is a boundary. Hence we have $H_1(\mathfrak{n},M)_{k+1}=0$. Now assume that the induced map $M^{(k+1)}\to E^{(k+1)}$ is an isomorphism. Since $H_1(\mathfrak{n},E)_{k+1}$ depends only on $E^{(k+1)}$, we see that $H_1(\mathfrak{n},E)_{k+1}=H_1(\mathfrak{n},M)_{k+1}=0$.

Conversely suppose that $H_1(\mathfrak{n},E)_{k+1}=0$. In order to prove $\rho: M^{(k+1)}\to E^{(k+1)}$ is an isomorphism, it suffices to show that $\rho: M_{k+1}\to E_{k+1}$ is injective. Suppose $\rho(\alpha)=0$ for an $\alpha\in M_{k+1}$. Since $\deg\alpha=k+1$, we can write

$$\alpha=\pi(\sum a\otimes x)\,,\qquad a\in E^{(k)}\,,\quad x\in\mathfrak{n}\,.$$

Then

$$\begin{aligned}\partial(\sum a\otimes x)&=\sum a\cdot x\\&=\rho\pi\sum a\otimes x=0\,.\end{aligned}$$

Therefore we can write

$$\sum a \otimes x = \partial(\sum b \otimes y \wedge z) .$$

Then

$$\begin{aligned} \alpha &= \pi\partial(\sum b \otimes y \wedge z) \\ &= \pi(\sum by \otimes z - bz \otimes y - b \otimes [y, z]) \\ &= \pi(\sum b \otimes yz - b \otimes zy - b \otimes [y, z]) \\ &= 0 , \end{aligned}$$

because $\pi(by \otimes z) = \pi(b \otimes yz)$, $\pi(bz \otimes y) = \pi(b \otimes zy)$ whenever $\deg(by)$, $\deg(bz) \le k$, which completes the proof.

Now let us recall the definition of quasi-regular basis. Given an admissible basis, in order to see if it is quasi-regular we must in general verify an infinite number of conditions. But if E is k-determined (in many instances of application we know beforehand the integer k to which E is k-determined), we have only to check a finite number of conditions. More precisely we have the following proposition. For convenience we assume $\mathfrak{n}_\mu \neq 0$, so that $\chi(1) = \mu$.

PROPOSITION 3.2. *Let E be a graded $U(\mathfrak{n})$-module satisfying $\mathscr{F}^{k+\mu}H_1(\mathfrak{n}, E) = 0$ for an integer k. If an admissible basis $\{u_1, \cdots, u_n\}$ of $\mathfrak{n}$ is injective to E_{k+j} for $0 \le j \le \mu - 1$, then it is a quasi-regular basis of $\mathfrak{n}$ relative to $\mathscr{F}^{k-\mu}E$.*

PROOF. We prove if $H_1(\mathfrak{n}, E)_{k+\mu} = 0$ and if $\{u_1, \cdots, u_n\}$ is injective to E_{k+j} $(0 \le j \le \mu - 1)$, then $\{u_1, \cdots, u_n\}$ is injective to $E_{k+\mu}$, that is,

$$(E/E(u_1, \cdots, u_i))_{k+\mu-d(u_{i+1})} \xrightarrow{u_{i+1}} (E/E(u_1, \cdots, u_i))_{k+\mu} \tag{3.1}$$

is injective for $i = 0, 1, \cdots, n-1$.

Now suppose that

$$\alpha u_{i+1} \equiv 0 \qquad \mathrm{mod}\ E(u_1, \cdots, u_i)$$

for an $\alpha \in E_{k+\mu-d(u_{i+1})}$. Then we can write:

$$\alpha u_{i+1} = \sum_{a \le i} \beta_a u_a ,$$

that is,

$$\partial(\alpha \otimes u_{i+1} - \sum \beta_a \otimes u_a) = 0 .$$

Since $H_1(\mathfrak{n}, E)_{k+\mu} = 0$, we can then find

$$\phi = \sum_{A < B \leq i+r} \phi_{AB} \otimes u_A \wedge u_B$$

such that

$$\alpha \otimes u_{i+1} - \sum_{a \leq i} \beta \otimes u_a = \partial \phi . \tag{3.2}$$

