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Spectral Flow and Maslov Index Arising from Lagrangian Intersections

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Dedicated to Professor Nobuhiko Tatsuuma on his 60-th birthday

§1. Introduction.

Let P be a $2n$-dimensional symplectic manifold with the symplectic structure ω, and L, L' be two Lagrangian submanifolds of P. We assume, in this paper, that L and L' intersect transversally with a non-empty intersection, and a smooth map

$$f: I\times I \to P, \qquad I=[0, 1]$$

is given with the following properties:

$$\begin{aligned} &f(\tau, 0)\in L, \quad f(\tau, 1)\in L' \qquad \text{for any} \quad \tau\in I, \\ &f(0, t)\equiv x, \quad f(1, t)\equiv y \qquad \text{for any} \quad t\in I, \end{aligned} \tag{1-1}$$

where $x, y\in L\cap L'$. Under these assumptions, two homotopic invariants arise. One is the spectral flow associated to a family of certain operators, and the other is the Maslov index of a curve which relates with the boundary conditions imposed on the operators in that family.

A. Floer has shown that these two are equal to relative Morse index from x to y, which is defined as the Fredholm index of a certain elliptic operator (see [F]).

Now let us explain the problem more precisely. Let g be a Riemannian metric on P which is adapted to the symplectic structure ω, that is, if we write $\omega(X, Y)=g(X, JY)$, for any vector fields X, Y on P, then J is an almost complex structure of P. Hence $J^2=-\mathrm{Id}$ and ${}^tJ=-J$ (${}^tJ$ is the transpose of J with respect to the metric g). We fix such a metric henceforth, and denote by ∇ the Riemannian connection of the metric g. Let Ω be a path space of P consisting of smooth paths $z: I\to P$ such that $z(0)\in L$ and $z(1)\in L'$. Consider, at least locally, a symplectic action functional $a: \Omega\to \boldsymbol{R}$ defined by

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the equation

$$\langle da(z), \xi\rangle = \int_0^1 \omega(\dot{z}, \xi)dt\,, \tag{1-2}$$

where ξ is a smooth section of the induced bundle $z^*(TP)$ on I, or ξ is a vector field along the curve $\{z(t)\}$, which can be regarded also as a tangent vector at z in Ω. Then we obtain an operator field $\mathscr{A}_z$ on Ω as "Hessian of the functional a":

$$\mathscr{A}_z(\xi) = J\nabla_{\dot{z}}\xi + (\nabla_\xi J)(\dot{z})\,, \qquad \xi \in C_0^\infty(z^*(TP))\,. \tag{1-3}$$

Let $f : I \times I \to P$ be a map as above, and we denote $\mathscr{A}_\tau = \mathscr{A}_{f_\tau}$ for the path $f_\tau(t) = f(\tau, t)$. We would like to associate two quantities, so-called spectral flow and Maslov index, to the family of operators $\{\mathscr{A}_\tau\}$ both of which depend only on the two points $x, y \in L \cap L'$ and the homotopy class of the map f. To do so, there are three problems to be made clear:

(1) It can be verified that, if $z \in \Omega$ is a constant path, i.e. $z(t) \equiv x$, $x \in L \cap L'$, then $\mathscr{A}_z$ is formally symmetric, or if P is Kähler and g is the Kähler metric, then $\mathscr{A}_z$ is symmetric for any path $z \in \Omega$. But the operator $\mathscr{A}_z$ is not even formally symmetric in general.

(2) Each operator $\mathscr{A}_z$ acts on a different space of sections. We must make the domains of the operators to be identical one, and give a definite meaning of the continuity of the family $\{\mathscr{A}_\tau\}$ with respect to the parameter τ.

(3) The family $\{\mathscr{A}_z\}$ does not form a loop, if $x \neq y$. We would like to make it a loop in a natural way.

Although A. Floer discussed on the coincidence of two quantities in his paper [F] by interceding with a single Fredholm operator ('desuspension' of one parameter family of selfadjoint Fredholm operators (see [APS])), it is not so clear for us about these points (1), (2) and (3). So it is our aim of this paper to make clear these points together with giving a definite description of the two homotopy invariants within the framework given by [AS], [BW] and [V]. By making clear these points, especially by clarifying the continuity of the family $\{\mathscr{A}_\tau\}$ with respect to the parameter τ, we get a direct proof (without 'desuspending' the family $\{\mathscr{A}_\tau\}$) of the coincidence of the two homotopy invariants. Hence we think it is worth while to write this paper.

In §2, we review the notion "spectral flow" by following [AS] and [BW], and note a slight generalization in the proposition 2.4, which is used to define the spectral flow for loops of non-selfadjoint Fredholm operators. Also we explain an example to which our case is reduced in the final step in the proof of the main Theorem 5.1, in §5.

In §3, we shall construct two loops in Lagrangian-Grassmannian manifold $\Lambda(n) = U(n)/O(n)$ from the data given by the map $f : I \times I \to P$ with the property (1-1) and show that the Maslov indices of these two loops coincide. One of the two loops is used to construct a loop of operators from the family $\{\mathscr{A}_\tau\}$.

In §4, we construct a continuous loop in the space of bounded selfadjoint Fredholm operators $\hat{\mathscr{F}}_*$ (see §2) from the family $\{\mathscr{A}_\tau\}$, and in §5 we show our main Theorem 5.1.

§ 2. Spectral flow.

In this section, we first describe the notion "spectral flow". For more details see [AS], [APS] and [BW].

Let H_C be a separable complex Hilbert space of infinite dimension, $\mathscr{F}=\mathscr{F}(H_C)$ the space of bounded Fredholm operators on H_C, and $\hat{\mathscr{F}}$ the subspace of $\mathscr{F}$ consisting of self-adjoint operators.

$\hat{\mathscr{F}}$ has three components $\hat{\mathscr{F}}_+$, $\hat{\mathscr{F}}_-$ and $\hat{\mathscr{F}}_*$ consisting of essentially positive operators, essentially negative operators, and others respectively. It is easy to see that $\hat{\mathscr{F}}_\pm$ are contractible.

Define a map

$$\alpha : \hat{\mathscr{F}}_* \to \Omega\mathscr{F}$$

by assigning to each $A \in \hat{\mathscr{F}}_*$ the path

$$\cos \pi t + \sqrt{-1}\, A \sin \pi t \qquad \text{for} \quad t \in I\,,$$

then we have the following theorem.

