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Positive Projections on C^*-Algebras

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§1. Introduction.

In recent years there has been considerable progress in the study of certain linear maps of C^*-algebras which preserve the natural partial ordering. Suppose A is a unital C^*-algebra and P a unital positive projection of A into itself. It is known [6, 22] that $P(A)$ is a C^*-algebra under the product $a\circ b=P(ab)$ if P is completely positive, and P is automatically completely positive if $P(A)$ is a C^*-algebra. E. Størmer [18] linked the decomposability of P, which is weaker than complete positivity, to the theory of JC-algebras. In [15], A. G. Robertson has showed that the decomposability of P is equivalent to the existence of decomposition of P as a sum of a 2-positive map and a 2-copositive map under some condition. The global structure of positive linear maps is, however, very complicated, even in the finite dimensional case [3, 4, 9, 18, 21].

In this paper we shall investigate the difference between complete positivity and positivity of contractive projections on C^*-algebras, particularly in case of matrix algebras. As an application, we shall describe general C^*-algebras for which n-positivity coincides with $(n+1)$-positivity in the class of projections.

Let A and B be C^*-algebras. We do not assume units for C^*-algebras. The $n\times n$ matrix space over A, that is, $M_n(A)$ naturally inherits the corresponding order as a C^*-algebra. A C^*-algebra is said to be n-subhomogeneous if every irreducible representation of the algebra is finite dimensional with dimension not greater than n. Let ϕ be a positive linear map of A into B. Recall that ϕ is said to be n-positive (respectively, n-copositive) if the n-multiplicity map $\phi(n)$ (respectively, the n-comultiplicity map $\phi^c(n)$),

$$\phi(n) : [a_{i,j}]\in M_n(A) \longrightarrow [\phi(a_{i,j})]\in M_n(B)$$

$$(\text{respectively, } \phi^c(n) : [a_{i,j}]\in M_n(A) \longrightarrow [\phi(a_{j,i})]\in M_n(B))$$

is positive. The map ϕ is completely positive if it is n-positive for every positive integer n. It is, however, known that every n-positive map on an n-subhomogeneous C^*-

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algebra, particularly on $M_n(C)$, is completely positive. Completely copositive maps are defined in a similar way and the saturation of copositivity on an n-subhomogeneous C^*-algebra also occurs. We call ϕ decomposable if ϕ can be decomposed into a sum of a completely positive map and a completely copositive map and ϕ is called a Schwarz map if it satisfies the Schwarz inequality; $\phi(a^*a) \geq \phi(a)^*\phi(a)$, $a \in A$. Our main result is the following:

THEOREM. *Let P be a contractive positive projection on $M_n(C)$. Then we have*

(i) *For $n=2$, 3, P is completely positive if and only if P is a Schwarz map.*

(ii) *For $n \geq 4$, P is completely positive if and only if P is $[n/2]$-positive, where [ ] means the Gauss's symbol.*

In general, a 2-positive linear map is a Schwarz map [2, Corollary 2.8], but the converse is false [5, appendix A]. By the above assertion (1), we know that there is a non-trivial case in which these properties are coincident [see 13, Lemma 2.4]. As an application, we obtain the following result:

THEOREM. *Let A be a C^*-algebra and consider the following assertions:*

(1) *Every k-positive projection on A is $(k+1)$-positive,*

(2) *Every k-positive contractive projection on A is completely positive,*

(3) *Every k-positive contractive projection on A is $(k+1)$-positive,*

(4) *A is $(2k+1)$-subhomogeneous which has at most one equivalent class of irreducible representations π's with* $\dim \pi \geq k+1$.

We have then the following implications;

$$\begin{tikzcd} (1) \arrow[r, Rightarrow] \arrow[dr, Rightarrow] & (4) \arrow[r, Rightarrow] & (2) \arrow[dl, Rightarrow] \\ & (3) & \end{tikzcd}$$

This result partially sharpens [24, Theorem 1.2].

§2. Positive projections on general C^*-algebras.