Note that in computing $\partial\phi$, $[u_A, u_B]$ can be always expressed as a linear combination of $\{u_C \mid C \leq \mathrm{Min}(A, B)\}$. Therefore if $r \leq 0$ it follows immediately from (3.2) that $\alpha = 0$. If $r = 1$, comparing the coefficients of u_{i+1} in (3.2), we have:

$$\alpha = \sum_{A \leq i} \phi_{A,i+1} u_A$$
$$\equiv 0 \qquad \bmod E(u_1, \cdots, u_i) ,$$

which shows that the map (3.1) is injective. Now in the case $r > 1$, let

$$A_0 = \mathrm{Max}\{A \mid \phi_{A,i+r} \neq 0\} .$$

Then, comparing the coefficients of u_{i+r} in (3.2), we have:

$$\phi_{A_0,i+r} u_{A_0} = - \sum_{A < A_0} \phi_{A,i+r} u_A .$$

Note that

$$\deg(\phi_{A_0,i+r} u_{A_0}) = \deg \phi - d(u_{i+r})$$
$$= k + \mu - d(u_{i+r}) .$$

Therefore $k \leq \deg(\phi_{A_0,i+r} u_{A_0}) \leq k + \mu - 1$. Hence by our assumption, we can write:

$$\phi_{A_0,i+r} = \sum_{A < A_0} \eta_A u_A .$$

Now set

$$\phi' = \phi - \partial \sum_{A < A_0} \eta_A \otimes u_A \wedge u_{A_0} \wedge u_{i+r} ,$$

and let ϕ' take the rôle of ϕ in (3.2). Then $A'_0 < A_0$, where A'_0 is the integer corresponding to ϕ' defined in the same way as A_0. Repeating this procedure, we can reduce to the case $r = 1$. We thus conclude that $\alpha \equiv 0 \bmod E(u_1, \cdots, u_i)$. Hence (3.1) is injective. q.e.d.

REMARK 1. Under the assumption of Prop. 3.2, we have actually $\mathscr{F}^k H_1(\mathfrak{n}, E) = 0$ as a consequence of Th. 2.1.

REMARK 2. In the case $\mu = 1$, the dual version of Prop. 3.2 is well known ([6] Prop. 3.3).

§4. So far we have shown that the existence of a quasi-regular sequence implies the vanishing of the homology group in certain range. In this section we consider whether there exists a quasi-regular sequence. In this and next sections we shall assume that the ground field $\boldsymbol{F}$ contains more than countable elements (e.g., $\boldsymbol{R}$ or $\boldsymbol{C}$), and that the graded module $E=\bigoplus_{p\in \boldsymbol{Z}} E_p$ satisfies $\dim E_p<\infty$ for $\forall p\in \boldsymbol{Z}$.

In the case where $\mathfrak{n}$ is abelian a slight modification of Serre's theorem (see Appendix of [2]) gives a necessary and sufficient condition for the existence of a quasi-regular sequence: Let $\mathfrak{a}$ be a finite dimensional graded abelian Lie algebra concentrated to degree $\nu>0$, namely $\mathfrak{a}=\mathfrak{a}_\nu$, and let $E=\bigoplus E_p$ be a graded $U(\mathfrak{a})$-module with $\dim E_p<\infty$. Then:

THEOREM 4.1 (Serre). *The following conditions are equivalent*:
(1) *There exists a quasi-regular sequence of* $\mathfrak{a}$ *of length s relative to* $\mathscr{F}^l E$.
(2) $\mathscr{F}^{l+p\nu}H_p(\mathfrak{a}, E)=0$, *for* $p=\dim\mathfrak{a}-s+1, \cdots, \dim\mathfrak{a}$.