THEOREM [AS]. *The map α is a homotopy equivalence, and so $\hat{\mathscr{F}}_*$ is a classifying space for the functor K^{-1}, i.e., for any compact space X, we have an isomorphism*

$$K^{-1}(X) = [X, \hat{\mathscr{F}}_*]\,, \tag{2-1}$$

where $[X, \hat{\mathscr{F}}_*]$ is the set of homotopy classes of continuous maps from X to $\hat{\mathscr{F}}_*$.

From the Bott periodicity theorem and this theorem, we have

$$\pi_1(\hat{\mathscr{F}}_*) \cong \boldsymbol{Z}\,. \tag{2-2}$$

The isomorphism (2-2) implies that a loop in $\hat{\mathscr{F}}_*$ has just a homotopy invariant and it can be characterized by an integer, called, "spectral flow".

We now explain this. Let $\ell : I \to \hat{\mathscr{F}}_*$ be a continuous map, and ℓ_0 and ℓ_1 have the same spectral set: $\sigma(\ell_0) = \sigma(\ell_1)$. Then, roughly speaking, the spectral flow of the family $\{\ell_t\}$ is the difference between the number of eigenvalues which change the sign from $-$ to $+$ on I and the number of eigenvalues which change the sign from $+$ to $-$.

To put it more precisely, let $\hat{F}(\infty)$ be a subspace of $\hat{\mathscr{F}}_*$ such that $A \in \hat{F}(\infty)$, if and only if,

> the intersection $\sigma(A) \cap (-1, 1)$ consists of finite isolated eigenvalues with finite multiplicities, $\sigma_{ess}(A) = \{-1, 1\}$ and the norm of A is equal to 1, $\|A\| = 1$,

where $\sigma_{ess}(A)$ is the set of essential spectra of the operator A (see [K] for the definitions).

PROPOSITION 2.1 (see [BW]). *The space $\hat{F}(\infty)$ is a deformation retract of $\hat{\mathscr{F}}_*$.*

Let $\ell : I \to \hat{F}(\infty)$ be a continuous loop ($\ell_0 = \ell_1$, and the topology of $\hat{\mathscr{F}}(\infty)$ is of course the uniform topology). Then the graph of the spectrum of ℓ can be parametrized through a finite monotone sequence of continuous functions

$$\lambda_j : I \to [-1, 1], \qquad j=1, \cdots, N,$$
$$-1 \leqq \lambda_1(t) \leqq \lambda_2(t) \leqq \cdots \leqq \lambda_{N-1}(t) \leqq \lambda_N(t) \leqq 1 \quad \text{for } t \in I.$$

Then we have

PROPOSITION 2.2 [BW]. *Let $\ell : I \to \hat{F}(\infty)$ be as above, then there exists an integer s such that $\lambda_{k+s}(0)=\lambda_k(1)$, for each k, where we regard $\lambda_{-|s|}=\lambda_{-|s|+1}=\cdots=\lambda_0=-1$, and $\lambda_N=\lambda_{N+1}=\cdots=\lambda_{N+|s|}=1$. Moreover the number s is homotopically invariant.*

From this proposition we can give the

DEFINITION. For a homotopy class of a loop $\ell : I \to \hat{F}(\infty)$, we can define $\mathrm{sf}\{\ell\}=s$, where s is an integer in the proposition 2.2 above, and we call it the spectral flow of the loop ℓ.

Hence by this definition together with the proposition 2.1 we have

PROPOSITION 2.3 ([BW]). *The map defined by the spectral flow gives an isomorphism* $\mathrm{sf} : \pi_1(\hat{\mathscr{F}}_*) \cong \boldsymbol{Z}$.

We will now explain some relations between real and complex cases briefly. Let $H_{\boldsymbol{R}}$ be a real Hilbert space, then $\mathscr{F}(H_{\boldsymbol{R}})$, $\hat{\mathscr{F}}(H_{\boldsymbol{R}})$, $\hat{\mathscr{F}}_{\pm}(H_{\boldsymbol{R}})$ and $\hat{\mathscr{F}}_*(H_{\boldsymbol{R}})$ are defined in the same ways as for the complex cases, and $\hat{\mathscr{F}}_*(H_{\boldsymbol{R}})$ is, in this case, a classifying space for the functor KR^{-7} (real K-group). Although $\hat{\mathscr{F}}_*(H_{\boldsymbol{R}})$ and $\hat{\mathscr{F}}_*(H_{\boldsymbol{R}} \otimes \boldsymbol{C})$ are not homotopic, we have

$$\pi_1(\hat{\mathscr{F}}_*(H_{\boldsymbol{R}})) \cong \pi_1(\hat{\mathscr{F}}_*(H_{\boldsymbol{R}} \otimes \boldsymbol{C}))$$

through the complexification $\hat{\mathscr{F}}_*(H_{\boldsymbol{R}}) \subset \hat{\mathscr{F}}_*(H_{\boldsymbol{R}} \otimes \boldsymbol{C})$. This can be done by defining an isomorphism

$$\mathrm{sf} : \pi_1(\hat{\mathscr{F}}_*(H_{\boldsymbol{R}})) \cong \boldsymbol{Z}$$

in the same way as in the complex case.

REMARK 1. In this paper we will work on real Hilbert spaces.

REMARK 2. Let $\hat{F}(H_{\boldsymbol{R}})$ be the space of skew-adjoint Fredholm operators on a real Hilbert space $H_{\boldsymbol{R}}$. Various subspaces $\hat{F}^{-k}(H_{\boldsymbol{R}})$ of $\hat{F}(H_{\boldsymbol{R}})$, for $k=1, \cdots, 7$, are defined through the $*$-representations of Clifford algebras on $H_{\boldsymbol{R}}$, each of which is a classifying space for the functor KR^{-k}, the k-th real K-group (see [AS]). For example $\hat{F}(H_{\boldsymbol{R}})$ itself is a classifying space for KR^{-1}. In this case the complexification

$$\hat{F}(H_{\boldsymbol{R}}) \subset \hat{F}(H_{\boldsymbol{R}} \otimes \boldsymbol{C}) \cong \sqrt{-1}\,\hat{\mathscr{F}}_*(H_{\boldsymbol{R}} \otimes \boldsymbol{C})$$

induces the trivial map $\pi_1(\hat{F}(H_{\boldsymbol{R}})) \cong \boldsymbol{Z}_2 \to \pi_1(\hat{\mathscr{F}}_*(H_{\boldsymbol{R}} \otimes \boldsymbol{C})) \cong \boldsymbol{Z}$. As for the case $k=7$, the space $\hat{F}^{-7}(H_{\boldsymbol{R}})$ is isomorphic to the space $\hat{\mathscr{F}}_*(H_{\boldsymbol{R}})$.