Let A be a C^*-algebra and let P be a contractive positive projection of A into itself. Let P^{**} denote the normal extension of P to the second dual A^{**} of A, and consider A as a C^*-subalgebra of A^{**}. Let e be the support projection of P^{**} and let $N=\{a \in A^{**} : P^{**}(a^*a)=0=P^{**}(aa^*)\}$. Suppose that P satisfies the Schwarz inequality: $P(a)^*P(a) \leq P(a^*a)$, $a \in A$, then P^{**} also satisfies the Schwarz inequality because of approximating elements of A in the σ-strong* topology. By [8, Theorem 2.3], $P^{**}(A^{**})+N$ is a von Neumann algebra. For each subset S of $B(H)$ the set of all bounded linear operators on a Hilbert space H, let $S'=\{x \in B(H) : xs=sx, \forall s \in S\}$.

Since $N=(1-e)A^{**}(1-e)$ and $e\in P(A)'$ by [7, Lemma 1.2], we know that $P^{**}(A^{**})e$ is a von Neumann algebra. Then we consider the following three linear maps

$$\varphi_1 : A^{**}\ni a \longrightarrow eae\in eA^{**}e\,,$$

$$\varphi_2 : eA^{**}e\ni eae \longrightarrow P^{**}(eae)e\in P^{**}(A^{**})e\,, \quad \text{and}$$

$$\varphi_3 : P^{**}(A^{**})e\ni P^{**}(a)e \longrightarrow P^{**}(a)\in P^{**}(A^{**})\,, \qquad a\in A^{**}\,.$$

It is obvious that $P(a)=(\varphi_3\circ\varphi_2\circ\varphi_1)(a)$ $(a\in A)$ and φ_1 is completely positive. Since $P^{**}(A^{**})e$ is a von Neumann algebra and φ_2 is a contractive positive projection, φ_2 is completely positive ([20, III Theorem 3.4], [22]). Therefore P is completely positive if and only if φ_3 is completely positive. Note that φ_3 is an order isomorphism (i.e. linear isomorphism and φ_3 and ${\varphi_3}^{-1}$ are positive) and ${\varphi_3}^{-1}$ is completely positive.

We summarize the above argument in the following,

PROPOSITION 2.1. *Let A be a C^*-algebra and P a contractive projection of A into itself. Then P is completely positive if and only if P is a Schwarz map and φ_3 is completely positive.*

COROLLARY 2.2. *Let A be a C^*-algebra and P a contractive projection of A into itself. Suppose P is faithful when restricted to the C^*-algebra $C^*(P(A))$ generated by $P(A)$. Then, if P is a Schwarz map, P is completely positive.*

PROOF. From the assumption, the map $P(a)e\rightarrow P(a)$ $(a\in A)$ extends to a $*$-isomorphism π of $C^*(P(A))e$ onto $C^*(P(A))$. Since $\varphi_3=\pi$ on $P(A)e$, φ_3 is completely positive. Hence the corollary follows from Proposition 2.1. Q.E.D.

A von Neumann algebra M is said to be a factor if $M\cap M'=\boldsymbol{C}1$.

COROLLARY 2.3. *Let A be a C^*-algebra and P a contractive projection of A into itself. Suppose the von Neumann algebra generated by $P(A)$ is a factor. Then if P is a Schwarz map, P is completely positive.*

PROOF. Let e be a support projection of P^{**}. Then $e\in P(A)'$. Since $P(A)''$ is a factor from the assumption, the map $P(a)\rightarrow P(a)e$ $(a\in A)$ extends to a $*$-isomorphism π of $P(A)''$ onto $P(A)''e$ [11, Corollary 2.6.8]. As in the argument of Corollary 2.2, we get the assertion. Q.E.D.

COROLLARY 2.4. *Let P be a contractive projection of $M_2(\boldsymbol{C})$ into itself. Then if P is a Schwarz map, P is completely positive.*

PROOF. The algebra $P(M_2(\boldsymbol{C}))''$ is either $M_2(\boldsymbol{C})$, $\boldsymbol{C}I$, or unitarily equivalent to $\boldsymbol{C}\oplus\boldsymbol{C}$. When $P(M_2(\boldsymbol{C}))''=M_2(\boldsymbol{C})$, the corollary follows from the previous result. In other cases, it is obvious that P is completely positive ([1, Theorem 7]). Q.E.D.

REMARK 2.5. We notice that the Schwarz property of P in Corollary 2.4 determines complete positivity of P. To clarify further situation, we give below the example of a unital positive projection which is not 2-positive.