In fact the implication (1) ⇒ (2) follows from Th. 2.1. The converse (2) ⇒ (1) follows from the induction as in the proof of Th. 2.1 and the following:

LEMMA 4.1. *The following conditions are equivalent*:
(1) *There exist a countable number of proper subspaces* $A_i\subset\mathfrak{a}$ *such that*

$$\mathscr{F}^l E \xrightarrow{\ u\ } \mathscr{F}^{l+\nu}E$$

is injective for all $u\in\mathfrak{a}\setminus\bigcup A_i$.
(2) $\mathscr{F}^{l+\nu n}H_n(\mathfrak{a}, E)=0$, *where* $n=\dim\mathfrak{a}$.

Note that in the above lemma we do not assume E is finitely generated as $U(\mathfrak{a})$-module, the proof is however similar to that of Serre. For the sake of completeness we will give the proof of Lemma 4.1: It is clear that (1) implies (2). To prove the converse, let us consider $\mathrm{Ass}(\mathscr{F}^l E)$, the set of all associated prime ideals to $\mathscr{F}^l E$; a prime ideal $\mathfrak{p}\subset U(\mathfrak{a})$ is said to belong to $\mathrm{Ass}(\mathscr{F}^l E)$ if and only if there exists a non-zero $\alpha\in\mathscr{F}^l E$ such that

$$\mathfrak{p}=\{x\in U(\mathfrak{a}) \mid \alpha x=0\}\,.$$

Then for $u\in\mathfrak{a}$, by a standard argument we see that the multiplication $\mathscr{F}^l E\xrightarrow{\ u\ }\mathscr{F}^{l+\nu}E$ is injective if and only if $u\notin\bigcup_{\mathfrak{p}\in\mathrm{Ass}(\mathscr{F}^l E)}\mathfrak{p}$.

On the other hand note that

$$\mathrm{Ass}(\mathscr{F}^l E)=\bigcup_{p\geq l}\mathrm{Ass}(E_p U(\mathfrak{a}))$$

and that $\mathrm{Ass}(E_p U(\mathfrak{a}))$ is a finite set because $E_p U(\mathfrak{a})$ is finitely generated (cf. [1]). Hence $\mathrm{Ass}(\mathscr{F}^l E)$ is a countable set.

Now suppose that $\mathscr{F}^{l+\nu n}H_n(\mathfrak{a}, E)=0$. Then the condition "$\alpha\cdot U(\mathfrak{a})=0$, $\alpha\in\mathscr{F}^l E$"

implies $\alpha=0$, that is to say $U(\mathfrak{a})\notin \mathrm{Ass}(\mathscr{F}^l E)$. Hence $\mathrm{Ass}(\mathscr{F}^l E)$ consists of a countable number of proper ideals of $U(\mathfrak{a})$. Therefore (2) implies (1).

Now returning to our nilpotent case, let $\mathfrak{n}=\bigoplus_{p=1}^{\mu}\mathfrak{n}_p$ be a graded Lie algebra and $E=\bigoplus_{p\in\mathbf{Z}}E_p$ a graded $U(\mathfrak{n})$-module. For each integer i, $E/(E\cdot\mathscr{F}^{i+1}\mathfrak{n})$ becomes a graded $U(\mathfrak{n}_i)$-module, where $\mathfrak{n}_i$ is regarded as an abelian graded Lie algebra concentrated to degree i, so that we can consider the homology group $H_p(\mathfrak{n}_i, E/(E\cdot\mathscr{F}^{i+1}\mathfrak{n}))$.

For an integer $r\geq 0$, define integer λ, s by

$$r=\dim\mathscr{F}^{\lambda+1}\mathfrak{n}+s \qquad (0\leq\lambda\leq\mu,\ 0\leq s<\dim\mathfrak{n}_\lambda)\,.$$

Then we have:

THEOREM 4.2. *There exists a quasi-regular sequence of* $\mathfrak{n}$ *relative to* $\mathscr{F}^l E$ *of length* r *if and only if*

$$\mathscr{F}^{l+\mu-i+pi}H_p(\mathfrak{n}_i, E/E\cdot\mathscr{F}^{i+1}\mathfrak{n})=0$$

for $\lambda\leq i\leq\mu$ *and*

$$p=\begin{cases}1,\cdots,\mathfrak{n}_i & \text{if}\quad i\geq\lambda+1\,,\\ \mathfrak{n}_\lambda-s+1,\cdots,\mathfrak{n}_\lambda & \text{if}\quad i=\lambda\,.\end{cases}$$