Next we give a proposition.

PROPOSITION 2.4. *Let $\mathscr{K}$ be the space of all compact operators on H (H is real or complex), then the inclusion*

$$i : \hat{\mathscr{F}}_* \to \hat{\mathscr{F}}_* + \mathscr{K}$$

is a homotopy equivalence.

PROOF. Let $A = S + K$, $S \in \hat{\mathscr{F}}_*$, $K \in \mathscr{K}$, then $(A + A^*)/2 = S + (K + K^*)/2 \in \hat{\mathscr{F}}_*$. So define a map

$$j : \hat{\mathscr{F}}_* + \mathscr{K} \to \hat{\mathscr{F}}_*$$

by $j(A) = (A + A^*)/2$, then $i \circ j$ is homotopic to Id through a homotopy $f_t(A) = tA + (1-t) i \circ j(A)$. Because, if $A = S + K$, $S \in \hat{\mathscr{F}}_*$, $K \in \mathscr{K}$, then

$$\begin{aligned} f_t(A) &= t(S+K) + (1-t)(S + (K+K^*)/2) \\ &= S + ((1+t)/2)K + ((1-t)/2)K^* \in \hat{\mathscr{F}}_* + \mathscr{K} \end{aligned}$$

for each $t \in I$.

Owing to this proposition we have an integer "spectral flow" for a loop $\{\ell_t\}$ in $\hat{\mathscr{F}}_* + \mathscr{K}$, which can be regarded as a total number of eigenvalues of $\{\ell_t\}$ across the imaginary axis with directions when t moves from 0 to 1. Especially we have

COROLLARY 2.5. *If a loop $\ell : I \to \hat{\mathscr{F}}_* + \mathscr{K}$ has the form $\ell_t = S_t + K_t$, where $S : I \to \hat{\mathscr{F}}_*$ and $K : I \to \mathscr{K}$ are both continuous loops, then* $\mathrm{sf}\{\ell_t\} = \mathrm{sf}\{S_t\}$.

Finally we discuss an important example to which our case will be reduced in §5. Let

$$J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$$

where I_n is the $n \times n$ unit matrix, and consider an operator

$$A = J \frac{d}{dt} : C_0^\infty(I) \otimes \boldsymbol{R}^{2n} \to L_2(I) \otimes \boldsymbol{R}^{2n} .$$

Here function spaces $C_0^\infty(I)$ and $L_2(I)$ (square integrable functions on the interval $I = [0, 1]$) are to be considered as those consisting of real valued functions. We will denote $L_2(I) \otimes \boldsymbol{R}^{2n}$ by $H_{\boldsymbol{R}}$ and by W_1, the first order Sobolev space, i.e., $f \in W_1$, if and only if, $f \in H_{\boldsymbol{R}}$ and $f' \in H_{\boldsymbol{R}}$ (the norm of $f \in W_1$ will be defined as $\|f\|_1^2 = \|f\|^2 + \|f'\|^2$).

Let $(\boldsymbol{R}^{2n}, \omega)$ be the symplectic vector space with the skew symmetric form ω defined by $J : \omega(x, y) = (x, Jy)$, where $(\cdot, \cdot)$ is the standard Euclidean inner product on $\boldsymbol{R}^{2n}$.

Henceforth we will use identifications:

$$\boldsymbol{R}^{2n} \cong \boldsymbol{C}^n \cong \boldsymbol{R}^n \oplus \sqrt{-1}\,\boldsymbol{R}^n$$

and

$$Sp(n, \boldsymbol{R}) \cap SO(2n) \cong U(n)\,.$$

We also use the decomposition $\boldsymbol{R}^n \oplus \sqrt{-1}\,\boldsymbol{R}^n \equiv \lambda_{\mathrm{Re}} \oplus \sqrt{-1}\,\lambda_{\mathrm{Im}}$. Let $\Lambda(n)$ be the set of all Lagrangian subspaces of $\boldsymbol{R}^{2n}$ (n-dim subspace on which ω vanishes). λ_{Re} and λ_{Im} are in $\Lambda(n)$. Under these identifications the unitary group $U(n)$ acts on $\Lambda(n)$ transitively, and we have $\Lambda(n) = U(n)/O(n)$ ($O(n)$ is the stationary subgroup of the point λ_{Re}).

Let λ be a Lagrangian subspace in $\boldsymbol{R}^{2n}$, and define a subspace of $H_{\boldsymbol{R}}$ as follows:

$$D_\lambda = \{f : f \in W_1,\ f(0) \in \lambda_{\mathrm{Re}},\ f(1) \in \lambda\}\,.$$

Then we have,

PROPOSITION 2.6. *The operator A can be extended to D_λ as a selfadjoint operator. We denote this operator by A_λ.*

Next, let $u : I \to U(n)$ be a C^2-class curve with $u_0 = \mathrm{Id}$, and define orthogonal operators $U_\tau : H_{\boldsymbol{R}} \to H_{\boldsymbol{R}}$ by

$$(U_\tau f)(t) = u_{t\tau}(f(t))\,.$$

Put $\lambda_\tau = u_\tau(\lambda_{\mathrm{Re}})$, and $A_\tau = A_{\lambda_\tau}$, then we have

PROPOSITION 2.7. *The domain of the operator $U_\tau^{-1} A_\tau U_\tau$ is $D_{\lambda_{\mathrm{Re}}}$, and*

$$U_\tau^{-1} \circ A_\tau \circ U_\tau = A_0 + B_\tau\,, \tag{2-3}$$

where B_τ is given by

$$(B_\tau \phi)(t) = \left(u_{t\tau}^{-1} \circ J \circ \frac{d}{dt}(u_{t\tau})\right)(\phi(t))\,. \tag{2-4}$$

Moreover $B_\tau(W_1) \subset W_1$, and B_τ and U_τ are continuous families of operators from W_1 to W_1 with respect to the parameter τ and, of course, also as operators $H_{\boldsymbol{R}} \to H_{\boldsymbol{R}}$.