EXAMPLE. Let $P: M_2(C)\to M_2(C)$ be defined by

$$P\left(\begin{bmatrix}\alpha & \beta\\ \gamma & \delta\end{bmatrix}\right)=\begin{bmatrix}\alpha & 1/2(\beta+\gamma)\\ 1/2(\beta+\gamma) & \delta\end{bmatrix}.$$

Then this map is a unital positive projection but not 2-positive.

PROOF. Let $\{E_{i,j}\}$ be canonical matrix units for $M_2(C)$. Then we have

$$[P(E_{i,j})]=\begin{bmatrix}1 & 0 & 0 & 1/2\\ 0 & 0 & 1/2 & 0\\ 0 & 1/2 & 0 & 0\\ 1/2 & 0 & 0 & 1\end{bmatrix}.$$

Let $\det([\alpha_{i,j}])$ be the determinant of $[\alpha_{i,j}]$, then we have $\det([P(E_{i,j})])=-3/16<0$. Hence, by [3, Theorem 2], P is not 2-positive. It is obvious that P is a unital projection. Let $[\bar{\alpha}_i\alpha_j]\in M_2(C)^+$. Then we have $\det(P([\bar{\alpha}_i\alpha_j]))\geq 0$ and $\bar{\alpha}_1\alpha_1\geq 0$, hence P is positive. Q.E.D.

In the end of this section, we consider a connection between complete positivity of contractive projections on C^*-algebra and JC-algebras as in the work of [13, 18]. A JC-algebra is a norm closed Jordan subalgebra of the self-adjoint part of a C^*-algebra, equipped with the product $a\circ b=1/2(ab+ba)$. A JC-algebra is said to be reversible if it is closed under arbitrary symmetric products $a_1a_2\cdots a_n+a_na_{n-1}\cdots a_1$, where each a_k is an element of the algebra and n is a positive integer.

The following lemma is indebted to [12].

LEMMA 2.6. *Let A be a C^*-algebra and ϕ a contractive Schwarz linear map of A into itself. Let A_h be the self-adjoint part of A and $A^\phi=\{a\in A_h: \phi(a)=a, \phi(a^2)=a^2\}$. Then A^ϕ is a reversible JC-algebra.*

PROOF. Let $B=\{a\in A: \phi(a)=a, \phi(a^*a)=a^*a, \phi(aa^*)=aa^*\}$. By [12] we know that B is a C^*-algebra.

It is obvious that $A^\phi\subset B$ and for any $a\in A^\phi$, $a^2\in A^\phi$. Therefore A^ϕ is a JC-algebra. Since A^ϕ is the self-adjoint part of B, A^ϕ is reversible. Q.E.D.

PROPOSITION 2.7. *Let A be a C^*-algebra and P a contractive positive projection of A into itself. Then $P(A)$ is a C^*-algebra if and only if $P(A_h)$ is a JC-algebra and P is a Schwarz map.*

PROOF. Suppose $P(A_h)$ is a JC-algebra and P is a Schwarz map. Since $P(A)$ is a JC-algebra, $P(A_h)=\{a\in A_h: P(a)=a, P(a^2)=a^2\}$. As in the proof of Lemma 2.6, $P(A_h)$

is the self-adjoint part of a C^*-algebra. Hence $P(A)$ is a C^*-algebra.

Conversely, if $P(A)$ is a C^*-algebra, P is automatically completely positive. Hence, it is obvious that $P(A_h)$ is a JC-algebra. Q.E.D.

REMARK 2.8. Compared with [13, Propositions 2.1 and 2.2], Lemma 2.6 seems to indicate that the property of 2-positivity is stronger than that of the decomposability in some sense. In general, JC-algebra is not necessarily the self-adjoint part of a C^*-algebra, but A^ϕ in Lemma 2.6 is the self-adjoint part of a C^*-algebra. Therefore, it seems to be natural to ask the following question.

QUESTION. What is the difference between the property of complete positivity and that of decomposability?

More explicitly, let A be a C^*-algebra and P be a contractive positive projection of A into itself. If P is decomposable and 2-positive, then is P completely positive?

Note that the Schwarz property can not replace the 2-positivity in the above problem. In fact, we can easily construct the following counter example.