In fact the existence of such a quasi-regular sequence is equivalent to the existence of quasi-regular bases of $\mathfrak{n}_i$ relative to $\mathscr{F}^{l+\mu-i}(E/E\cdot\mathscr{F}^{i+1}\mathfrak{n})$ for $\lambda<i\leq\mu$ and quasi-regular sequence of $\mathfrak{n}_\lambda$ of length s relative to $\mathscr{F}^{l+\mu-\lambda}(E/E\cdot\mathscr{F}^{i+1}\mathfrak{n})$. Hence Theorem 4.2 follows from Th. 4.1.

As a consequence of Th. 4.2, we see that there does not necessarily exist quasi-regular sequence. In fact, if there exists one which covers $\mathfrak{n}_\mu$, then we should have $\mathscr{F}^l H_{\dim\mathfrak{n}_\mu}(\mathfrak{n}_\mu, E)=0$ for large l. But E is not necessarily finitely generated as $U(\mathfrak{n}_\mu)$-module even if so is as $U(\mathfrak{n})$-module. Therefore $\mathscr{F}^l H_{\dim\mathfrak{n}_\mu}(\mathfrak{n}_\mu, E)$ does not vanish in general (cf. Th.1.1). For example, if $\mathfrak{n}_\mu$ acts trivially on E, then $H(\mathfrak{n}_\mu, E)=E\otimes\bigwedge\mathfrak{n}_\mu$, which has non-zero elements of arbitrary high degree provided that $\dim E=\infty$. Hence for such E, there does not exist any quasi-regular sequence of $\mathfrak{n}$ relative to $\mathscr{F}^l E$, however large l may be. This is a typical phenomenon of nilpotent cases contrary to abelian cases.

§5. Here we discuss how to compute the homology group $H(\mathfrak{n}, E)$ in general. First we treat one extreme case.

THEOREM 5.1. *If the* $U(\mathfrak{n})$*-module* E *satisfies the following conditions*:

(1) $\mathfrak{n}_\mu$ *acts trivially on* $\mathscr{F}^{l-\mu}E$,

(2) $\mathscr{F}^{l+\chi(\mathfrak{n}/\mathfrak{n}_\mu, q)}H_q(\mathfrak{n}/\mathfrak{n}_\mu, \mathscr{F}^{l-\mu}E)=0$ *for* $q=\dim\mathfrak{n}/\mathfrak{n}_\mu-r+1,\cdots,\dim\mathfrak{n}/\mathfrak{n}_\mu$ (*this condition is satisfied if there exists a quasi-regular sequence of length* r *of* $\mathfrak{n}/\mathfrak{n}_\mu$ *relative to* $\mathscr{F}^l E$),

then $\mathscr{F}^{l+\chi(\mathfrak{n},p)}H_p(\mathfrak{n}, E)=0$ *for* $p=\dim\mathfrak{n}-r+1,\cdots,\dim\mathfrak{n}$.

PROOF. We proceed by induction on dim $\mathfrak{n}_\mu$. Let u be a non-zero element of $\mathfrak{n}_\mu$ and put $\bar{\mathfrak{n}}=\mathfrak{n}/\langle u\rangle$. Then $\mathscr{F}^{l-\mu}E$, which will be denoted by E', is a $U(\bar{\mathfrak{n}})$-module satisfying the conditions stated above. Hence by induction assumption we have:

(5.1) $$\mathscr{F}^{l+\chi(\bar{\mathfrak{n}},q)}H_q(\bar{\mathfrak{n}}, E')=0 \qquad \text{for} \quad q=\dim\bar{\mathfrak{n}}-r+1, \cdots, \dim\bar{\mathfrak{n}} .$$

Now identifying $\bar{\mathfrak{n}}$ with a subspace of $\mathfrak{n}$ complementary to $\langle u\rangle$, we have the following exact sequence:

$$0 \longrightarrow \bigwedge\bar{\mathfrak{n}} \xrightarrow{\wedge u} \bigwedge\mathfrak{n} \longrightarrow \bigwedge\bar{\mathfrak{n}} \longrightarrow 0 .$$