PROOF. By the definition $(U_\tau f)(0) = f(0)$ and $(U_\tau f)(1) = u_\tau(f(1))$. So $D_{\lambda_\tau} = U_\tau(D_{\lambda_{\mathrm{Re}}})$. Therefore the domain of $U_\tau^{-1} \circ A_\tau \circ U_\tau$ is $D_{\lambda_{\mathrm{Re}}}$. By noting $J \circ u_\tau = u_\tau \circ J$, a direct calculation gives us the formula (2.3) and (2.4). Since B_τ is a matrix operator, we can easily verify the remainder assertions.

Since the family $\{A_\tau\}$ consists of unbounded operators, we transform each A_τ as follows to obtain a continuous path in $\mathscr{F}_*$: let

$$\tilde{A}_\tau = A_\tau \circ (1 + A_\tau^2)^{-1/2}\,,$$

where we regard that the operator $(1 + A_\tau^2)^{-1/2}$ is defined by an integral

$$\int_{\Gamma}(1+\lambda)^{-1/2}(\lambda-A_\tau^2)^{-1}d\lambda\,,$$

on a suitably taken contour Γ in $\boldsymbol{C}$ (note here that A_τ is a real operator for each τ). Then each $\tilde{A}_\tau$ is a bounded selfadjoint Fredholm operator on $H_{\boldsymbol{R}}$ with eigenvalues of the form $\lambda_k(\tau)(1+\lambda_k(\tau)^2)^{-1/2}$, where $\lambda_k(\tau)$ is the k-th eigenvalue of A_τ. Then we have

PROPOSITION 2.8. *The family $\{\tilde{A}_\tau\}$ is a continuous path in $\hat{\mathscr{F}}_*(H_{\boldsymbol{R}})$.*

PROOF. Put $T_\tau=A_0+B_\tau$. First recall an inequality

$$\|f\|_1\leqq C(\|T_\tau f\|+\|f\|) \tag{2-5}$$

for any $f\in W_1\otimes \boldsymbol{C}$, where the constant C does not depend on the parameter τ.

If we regard the resolvent operator $(\lambda-T_\tau^2)^{-1}$ as an operator

$$(\lambda-T_\tau^2)^{-1}:H_{\boldsymbol{R}}\otimes\boldsymbol{C}\to W_1\otimes\boldsymbol{C}\,,$$

then by the inequality (2-5) we obtain the following estimate:

$$\|(\lambda-T_\tau^2)^{-1}\|_{H_{\boldsymbol{R}}\otimes \boldsymbol{C},W_1\otimes\boldsymbol{C}}=O(|\lambda|^{-1/2})\,, \tag{2-6}$$

which is valid uniformly with respect to the parameter τ on the line $\arg\lambda=\text{const}\neq 0$.

For $\tau,\tau'\in I$, the differences $T_\tau^2-T_{\tau'}^2$ are first order differential operators with C^1-coefficients and the operator norms as operators from W_1 to $H_{\boldsymbol{R}}$ satisfy

$$\|T_\tau^2-T_{\tau'}^2\|_{W_1,W_{\boldsymbol{R}}}=O(|\tau-\tau'|)\,. \tag{2-7}$$

These properties (2-6) and (2-7) together with a resolvent equation

$$(\lambda-T_\tau^2)^{-1}-(\lambda-T_{\tau'}^2)^{-1}=(\lambda-T_\tau^2)^{-1}(T_{\tau'}^2-T_\tau^2)(\lambda-T_{\tau'}^2)^{-1}$$

give us an estimate of the operator norm from $H_{\boldsymbol{R}}$ to W_1:

$$\|(\mathrm{Id}+T_\tau^2)^{-1/2}-(\mathrm{Id}+T_{\tau'}^2)^{-1/2}\|_{W_{\boldsymbol{R}},W_1}=O(|\tau-\tau'|)\,. \tag{2-8}$$

From this estimate (2-8) and the fact that $\{T_\tau\}$ is a continuous family of operators from W_1 to $H_{\boldsymbol{R}}$, we have the continuity of the family $\{T_\tau\circ(\mathrm{Id}+T_\tau^2)^{-1/2}\}$ in $\hat{\mathscr{F}}_*$ with respect to the parameter τ, and also that of the family $\{\tilde{A}_\tau\}$, since $\tilde{A}_\tau=U_\tau\circ T_\tau\circ(\mathrm{Id}+T_\tau^2)^{-1/2}\circ U_\tau^{-1}$. This completes the proof.

Especially take a path $s:I\to U(n)$,

$$s(\tau)=\begin{pmatrix} e^{ik\tau\pi} & & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{pmatrix}$$

for a fixed integer k. Then the path $\{\tilde{A}_\tau\}$ corresponding to the curve s forms a loop in $\hat{\mathscr{F}}_*$ and by an explicit calculation of the eigenvalues of this operator A_τ for each τ, we have

PROPOSITION 2.9. *The spectral flow of this family is* k: $\mathrm{sf}\{\tilde{A}_\tau\}=k$.

REMARK 3. Let take a C^2-map $u: I\times I\to U(n)$ such that $u(0,s)=\mathrm{Id}$ for any $s\in I$, instead of the path u in the propositions 2.7 and 2.8, and denote by $\{A_{\tau,s}\}$ the selfadjoint extension of the operator $A=J\,d/dt$ to the domain

$$D_{\tau,s}=\{f\in W_1 : f(0)\in\lambda_{\mathrm{Re}},\ f(1)\in u_{\tau,s}(\lambda_{\mathrm{Re}})\}\ .$$

Then by the same method as in the proof of the proposition 2.8, the family $\{\tilde{A}_{\tau,s}\}$, $\tilde{A}_{\tau,s}=A_{\tau,s}\circ(\mathrm{Id}+A_{\tau,s}{}^2)^{-1/2}$, can be shown to be a continuous family of the parameters τ and s. We will use this fact in §5.

§3. Lagrangian intersection and Maslov index.

Let $\pi: E\to X$ be a symplectic vector bundle on a topological space X. The structure group of this bundle is $Sp(n, \boldsymbol{R})$, the symplectic group, where $2n$ is the fiber dimension of E, and is reduced to its maximal compact subgroup $U(n)$. This means that E can be regarded as a complex vector bundle on X. We denote by $\Lambda(E)$ the Lagrangian-Grassmannian bundle on X associated to E with the fiber $\Lambda(n)=U(n)/O(n)$.