EXAMPLE. Define $\phi : M_2(C)\to M_2(C)$ by

$$\phi(X)=X^{\mathrm{tr}}/2+\mathrm{tr}(X)1/4$$

where X^{tr} stands for the transpose of X and $\mathrm{tr}(X)$ is the canonical trace of X. It is known that ϕ is a unital Schwarz map and decomposable, but not 2-positive [5, Appendix A]. Let $P: M_4(C)\to M_4(C)$ be defined by

$$P\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right)=\begin{bmatrix} a & 0 \\ 0 & \phi(a) \end{bmatrix}$$

where a, b, c, and $d\in M_2(C)$. It is obvious that P is a unital projection with Schwarz property and moreover decomposable, but this map is not 2-positive.

§ 3. Positive projections on matrix algebras.

In this section we study the difference between complete positivity and positivity of contractive projections on matrix algebras. The following assertion is our main theorem.

THEOREM 3.1. *Let P be a contractive positive projection on $M_n(C)$. Then we have*

(i) *For $n=2$ or 3, P is completely positive if and only if P is a Schwarz map.*

(ii) *For $n\geq 4$, P is completely positive if and only if P is $[n/2]$-positive, where [ ] means the Gauss's symbol.*

PROOF. Let e be a support projection for P. We assume P is a Schwarz map. As in the previous section, we consider three positive linear maps φ_1, φ_2 and φ_3 suth that $P=\varphi_3\circ\varphi_2\circ\varphi_1$. Since $e\in P(M_n(C))'$, we have

$$\begin{aligned}\varphi_3(x) &= e\varphi_3(x)e + (1-e)\varphi_3(x)(1-e)\\ &= x + (1-e)\varphi_3(x)(1-e)\end{aligned}$$

for any $x \in P(M_n(C))e$. Therefore, in both cases of (i) and (ii), we have only to show that $(1-e)\varphi_3(\ )(1-e)$ is completely positive.

If $n=3$, then $\dim(e)$ ($=$the dimension of eC^3) is either 3, or 2, or 1. When $\dim(e)=3$, then P is faithful, hence P is completely positive by Corollary 2.2. When $\dim(e)=1$, then $P(M_3)e=C$, hence φ_3 is completely positive. When $\dim(e)=2$, then $\dim(1-e)=1$ and the map $(1-e)\varphi_3(\ )(1-e)$ is a positive linear functional. Hence φ_3 is completely positive. Therefore, the proof of case (i) is completed combining with Corollary 2.4.

Case (ii): When $[n/2]+1 \leq \dim(e) \leq n$, the map $(1-e)\varphi_3(\ )(1-e)$ is completely positive [1, Theorem 5]. It follows that φ_3 is completely positive. Suppose $\dim(e) \leq [n/2]$. There is a projection of norm one $E: eM_n(C)e \to P(M_n(C))e$ because $P(M_n(C))e$ is an injective C^*-subalgebra of $eM_n(C)e$ ([25]). Considering the map $(1-e)\varphi_3(\ )(1-e)\circ E: eM_n(C)e \to (1-e)M_n(C)(1-e)$, we know that it is completely positive [1, Theorem 6]. Since the map

$$(1-e)\varphi_3(1-e)\circ E\,|\,P(M_n(C))e:\ P(M_n(C))e \longrightarrow (1-e)M_n(C)(1-e)$$

is equal to the map $(1-e)\varphi_3(\ )(1-e)$, φ_3 is completely positive. Therefore, we complete the proof of case (ii). Q.E.D.

REMARK 3.2. We remark that the condition of Theorem 3.1 is the best condition for deciding a contractive positive projection to be completely positive. We give below examples as in Remark 2.6.

EXAMPLES. Let $P: M_2(C) \to M_2(C)$ be a unital positive projection in Remark 2.6 which is not 2-positive. We define, then, the map $\tilde{P}: M_3(C) \to M_3(C)$ by

$$\tilde{P}\left(\begin{bmatrix}\alpha_{1,1} & \alpha_{1,2} & \alpha_{1,3}\\ \alpha_{2,1} & \alpha_{2,2} & \alpha_{2,3}\\ \alpha_{3,1} & \alpha_{3,2} & \alpha_{3,3}\end{bmatrix}\right) = \begin{bmatrix} P\left(\begin{bmatrix}\alpha_{1,1} & \alpha_{1,2}\\ \alpha_{2,1} & \alpha_{2,2}\end{bmatrix}\right) & \begin{matrix}0\\0\end{matrix}\\ \begin{matrix}0 & 0\end{matrix} & \alpha_{3,3}\end{bmatrix}.$$

It is obvious that $\tilde{P}$ is a unital positive projection but not 2-positive.