Since u acts trivially on E', we have the exact sequence of complexes:

$$0 \longrightarrow E'\otimes\bigwedge\bar{\mathfrak{n}} \xrightarrow{\wedge u} E'\otimes\bigwedge\mathfrak{n} \longrightarrow E'\otimes\bigwedge\bar{\mathfrak{n}} \longrightarrow 0 ,$$

which yields a long exact sequence:

(5.2) $$\longrightarrow \mathscr{F}^k H_{p-1}(\bar{\mathfrak{n}}, E') \longrightarrow \mathscr{F}^{k+\mu}H_p(\mathfrak{n}, E') \longrightarrow \mathscr{F}^{k+\mu}H_p(\bar{\mathfrak{n}}, E') \longrightarrow \cdots .$$

Let $k=l+\chi(\mathfrak{n}, p)-\mu$ and let $\dim\mathfrak{n}-r+1\leq p\leq\dim\mathfrak{n}$. Then from (5.1), we see that

$$\mathscr{F}^k H_{p-1}(\bar{\mathfrak{n}}, E')=\mathscr{F}^{k+\mu}H_p(\bar{\mathfrak{n}}, E')=0 ,$$

because

$$p-1\geq\dim\bar{\mathfrak{n}}-r+1 ,$$
$$k-(l+\chi(\bar{\mathfrak{n}}, p-1))=\chi(\mathfrak{n}, p)-\mu\chi(\bar{\mathfrak{n}}, p-1)=0 ,$$
$$k+\mu-(l+\chi(\bar{\mathfrak{n}}, p))=\chi(\mathfrak{n}, p)-\chi(\bar{\mathfrak{n}}, p)\geq 0 .$$

Hence by (5.2) we have $\mathscr{F}^{k+\mu}H_p(\mathfrak{n}, E')=0$. But note that

$$\mathscr{F}^{k+\mu}(E'\otimes\bigwedge^i\mathfrak{n})=\mathscr{F}^{k+\mu}(E\otimes\bigwedge^i\mathfrak{n}) \qquad \text{for} \quad i\leq p+1 ,$$

because

$$k+\mu-\chi(i)\geq k+\mu-\chi(p+1)$$
$$=l+\chi(p)-\chi(p+1)\geq l-\mu .$$

Therefore

$$\mathscr{F}^{k+\mu}H_p(\mathfrak{n}, E)=\mathscr{F}^{k+\mu}H_p(\mathfrak{n}, E')=0 ,$$

which completes the proof.

Now let us sketch a general idea to compute $H(\mathfrak{n}, E)$. We assume that E is finitely generated as $U(\mathfrak{n})$-module.

Case 1, where there exists $u\in\mathfrak{n}_\mu$ such that $E \xrightarrow{u} E$ is injective. In this case, using the exact sequences (2.4), (2.5), we can reduce our computation to that of

$H(\mathfrak{n}/\langle u\rangle, E/E(u))$.

Case 2, where $\mathfrak{n}_\mu$ acts trivially on E. By Prop. 5.1, the computation is reduced to that of $H(\mathfrak{n}/\mathfrak{n}_\mu, E)$.

General case: We define an ascending sequence of $U(\mathfrak{n})$-submodules $\{K^i\}$ by

$$K^1=\{\alpha\in E \mid \alpha\mathfrak{n}_\mu=0\}\ ,$$

$$K^{i+1}=\{\alpha\in E \mid \alpha\mathfrak{n}_\mu\subset K^i\} \qquad \text{for} \quad i\geq 1\ .$$

Then by Noetherian property this sequence stabilizes, say, $K^s=K^{s+1}=\cdots$, for an s. We have the following exact sequences:

$$0\to K^s\to E\to E/K^s\to 0$$

$$0\to K^{s-1}\to K^s\to K^s/K^{s-1}\to 0$$

$$\cdots\cdots\cdots$$

$$0\to K^2\to K^3\to K^3/K^2\to 0$$

$$0\to K^1\to K^2\to K^2/K^1\to 0\ .$$

Here $K^1, K^2/K^1, \cdots, K^s/K^{s-1}$ are $U(\mathfrak{n})$-modules on which $\mathfrak{n}_\mu$ acts trivially (case 2), and E/K^s is a module which satisfies $H_{\dim \mathfrak{n}_\mu}(\mathfrak{n}_\mu, E/K^s)=0$, hence is of type treated in case 1 by virtue of Lemma 4.1. Thus by using the above exact sequences we may reduce our computation to case 1 and case 2 (up to long exact sequences).