Put $G(n)=\{U\in U(n): \det^2 U=1\}$, then the structure group of E can be reduced to the group $G(n)$, if and only if, $2c_1(E)=0$, where $c_1(E)$ is the first Chern class of E as a complex vector bundle. If this condition is satisfied, then there is a map $\alpha: \Lambda(E)\to U(1)$, which depends on the way of a reduction of the structure group of E from $U(n)$ to $G(n)$. Let $\mathrm{Det}^2: \Lambda(n)\to U(1)$ be the map induced from the map $\det^2: U(n)\to U(1)$, and denote by $[d\theta]$ the generator of $H^1(U(1), \boldsymbol{Z})$, then the class $(\mathrm{Det}^2)^*([d\theta])$ is, so called, Keller-Maslov-Arnold characteristic class. If we restrict the class $\alpha^*([d\theta])$ to each fiber of $\Lambda(E)$, then it is identified with $(\mathrm{Det}^2)^*([d\theta])$.

Also let $\beta: \Lambda(E)\to U(1)$ be defined by another reduction of the structure group from $U(n)$ to $G(n)$ of the symplectic vector bundle E. We have the difference

$$\alpha^*([d\theta])-\beta^*([d\theta])=\pi^*(c)$$

with a certain $c\in H^1(X, \boldsymbol{Z})$ (see the appendices of [V]).

In particular, if both $H^1(X, \boldsymbol{Z})$ and $H^2(X, \boldsymbol{Z})$ vanish, then we have a unique class $m\in H^1(\Lambda(E), \boldsymbol{Z})$, $m=\alpha^*([d\theta])$, through a map $\alpha: \Lambda(E)\to U(1)$.

Based on these facts, we now proceed to our problem. Let (P, ω) be a symplectic manifold of $2n$-dimension. Let L and L' be two Lagrangian submanifolds of P, which intersect transversally with a non empty intersection. Then the intersection $L\cap L'$ consists of a discrete set of points in P. Let points x and y be in $L\cap L'$ and a smooth map $f: I\times I\to P$ satisfy $f(\tau, 0)\in L$, $f(\tau, 1)\in L'$, $f(0, t)\equiv x$ and $f(1, t)\equiv y$ for all $\tau, t\in I$.

We take a Riemannian metric g on P which is adapted to the symplectic form ω, and denote by J the almost complex structure defined by g and ω as in the introduction. Since $T_x(L)$ and $J_x(T_x(L))$ are orthogonal, we will use the following identification:

$$T_x(L) \oplus J_x(T_x(L)) \equiv \boldsymbol{R}^n \oplus \sqrt{-1}\,\boldsymbol{R}^n \tag{3-1}$$

and by taking a suitable orthonormal basis in $T_x(L)$, we have

$$J_x = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}.$$

When two subspaces λ and μ in $\boldsymbol{R}^{2n}$ are transversal, we will write as follows: $\lambda \pitchfork \mu$.

Let $P_{\tau,t}$ be the parallel displacement along the path $\{f(s, t)\}$ $(0 \leqq s \leqq \tau)$ for each fixed $t \in I$:

$$P_{\tau,t} : T_x P \to T_{f(\tau,t)} P$$

is an isometry, and put

$$J_{\tau,t} = P_{\tau,t}{}^{-1} \circ J_{f(\tau,t)} \circ P_{\tau,t}\,.$$

Then we have a smooth map $g : I \times I \to SO(2n)$ such that

$$g_{\tau,t}{}^{-1} \circ J_{\tau,t} \circ g_{\tau,t} = J_x\,.$$

Put

$$P_{\tau,0}{}^{-1}(T_{f(\tau,0)}(L)) = \tilde{\lambda}_0(\tau)\,,$$
$$P_{\tau,1}{}^{-1}(T_{f(\tau,1)}(L')) = \tilde{\lambda}_1(\tau)$$

and

$$g_{\tau,0}(\tilde{\lambda}_0(\tau)) = \lambda_0(\tau)\,,$$
$$g_{\tau,1}(\tilde{\lambda}_1(\tau)) = \lambda_1(\tau)\,.$$

We have the trivialization of $f^*(TP)$ by using the parallel displacement $\{P_{\tau,t}\}$:

$$f^*(TP) \cong I \times I \times T_x(P) \cong I \times I \times \boldsymbol{R}^{2n}\,. \tag{3-2}$$

Then $\lambda_0(\tau)$ and $\lambda_1(\tau)$ are Lagrangian subspaces in $T_x(P) \cong \boldsymbol{R}^{2n}$, i.e. $\{\lambda_0(\tau)\}$ and $\{\lambda_1(\tau)\}$ are two curves in $\Lambda(n)$. Notice that

$$\lambda_0(0) \pitchfork \lambda_1(0)\,, \qquad \lambda_0(1) \pitchfork \lambda_1(1)\,. \tag{3-3}$$

There are two ways of constructing loops in $\Lambda(n)$ from these two curves $\{\lambda_0\}$ and $\{\lambda_1\}$, that have the same Maslov index.

We will now explain these ways: Since $\lambda_0(0) \pitchfork \lambda_1(0)$ and $\lambda_0(1) \pitchfork \lambda_1(1)$, there is a symplectic transformation $\sigma \in Sp(n, \boldsymbol{R})$ such that

$$\sigma(\lambda_0(0)) = \lambda_0(1) \quad \text{and} \quad \sigma(\lambda_1(0)) = \lambda_1(1)\,. \tag{3-4}$$

Take a path $C(\tau)$ $(0 \leqq \tau \leqq 1)$ in $\Lambda(n)$ with

$$C(0)=\lambda_0(0), \qquad C(1)=\lambda_1(0). \tag{3-5}$$

Then four curves

$$\lambda_0(\tau), \quad \sigma(C(\tau)), \quad \lambda_1(1-\tau) \quad \text{and} \quad C(1-\tau) \qquad (0 \leqq \tau \leqq 1)$$

form a loop in $\Lambda(n)$. We denote this loop by $b(s)$ $(0 \leqq s \leqq 4)$. Since the Maslov class $m \in H^1(\Lambda(n), \mathbf{Z})$ is invariant under the action of the group $Sp(n, \mathbf{R})$ on $\Lambda(n)$, the integer $\int_b m$ is independent of the choice of the path C.