When $n \geq 4$ and n is even, there is a unital positive map Φ of $M_{(n/2)}(C)$ into itself which is $((n/2)-1)$-positive but not $(n/2)$-positive [1, Theorem 1]. We define the map $P: M_n(C) \to M_n(C)$ by

$$P\left(\begin{bmatrix} a & b\\ c & d\end{bmatrix}\right) = \begin{bmatrix} a & 0\\ 0 & \Phi(a)\end{bmatrix}, \qquad \text{where} \quad a, b, c, d \in M_{(n/2)}(C),$$

then it is obvious that P is a unital $((n/2)-1)$-positive projection but not $(n/2)$-positive.

When $n \geq 4$ and n is odd, we take the map P on $M_{(n-1)}(C)$ as a unital $([n/2]-1)$-positive projection but not $[n/2]$-positive by the above construction. We then define the map $\tilde{P}: M_n(C) \to M_n(C)$ by

$$\tilde{P}([\alpha_{i,j}]) = \begin{bmatrix} P\left(\begin{bmatrix} \alpha_{1,1} & \cdots & \alpha_{1,(n-1)} \\ \vdots & & \vdots \\ \alpha_{(n-1),1} & \cdots & \alpha_{(n-1),(n-1)} \end{bmatrix}\right) & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \\ 0 \quad \cdots\cdots\cdots \quad 0 & \alpha_{n,n} \end{bmatrix} .$$

It is obvious that $\tilde{P}$ is a unital $([n/2]-1)$-positive projection but not $[n/2]$-positive.

§4. Applications.

In this section we shall apply the preceding result to study the link of finite multiplicity with the complete positivity of projections in general C^*-algebras. Previous results in this direction are the theorems by Stinespring [16], Choi [1], and Takasaki and Tomiyama [19] which state that for those linear maps between C^*-algebras, or between spaces associated to operator algebras one-positivity coincides with two-positivity if and only if either of C^*-algebras, or either of spaces associated to operator algebras, is commutative. On the other hand, Tomiyama [24] has shown the difference between n-positivity and complete positivity in C^*-algebras. Let M be a von Neumann algebra. At first we investigate the connections between the algebraic structure of M and the class of positive projections on M.

PROPOSITION 4.1. *Let M be a von Neumann algebra, then the following assertions are equivalent*:

(1) *Every k-positive contractive projection of M into itself is $(k+1)$-positive.*

(2) *Every k-positive contractive projection of M into itself is completely positive.*

(3) $M=\sum_{n=1}^{k} M_n(A_n)\oplus B$, *where $\{A_n\}$ are commutative von Neumann algebras and $B=M_l(C)$ $(k+1\leq l\leq 2k+1)$ (Some of $\{A_n, B\}$ may be zero.).*

PROOF. (1)$\to$(3). As in the argument similar to the below, we know that by Theorem 3.1 M is $(2k+1)$-subhomogeneous (See [24, Theorem 1.2].).

Suppose that M has $M_l(A)$ $(k+1\leq l\leq 2k+1)$ as a direct summand where A is a non trivial commutative von Neumann algebra. Then $M_l(C)\oplus M_l(C)$ is regarded as a C^*-subalgebra of M. Since $M_l(C)\oplus M_l(C)$ is injective, there is a projection of norm one E of M onto $M_l(C)\oplus M_l(C)$. Let ϕ be a k-positive unital linear map of $M_l(C)$ into itself but not $(k+1)$-positive ([1, Theorem 1]). We define $P: M_l(C)\oplus M_l(C)\to M_l(C)\oplus M_l(C)$ by $P(a\oplus b)=a\oplus\phi(a)$, then P is a k-positive projection but not $(k+1)$-positive. Considering the map $\tilde{P}=P\circ E: M\to M$, $\tilde{P}$ is a k-positive contractive projection but not $(k+1)$-positive. This is a contradiction.