§6. Let $\mathfrak{g}=\bigoplus_{p\in \boldsymbol{Z}}\mathfrak{g}_p$ be a transitive graded Lie algebra of depth $\mu>0$, which is by definition a graded Lie algebra satisfying the following conditions:

(1) $\dim \mathfrak{g}_p<\infty$ for all $p\in \boldsymbol{Z}$,

(2) If $[x_p, \mathfrak{g}_-]=0$ for $x_p\in\mathfrak{g}_p$ $(p\geq 0)$, then $x_p=0$, where we set $\mathfrak{g}_-=\bigoplus_{q<0}\mathfrak{g}_q$,

(3) $\mathfrak{g}_q=0$ for $q<-\mu$.

Then we have the cohomology group $H(\mathfrak{g}_-, \mathfrak{g})$ associated with the adjoint representation of $\mathfrak{g}_-$ on $\mathfrak{g}$, that is the cohomology group of the cochian complex $\mathrm{Hom}(\bigwedge\mathfrak{g}_-, \mathfrak{g})$ with the coboundary operator $\mathrm{Hom}(\bigwedge^p\mathfrak{g}_-, \mathfrak{g})\xrightarrow{\delta}\mathrm{Hom}(\bigwedge^{p+1}\mathfrak{g}_-, \mathfrak{g})$ defined by

$$\delta\phi(x_1, \cdots, x_{p+1})=\sum_{i=1}^{p+1}(-1)^{i-1}[x_i, \phi(x_1, \cdots, \hat{x}_i, \cdots, x_{p+1})]$$

$$+\sum_{i<j}(-1)^{i+j}\phi([x_i, x_j], x_1, \cdots, \hat{x}_i, \cdots, \hat{x}_j, \cdots, x_{p+1})$$

for $\phi\in\mathrm{Hom}(\bigwedge^p\mathfrak{g}_-, \mathfrak{g})$, $x_1, \cdots, x_{p+1}\in\mathfrak{g}_-$. Let $\mathrm{Hom}(\bigwedge\mathfrak{g}_-, \mathfrak{g})_r$ be the subspace of $\mathrm{Hom}(\bigwedge\mathfrak{g}_-, \mathfrak{g})$ consisting of all the elements of degree r (an element $\phi\in\mathrm{Hom}(\bigwedge\mathfrak{g}_-, \mathfrak{g})$ is said to be of degree r if $\phi(\mathfrak{g}_{i_1}\wedge\cdots\wedge\mathfrak{g}_{i_p})\subset\mathfrak{g}_{i_1+\cdots+i_p+r}$ for any $i_1, \cdots, i_p<0$). As easily seen, $\mathrm{Hom}(\bigwedge\mathfrak{g}_-, \mathfrak{g})_r$ is a subcomplex of $\mathrm{Hom}(\bigwedge\mathfrak{g}_-, \mathfrak{g})$. Denoting by $H(\mathfrak{g}_-, \mathfrak{g})_r=\bigoplus H^p(\mathfrak{g}_-, \mathfrak{g})_r$ its cohomology group, we obtain the direct sum decomposition

$$H(\mathfrak{g}_-, \mathfrak{g}) = \bigoplus_{p,r} H^p(\mathfrak{g}_-, \mathfrak{g})_r .$$

The cohomology group, endowed with this bi-gradation, is called the generalized Spencer cohomology group of the graded Lie algebra $\mathfrak{g}$.