For constructing another loop, take a curve $U: I \to U(n)$ such that

$$U_\tau(\lambda_0(0))=\lambda_0(\tau) \quad \text{and} \quad U_0=\mathrm{Id}, \tag{3-6}$$

where we used the identification (3-1). Put

$$U_\tau^{-1}(\lambda_1(\tau))=\lambda_2(\tau), \tag{3-7}$$

then we have

$$\lambda_2(0) \pitchfork \lambda_0(0) \quad \text{and} \quad \lambda_2(1) \pitchfork \lambda_0(0). \tag{3-8}$$

So we join $\lambda_2(0)$ and $\lambda_2(1)$ by a curve $\tilde{C}(\tau)$ $(0 \leqq \tau \leqq 1)$ with the property

$$\tilde{C}(\tau) \pitchfork \lambda_0(0) \tag{3-9}$$

for any $\tau \in I$. Notice that two such curves with the property (3-9) are homotopic to each other. As before, the curves $\tilde{C}(\tau)$ and $\lambda_2(1-\tau)$ $(0 \leqq \tau \leqq 1)$ define the loop $\{\tilde{b}(s)\}$ $(0 \leqq s \leqq 2)$. Then we have

PROPOSITION 3.1.

$$\int_b m = \int_{\tilde{b}} m. \tag{3-10}$$

PROOF. Let $H: [0, 1] \times [0, 4] \to \Lambda(n)$ be a continuous map such that

$$H(s, \tau)=\begin{cases} U_{s\tau}^{-1}(\lambda_0(\tau)) & 0 \leqq \tau \leqq 1, \quad 0 \leqq s \leqq 1, \\ U_s^{-1}(\sigma(C(\tau-1))) & 1 \leqq \tau \leqq 2, \quad 0 \leqq s \leqq 1, \\ U_{s(3-\tau)}^{-1}(\lambda_1(3-\tau)) & 2 \leqq \tau \leqq 3, \quad 0 \leqq s \leqq 1, \\ C(4-\tau) & 3 \leqq \tau \leqq 4, \quad 0 \leqq s \leqq 1. \end{cases}$$

Then H is a homotopy between the loop $\{b(\tau)\}$ and the loop $\{H(1, \tau)\}$, where $U: [0, 1] \to U(n)$ is the map given in (3-6).

Next, owing to the fact that the group $Sp(n, \mathbf{R})$ acts transitively on the space of pairs consisting of transversally intersecting Lagrangian subspaces in $\mathbf{R}^{2n}$, we have a map $V: I \to Sp(n, \mathbf{R})$ with the properties:

$$V_0 = \mathrm{Id}\,, \tag{3-11}$$

$$V_\tau(\lambda_0(0)) = \lambda_0(0) \qquad \text{for any} \quad \tau \in I\,, \tag{3-12}$$

$$V_\tau(\lambda_2(0)) = \lambda_2(\tau) \qquad \text{for any} \quad \tau \in I\,, \tag{3-13}$$

$$V_1 = U_1{}^{-1} \circ \sigma\,. \tag{3-14}$$

Note here that, if the map $V_1{}^{-1} \circ U_1{}^{-1} \circ \sigma$ restricted to the subspace $\lambda_0(0)$ has determinant -1, then σ should be replaced by another $\tilde{\sigma} \in Sp(\mathrm{n}, \boldsymbol{R})$ such that

$$\tilde{\sigma}(\lambda_0(0)) = \lambda_0(1)\,, \qquad \tilde{\sigma}(\lambda_1(0)) = \lambda_1(1)\,,$$

and

$$\det(\tilde{\sigma}^{-1}\sigma_{|\lambda_0(0)}) = -1\,,$$

for the sake that the map V satisfies the property (3-11).

Let $G : [0, 1] \times [0, 1] \to \Lambda(n)$ be a continuous map defined by: $G(s, \tau) = V_s(C(\tau))$. Then

$$\int_{\partial G} m = 0\,,$$

which implies

$$\int_{\tilde{b}} m = \int_{U_1{}^{-1} \circ \sigma(c)} m - \int_{\lambda_2} m - \int_C m\,.$$

Hence we have

$$\int_b m = \int_{\tilde{b}} m\,.$$

Remark 4. Let $\tilde{b}$ be the above loop. If there exists another Lagrangian subspace μ satisfying $\mu \pitchfork \lambda_2(\tau)$ for every $\tau \in I$, then the integer $\int_{\tilde{b}} m$ equals the Hörmander index $\sigma(\lambda_0(0), \mu\,; \lambda_2(0), \lambda_2(1))$ (see [H]), because of the very definition of it.

Remark 5. The following example satisfies the situation in Remark 4: Let ϕ be a Morse function on a smooth manifold M, the symplectic manifold P be $T^*(M)$ in this case, $L = M$, i.e., M is seen as zero sections in $T^*(M)$, and $L' = d\phi(M)$. Let x and y be in $L \cap L'$, that is, $d\phi_x = 0$ and $d\phi_y = 0$. Since the vertical foliation of $T^*(M)$ is transversal to $d\phi(M)$, we can take $T_x(T_x^*(M))$ as the Lagrangian subspace μ satisfying the above condition in Remark 4. Now let $f : I \times I \to T^*(M)$ be a smooth map satisfying conditions (1-1). Then we have a formula:

$$\int_{\tilde{b}} m = \operatorname{sign} H_\phi(y) - \operatorname{sign} H_\phi(x)\,, \tag{3-15}$$

where $H_{\phi}(\cdot)$ denotes the Hessian of ϕ at critical points. Note that the right hand side of this formula is independent of the map f.

§4. Lagrangian intersections and spectral flow.

Let (P, ω), L and L' be as in §3. Let Ω be the path space of P consisting of smooth paths $z : I \to P$ such that $z(0) \in L$ and $z(1) \in L'$. We consider a smooth path $\{f_\tau\}$ in Ω:

$$f : I \times I \to P, \qquad f_\tau(t) \equiv f(\tau, t).$$

Along the path $\{f_\tau\}$, we define a family of operators $\{\mathscr{A}_\tau\}$ as follows:

$$\mathscr{A}_\tau(\xi)(t) = J_{f_\tau(t)}(\nabla_{\dot{f}_\tau(t)}(\xi)(t)) + (\nabla_\xi(J_{f_\tau(t)}))(\dot{f}_\tau(t)), \qquad \xi \in C_0^\infty(f_\tau^*(TP)).$$

In the sequel we will construct a loop of operators in $\hat{\mathscr{F}}_*$ starting from the family $\{\mathscr{A}_\tau\}$.