By the structure theorem of von Neumann algebras of type I we obtain the assertion

(1)$\to$(3).

(3)$\to$(2). If B is zero, the assertion is obvious from [24, Theorem 1.2], so that we assume B is not zero. Let P be a k-positive contractive projection of M into itself and P^{**} be the double transpose of P. Let e be the support projection of P^{**}.

As in §1, we consider three positive linear maps φ_1, φ_2 and φ_3 such that $P=\varphi_3\circ\varphi_2\circ\varphi_1$ on M. Since φ_1 and φ_2 are completely positive, we have only to show that φ_3 is completely positive. Since $e\in P^{**}(M)'$, we have

$$\varphi_3(x)=e\varphi_3(x)e+(1-e)\varphi_3(x)(1-e)\,,\qquad x\in P^{**}(M^{**})e\,.$$

The map $e\varphi_3(\ )e$, moreover, is the identity map on $P^{**}(M^{**})e$, and we have only to show that the map $(1-e)\varphi_3(\ )(1-e): P^{**}(M^{**})e\to P^{**}(M^{**})(1-e)$ is completely positive.

Since $B=M_l(C)$ $(k+1\le l\le 2k+1)$ is also a direct summand of M^{**}, there is a central projection $z\in M^{**}$ such that $M^{**}z=M_l(C)$. If the dimension of $(1-e)z$ $(=\dim((1-e)z))\le k$, then $P^{**}(M^{**})(1-e)$ is k-subhomogeneous. We have, then, $(1-e)\varphi_3(\ )(1-e)$ is completely positive from [24, Theorem 1.2]. If $\dim((1-e)z)\ge k+1$, then $\dim(ez)\le k$ and $eM^{**}e$ is k-subhomogeneous. There is a projection of norm one $E: eM^{**}e\to P^{**}(M^{**})e$ because $P^{**}(M^{**})e$ is an injective von Neumann algebra. Considering the map $(1-e)\varphi_3(\ )(1-e)\circ E: eM^{**}e\to P^{**}(M^{**})(1-e)$, we know that it is completely positive. Since the map

$$(1-e)\varphi_3(\ )(1-e)\circ E\,|P^{**}(M^{**})e:\ P^{**}(M^{**})e\longrightarrow P^{**}(M^{**})(1-e)$$

is equal to the map $(1-e)\varphi_3(\ )(1-e)$, φ_3 is completely positive. Hence P is completely positive.

(2)$\to$(1). It is trivial. Q.E.D.

Next, we consider the case of C^*-algebras.

THEOREM 4.2. *Let A be a C^*-algebra and consider the following assertions:*

(1) *Every k-positive projection on A is $(k+1)$-positive,*

(2) *Every k-positive contractive projection on A is completely positive,*

(3) *Every k-positive contractive projection on A is $(k+1)$-positive,*

(4) *A is $(2k+1)$-subhomogeneous which has at most one equivalent class of irreducible representations π's with* $\dim\pi\ge k+1$.

Then we have the following implications;

$$\begin{array}{ccccc} (1) & \Longrightarrow & (4) & \Longrightarrow & (2) \\ & \searrow & & \swarrow & \\ & & (3) & & \end{array}$$

PROOF. (1)→(4). As in the argument of (1)→(3) in Proposition 4.1, it is obvious that A is $(2k+1)$-subhomogeneous. Suppose that there are disjoint irreducible representations π_1, π_2 of $\hat{A}$ with $\dim \pi_i \geq k+1$ $(i=1, 2)$. Let $\pi_i(A)=M_{l_i}(C)$ $(i=1, 2;\ k+1 \leq l_1 \leq l_2 \leq 2k+1)$. Let $\{e_{i,j}\}$ $(1 \leq i, j \leq l_1)$ and $\{f_{p,q}\}$ $(1 \leq p, q \leq l_2)$ be matrix units for $M_{l_1}(C)$ and $M_{l_2}(C)$, respectively. Since π_1 and π_2 are disjoint, for $1 \leq \forall i \leq l_1$ and $1 \leq \forall p \leq l_2$ there exist elements a_i and b_p in A such that $\pi_1(a_i)=e_{1,i}$, $\pi_2(a_i)=0$ and $\pi_1(b_p)=0$, $\pi_2(b_p)=f_{1,p}$ (see [23]). We have, then,