Let $\mathfrak{n} = \bigoplus_{p>0} \mathfrak{n}_p$ and $E = \bigoplus E_p$ be given by $\mathfrak{n}_p = \mathfrak{g}_{-p}$ and $E_p = (\mathfrak{g}_p)^*$. Then E is a graded $U(\mathfrak{n})$-module generated by $E_- = \bigoplus_{p<0} E_p$, and the p-th homology group of degree r, $H_p(\mathfrak{n}, E)_r$ is dual to the p-th cohomology group of degree r, $H^p(\mathfrak{g}_-, \mathfrak{g})_r$. Moreover $M(E^{(k)}) = E$ if and only if $\mathfrak{g}$ is the prolongation of $\bigoplus_{p \leq k} \mathfrak{g}_p$ (for the definition of the prolongation see e.g., [3]).

Therefore by passing to the dual, we can apply all the result of the preceding sections to the cohomology groups of graded Lie algebras. Now let us translate the main part of them.

Let $\mathfrak{g} = \bigoplus_{p \geq -\mu} \mathfrak{g}_p$ be a transitive graded Lie algebra of depth μ. A sequence $\{e_1, \cdots, e_r\}$ in $\mathfrak{g}_-$ is called admissible if it is linearly independent and if $e_i \in \mathfrak{g}_k$ for all i such that $\dim \bigoplus_{p<k} \mathfrak{g}_p < i \leq \dim \bigoplus_{p \leq k} \mathfrak{g}_p$, and called an admissible basis if moreover it forms a basis of $\mathfrak{g}_-$. By taking an admissible basis $\{e_1, \cdots, e_n\}$, we define:

$$\chi(p) = \chi(\mathfrak{g}_-, p) = \sum_{i=1}^{p} \deg e_i .$$

For an admissible sequence $\{e_1, \cdots, e_r\}$, we set

$$(\mathfrak{g}_p)_i = \{A \in \mathfrak{g}_p \mid [e_j, A] = 0 \text{ for } j \leq i\} .$$

Then we have the following exact sequence:

$$0 \longrightarrow (\mathfrak{g}_p)_{i+1} \longrightarrow (\mathfrak{g}_p)_i \xrightarrow{e_{i+1}} (\mathfrak{g}_{p+\deg e_{i+1}})_i$$

for $0 \leq i \leq r-1$, where we denote also by e_{i+1} the map: $A \mapsto [e_{i+1}, A]$.

DEFINITION 6.1. An admissible sequence $\{e_1, \cdots, e_r\}$ of $\mathfrak{g}_-$ is called "surjective from $\mathfrak{g}_m$" if the following maps are surjective for $i = 0, \cdots, r-1$:

$$(\mathfrak{g}_m)_i \xrightarrow{e_{i+1}} (\mathfrak{g}_{m+\deg e_{i+1}})_i .$$

We say that $\{e_1, \cdots, e_r\}$ is a quasi-sequence of $\mathfrak{g}_-$ with respect to $\mathscr{F}^l\mathfrak{g}$ if it is surjective from $\mathfrak{g}_m$ for all $m \geq l + \chi(1)$.

PROPOSITION 6.1. *Assume that*

(1) $\mathfrak{g}$ *is the prolongation of* $\bigoplus_{p<k+\mu} \mathfrak{g}_p$,

(2) *An admissible basis* $\{e_1, \cdots, e_n\}$ *of* $\mathfrak{g}_-$ *is surjective from* $\mathfrak{g}_{k+i}$ *for* $i = 0, 1, \cdots, \mu-1$.

Then $\{e_1, \cdots, e_n\}$ *is a quasi-regular basis with respect to* $\mathscr{F}^{k-\mu}\mathfrak{g}$.

THEOREM 6.1. *If there exists a quasi-regular sequence of* $\mathfrak{g}_-$ *of length r relative to* $\mathscr{F}^l\mathfrak{g}$, *then we have*

$$\mathscr{F}^{l+\chi(p)}H^p(\mathfrak{g}_-, \mathfrak{g})=0$$

for $p=\dim \mathfrak{g}_- - r+1, \cdots, \dim \mathfrak{g}_-$.

We can also dualize the discussion of §4 and §5.

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