Let $\|\cdot\|_0$ and $\|\cdot\|_1$ be the norms on $C^\infty(f_\tau^*(TP))$ defined by

$$\|\xi\|_0{}^2 = \int_0^1 g_{f_\tau(t)}(\xi(t), \xi(t))dt$$

and

$$\|\xi\|_1{}^2 = \int_0^1 g_{f_\tau(t)}(\nabla_{\dot{f}_\tau(t)}\xi(t), \nabla_{\dot{f}_\tau(t)}\xi(t))dt + \|\xi\|_0{}^2,$$

respectively. We denote the closure of $C^\infty(f_\tau^*(TP))$ with respect to the norms $\|\cdot\|_0$ and $\|\cdot\|_1$ by H_τ and W_τ respectively. Also let denote by D_τ a subspace of W_τ such that

$$D_\tau = \{\xi \in W_\tau : \xi(0) \in T_{f(\tau,0)}(L),\ \xi(1) \in T_{f(\tau,1)}(L')\}.$$

Then we can extend the operator $\mathscr{A}_\tau$ to a closed operator in H_τ with the domain D_τ, denoting it by A_τ.

Henceforth, we first transform operators A_τ without changing the spectral set $\sigma(A_\tau)$ in several steps corresponding to the deformations of the paths in §3. Consequently we get a family of operators with the same principal part.

Let P_τ be the isomorphism from H_0 to H_τ defined by

$$(P_\tau f)(t) = P_{\tau,t}(f(t)), \qquad f \in H_0,$$

where $P_{\tau,t}$ is the parallel displacement cited in §3. Put $A_\tau^{(1)} = P_\tau{}^{-1} \circ A_\tau \circ P_\tau$ and $D_\tau^{(1)} = P_\tau{}^{-1}(D_\tau)$, then $D_\tau^{(1)}$ is the domain of $A_\tau^{(1)}$. By recalling

$$J_{\tau,t} = P_{\tau,t}{}^{-1} \circ J_{f(\tau,t)} \circ P_{\tau,t}$$

we get

$$A_\tau^{(1)} = J_{\tau,t}\frac{d}{dt} + B_\tau^{(1)},$$

$$D_\tau^{(1)} = \{\xi \in W_0 : \xi(0) \in \tilde{\lambda}_0(\tau),\ \xi(1) \in \tilde{\lambda}_1(\tau)\}\,,$$

where $B_\tau^{(1)}$ is a matrix operator with smooth coefficients with respect to τ and t.

Next, let g_τ be the orthogonal transformations on H_0 defined by

$$(g_\tau f)(t) = g(\tau, t)(f(t))\,, \qquad f \in H_0\,,$$

where $g : I \times I \to SO(2n)$ is satisfying

$$g_{\tau,t}{}^{-1} \circ J_{\tau,t} \circ g_{\tau,t} = J_x\,.$$

Put $A_\tau^{(2)} = g_\tau{}^{-1} \circ A_\tau^{(1)} \circ g_\tau$ and $D_\tau^{(2)} = g_\tau{}^{-1}(D_\tau^{(1)})$, then $D_\tau^{(2)}$ is the domain of the operator $A_\tau^{(2)}$ and also we have

$$D_\tau^{(2)} = \{\xi \in W_0 : \xi(0) \in \lambda_0(\tau),\ \xi(1) \in \lambda_1(\tau)\}\,,$$

$$A_\tau^{(2)} = J_x \frac{d}{dt} + B_\tau^{(2)}\,,$$

where $B_\tau^{(2)}$ is a matrix operator with smooth coefficients with respect to τ and t.

Let $U : I \to U(n) = Sp(n, \mathbf{R}) \cap SO(2n)$ be a smooth curve such that $U_0 = \mathrm{Id}$, $U_\tau(\lambda_0(0)) = \lambda_0(\tau)$ as in §3. Put $A_\tau^{(3)} = U_\tau{}^{-1} \circ A_\tau^{(2)} \circ U_\tau$ and $D_\tau^{(3)} = U_\tau{}^{-1}(D_\tau^{(2)})$, where U_τ is regarded as an orthogonal transformation on H_0 defined as usual. Then $D_\tau^{(3)}$ is the domain of $A_\tau^{(3)}$ and since U_τ and J_x commute with each other, we get

$$A_\tau^{(3)} = J_x \frac{d}{dt} + B_\tau^{(3)}\,,$$

$$D_\tau^{(3)} = \{\xi \in W_0 : \xi(0) \in \lambda_0(0),\ \xi(1) \in \lambda_2(\tau)\}$$

where $B_\tau^{(3)}$ is a matrix operator with smooth coefficients with respect to τ and t.

Let $\tilde{D}_\tau^{(3)}$ be a subspace of W_0 such that

$$\tilde{D}_\tau^{(3)} = \{\xi \in W_0 : \xi(0) \in \lambda_0(0),\ \xi(1) \in \tilde{C}(\tau)\}\,,$$

where $\tilde{C}(\tau)$ is the (smooth) curve defined in §3, and denote by $\tilde{A}_\tau^{(3)}$ a closed extension of the formal differential operator

$$(P_\tau \circ g_\tau \circ U_\tau)^{-1} \circ \mathscr{A}_\tau \circ P_\tau \circ g_\tau \circ U_\tau$$

to the domain $\tilde{D}_\tau^{(3)}$. Then, at this stage, we have a loop of unbounded closed operators (not necessarily selfadjoint) corresponding to the loop $\{\tilde{\ell}(\tau)\}$ in $A(n)$ (see §3). Let denote this loop of operators by $\{\ell_\tau\}$:

$$\ell_\tau = \begin{cases} \tilde{A}_\tau^{(3)} & 0 \leqq \tau \leqq 1 \\ A_{2-\tau}^{(3)} & 1 \leqq \tau \leqq 2\,. \end{cases}$$

Since the domain of the adjoint operator ℓ_τ^* is the same with it of ℓ_τ, the operator $\ell_\tau + \ell_\tau^*$ has the domain of definition: $\tilde{D}_\tau^{(3)}$, $0 \leqq \tau \leqq 1$, and $D_{2-\tau}^{(3)}$, $1 \leqq \tau \leqq 2$, and is selfadjoint. Put $S_\tau = 1/2(\ell_\tau + \ell_\tau^*)$. Then S_τ has the form

$$S_\tau = T_\tau + R_\tau ,$$

where $T_\tau = J_x d/dt$ with the domain $\tilde{D}_\tau^{(3)}$ for $0 \leqq \tau \leqq 1$, $D_{2-\tau}^{(3)}$ for $1 \leqq \tau \leqq 2$. The zeroth order term R_τ is a selfadjoint matrix operator with smooth coefficients and forms a continuous family with respect to τ. Finally we have a proposition whose proof can be reduced to the proposition 2.8.