$$e_{i,j} \oplus f_{p,q}=(\pi_1 \oplus \pi_2)(a_i^*a_j+b_p^*b_q)\,,$$

where $1 \leq i, j \leq l_1$ and $1 \leq p, q \leq l_2$. Define the map ρ of $M_{l_1}(C) \oplus M_{l_2}(C)$ into A by

$$\rho([\alpha_{i,j}] \oplus [\beta_{p,q}])=\sum_{i=1}^{l_1} \sum_{j=1}^{l_1} \alpha_{i,j} a_i^* a_j+\sum_{p=1}^{l_2} \sum_{q=1}^{l_2} \beta_{p,q} b_p^* b_q\,,$$

then ρ is completely positive. Moreover for every element $[\alpha_{i,j}] \oplus [\beta_{p,q}] \in M_{l_1}(C) \oplus M_{l_2}(C)$, we have the equality

$$(\pi_1 \oplus \pi_2) \circ \rho([\alpha_{i,j}] \oplus [\beta_{p,q}])=[\alpha_{i,j}] \oplus [\beta_{p,q}]\,.$$

Let ϕ be a unital k-positive linear map of $M_{l_1}(C)$ into itself but not $(k+1)$-positive. We define P of $M_{l_1}(C) \oplus M_{l_2}(C)$ into itself by

$$P([\alpha_{i,j}] \oplus [\beta_{p,q}])=[\alpha_{i,j}] \oplus \left[\begin{array}{c:c} \phi([\alpha_{i,j}]) & 0 \\ \hdashline 0 & 0 \end{array}\right],$$

where $[\alpha_{i,j}] \oplus [\beta_{p,q}] \in M_{l_1}(C) \oplus M_{l_2}(C)$. Then P is k-positive projection but not $(k+1)$-positive. By the properties of $\pi_1 \oplus \pi_2$ and ρ one may easily verify that the composed map $\rho \circ P \circ (\pi_1 \oplus \pi_2)$ is a k-positive projection of A into itself but not $(k+1)$-positive. This is a contradiction.

(4) → (2). From the assumption, we see that A^{**} is $(2k+1)$-subhomogeneous, too. If A has no irreducible representation π with $\dim \pi \geq k+1$, then A is k-subhomogeneous and the assertion is trivial. Thus, we assume that A has an irreducible representation π with $\dim \pi=l$ $(k+1 \leq l \leq 2k+1)$. By the structure theorem of von Neumann algebras of type I, we have

$$A^{**}=\sum_{n=1}^{k} M_n(A_n) \oplus M_l(A_l)\,,$$

where $\{A_n\}$ and A_l are commutative von Neumann algebras (Some of A_n may be zero.).

Suppose A_l is a non trivial von Neumann algebra. Since π is irreducible, then there is a minimal central projection $z \in A^{**}$ such that $A^{**}z=M_l(C)$. Therefore, $A^{**}z$ is a non-trivial direct summand of $M_l(A_l)$ and we can write; $M_l(A_l)=A^{**}z \oplus M_l(A_l)z_l$. Since $M_l(A_l)z_l$ is l-subhomogeneous, there is a normal irreducible representation ρ' of A^{**} with $\rho'(z)=0$ and $\dim \rho'=l$. Let $\rho=\rho'|A$, then ρ is an irreducible representation of A

and disjoint of π. This is a contradiction. We have, therefore,

$$A^{**}=\sum_{n=1}^{k} M_n(A_n)\oplus M_l(C),$$

where $\{A_n\}$ are commutative von Neumann algebras (Some of them may be zero.).

Let P be a k-positive contractive projection of A into itself. Then the double transpose P^{**} of P is a k-positive contractive projection of A^{**} into itself. By Proposition 4.1, P^{**} is completely positive, hence P is completely positive.

The implications (2)→(3) and (1)→(3) are trivial. Q.E.D.

REMARK 4.3. In case of general positive maps, the assertions (3) and (4) in Theorem 4.2 are equivalent ([24, Theorem 1, 2]). Unfortunately, we could not prove the implication (3)→(4) in case of contractive projections.

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