PROPOSITION 4.1. *Put $\tilde{S}_\tau = S_\tau \circ (\mathrm{Id} + S_\tau{}^2)^{-1/2}$, then the family $\{\tilde{S}_\tau\}$ is a continuous loop in $\hat{\mathscr{F}}_*$. Moreover $\{R_\tau\}$ is itself a continuous loop of bounded selfadjoint operators.*

By means of this proposition and the proposition 2.4, we define the number $\mathrm{sf}\{\tilde{S}_\tau\}$ as the spectral flow for the family $\{\mathscr{A}_\tau\}$. It should be noted that the definition of $\mathrm{sf}\{\mathscr{A}_\tau\}$ does not depend on the choice of the curve $\{\tilde{C}(\tau)\}$ owing to the remark 3 in §2.

REMARK 6. The Riemannian metric g on P was taken so that it adapts to the symplectic form ω. If we take a Riemannian metric g which satisfies moreover that the spaces T_pL and T_pL' ($p = x, y$) are orthogonal, then the curve $\{\lambda_2(\tau)\}$ is already a loop, and so is the family $\{A_\tau\}$.

§5. Coincidence of spectral flow and Maslov index.

At this stage the coincidence of the two quantities is easily shown:

THEOREM 5.1. *Let $\{\tilde{S}_\tau\}$ be as above. Then*

$$\mathrm{sf}\{\mathscr{A}_\tau\} = \mathrm{sf}\{\tilde{S}_\tau\} = \int_{\tilde{b}} m .$$

In particular $\mathrm{sf}\{\mathscr{A}_\tau\}$ *is independent of the choice of Riemannian metrics adapted to the sympletic form ω and is a homotopy invariant of the smooth map $f : I \times I \to P$ with the properties* (1-1).

PROOF. Let $h_{s,\tau} = (T_\tau + sR_\tau) \circ (\mathrm{Id} + (T_\tau + sR_\tau)^2)^{-1/2}$ ($0 \leqq s \leqq 1$, $0 \leqq \tau \leqq 2$), then we see the map

$$\begin{aligned} h : [0, 1] \times [0, 2] &\to \hat{\mathscr{F}}_* \\ (s, \tau) &\mapsto h_{s,\tau} \end{aligned}$$

is continuous according to the proposition 2.8. Note here that the continuity of the map h can be proved separately on $[0, 1] \times [0, 1]$ and $[0, 1] \times [1, 2]$. Hence we have a homotopy between the loop $\{\tilde{S}_\tau\}$ and $\{\tilde{T}_\tau\}$, where $\tilde{T}_\tau = T_\tau \circ (\mathrm{Id} + T_\tau{}^2)^{-1/2}$, and T_τ is as follows:

$$T_\tau = J_x \frac{d}{dt} \quad \text{on} \quad \tilde{D}_\tau^{(3)} \qquad (0 \leqq \tau \leqq 1)\,,$$

$$T_\tau = J_x \frac{d}{dt} \quad \text{on} \quad D_{2-\tau}^{(3)} \qquad (1 \leqq \tau \leqq 2)\,.$$

Since the spectral flow is homotopically invariant, it is enough to show that the spectral flow of $\{\tilde{T}_\tau\}$ equals

$$\int_{\tilde{b}} m\,.$$

Take a path $d\colon I \to U(n)$ such that

$$d(\tau) = \begin{pmatrix} e^{i(\pi/2)k\tau} & & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{pmatrix}, \qquad 0 \leqq \tau \leqq 2\,,$$

where $k = \int_{\tilde{b}} m$. Then there exist C^∞-maps $f^{(0)}$ and $f^{(1)}$

$$f^{(i)} : I \times I \to U(n) \qquad (i = 0,\, 1)$$

such that for $0 \leqq \tau \leqq 1$

$$f^{(0)}(0, \tau) = d(\tau)\,, \qquad f^{(1)}(0, \tau) = d(1 + \tau)\,,$$

$$f^{(0)}(1, \tau) = \tilde{C}(\tau)\,, \qquad f^{(1)}(1, \tau) = \lambda_2(1 - \tau)\,,$$

and for any s $(0 \leqq s \leqq 1)$

$$f^{(0)}(s, 0) = f^{(1)}(s, 0) = \mathrm{Id}\,,$$

$$f^{(0)}(s, 1)(\lambda_0(0)) = f^{(1)}(s, 1)(\lambda_0(0))\,.$$

Again, owing to the proposition 2.8 and the remark 3 (§2), we have the loops $\{\tilde{T}_\tau\}$ and $\{\tilde{A}_\tau\}$ (in §2) are homotopic in $\hat{\mathscr{F}}_*$. So we have

$$\mathrm{sf}\{\tilde{T}_\tau\} = \mathrm{sf}\{\tilde{A}_\tau\} = k = \mathrm{sf}\{\mathscr{A}_\tau\}\,.$$

REMARK 7. Let T be a densely defined closed symmetric operator on a Hilbert space with the same defect indices: $\dim \mathrm{Im}(T - i)^\perp = \dim \mathrm{Im}(T + i)^\perp = N < +\infty$. If we fix orthonormal basis of $\mathrm{Im}(T - i)^\perp$ and $\mathrm{Im}(T + i)^\perp$, then selfadjoint extensions of T are parametrized (via Cayley transformation) by $N \times N$ unitary matrices. We denote by T_θ the selfadjoint extension corresponding to a matrix $\theta \in U(N)$.

Now we assume that an extension T_{θ_0} has a compact resolvent (equivalently, all resolvents of any extension are compact), and let $\tilde{T}_\theta = T_\theta \circ (\mathrm{Id} + T_\theta{}^2)^{-1/2}$. Then we have a map

$$\varepsilon : U(N) \to \hat{\mathscr{F}} , \qquad \varepsilon(\theta) = \tilde{T}_\theta .$$

In this paper, we have shown, for the case $T = J\,d/dt$ (here, $N = 2n$ and we regard that it is complexified), that some curves in $U(2n)$ with a suitable differentiability condition are mapped by ε to continuous curves in $\hat{\mathscr{F}}_*$.

Whereas the spectrum of T_θ changes 'continuously', when θ moves continuously (see [K]) and also the image of the map ε is contained in a component of $\hat{\mathscr{F}}$, it seems that the map ε itself will not be continuous in general. We are interested in what case the map ε is continuous and how the map $\varepsilon_* : \pi_1(U(N)) \to \pi_1(\hat{\mathscr{F}})$ is characterized.

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