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On the Divisibility Properties of the Orders of $K_2\mathcal{O}_F$ for Certain Totally Real Abelian Fields F

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Introduction.

In [2], Hettling has proved that for any prime number q there exist infinitely many totally real abelian fields F such that q divides the orders of $K_2\mathcal{O}_F$, Milnor's K_2-groups of the rings of integers in F (cf. [4]), in discussing the divisibility properties of the orders of these groups in certain cases. In this paper, we shall show that the prime q in this proposition can be replaced by any integer $n\in N$. We shall use the same notations as in [2], as explained in the following paragraph for completeness' sake.

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§1. Notations and preliminaries.

For $m\in N$, let ζ_m be a primitive m-th root of unity, and $Q(\zeta_m)^+:=Q(\zeta_m+\zeta_m^{-1})$ the maximal totally real subfield of the full cyclotomic field $Q(\zeta_m)$. For an arbitrary abelian number field F, $\mathcal{O}_F$ denotes its ring of integers, ζ_F the Dedekind zeta-function associated to F, and H the Dirichlet character group associated to F. For a character $\chi\in H$, let $L(s,\chi)$ be the Dirichlet L-series associated to χ and $B_{i,\chi}$, $i=1,2,3,\cdots$ the generalized Bernoulli numbers. The ordinary Bernoulli numbers $B_i=B_{i,1}$ belong to the principal character $\chi=1$, refer to [7].

The Birch-Tate conjecture (cf. [1], [5]) states that

$$\sharp K_2\mathcal{O}_F=|\,W_2(F)\cdot\zeta_F(-1)\,|$$

for any totally real number field F, where

$$W_2(F):=\max\{m\in N\mid g^2=1 \text{ for any element } g\in \mathrm{Gal}(F(\zeta_m)/F)\}\ .$$

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Equivalently,

$$W_2(F)=2\prod_p p^{n_p(F)},$$

where the product is taken over all prime numbers p, and

$$n_p(F):=\max\{n \mid \boldsymbol{Q}(\zeta_{p^n})^+\subset F\}.$$

For totally real abelian number fields, the work of Mazur and Wiles [3] on the "Iwasawa Main Conjecture" implies the Birch-Tate conjecture up to the 2-primary part. That is to say, *for any odd prime number q and for any totally real abelian number field F,*

(1) $$q^k \mid \#K_2\mathcal{O}_F \iff q^k \mid W_2(F)\cdot\zeta_F(-1).$$

We will work with subfields F of $\boldsymbol{Q}(\zeta_p)^+$, p being an odd prime. It is easy to see that

(2) $$W_2(F)=\begin{cases} 24, & \text{if } F\subsetneq \boldsymbol{Q}(\zeta_p)^+ \\ 24p, & \text{if } F=\boldsymbol{Q}(\zeta_p)^+ . \end{cases}$$

Next we give a finite expression for $|\zeta_F(-1)|$. Let F be the subfield of $\boldsymbol{Q}(\zeta_p)^+$ with $[F:\boldsymbol{Q}]=n$. Then every nontrivial character $\chi\in H$ is of conductor p. From the identities (cf. [7])

$$\zeta_F(s)=\prod_{\chi\in H} L(s,\chi)$$

and

$$L(-1,\chi)=-\frac{1}{2}B_{2,\chi},$$

we obtain

$$|\zeta_F(-1)|=\frac{1}{2^n}\prod_{\chi\in H} B_{2,\chi}.$$

Furthermore, since $B_{2,1}=B_2=1/6$ and

$$B_{2,\chi}=\frac{1}{p}\sum_{t=1}^{p-1}\chi(t)t(t-p), \qquad \chi\neq 1,$$

we obtain

$$|\zeta_F(-1)|=\frac{1}{12\cdot(2p)^{n-1}}\prod_{\substack{\chi\in H\\ \chi\neq 1}}\sum_{t=1}^{p-1}\chi(t)t(t-p).$$

Letting $S_\chi := \sum_{t=1}^{p-1} \chi(t)t(t-p)$ and considering (1), (2) we obtain

$$q^k \mid \# K_2\mathcal{O}_F \iff q^k \mid \prod_{\substack{\chi\in H\\ \chi\neq 1}} S_\chi \tag{3}$$

for any prime number $q \neq 2, p$.

§2. The main theorem.

THEOREM. *Let F be a subfield of $\boldsymbol{Q}(\zeta_p)^+$, $\mathcal{O}_F$ its ring of integers. For an integer $k \geq 1$,*
(i) *If $q \geq 5$ is a prime number and q^k divides $[F:\boldsymbol{Q}]$, then q^k divides $\# K_2\mathcal{O}_F$.*
(ii) *If 3^2 divides $p-1$ and 3^k divides $[F:\boldsymbol{Q}]$, then 3^k divides $\# K_2\mathcal{O}_F$.*

PROOF. By (3), we have only to show that

$$q^k \mid \prod_{\substack{\chi\in H\\ \chi\neq 1}} S_\chi\,. \tag{4}$$

Let F_k be the subfield of F with $[F_k:\boldsymbol{Q}] = q^k$ and H_k the Dirichlet character group associated to F_k. Since $H_k \subset H$, it is enough to show (4) in which H is replaced by H_k. We shall write

$$q^k \mid \prod_{\substack{\chi\in H_k\\ \chi\neq 1}} S_\chi \tag{4-k}$$

and prove this by induction on k.

In case $k=1$, the order of H_1 is q, all $\chi(t)$ are q-th roots of unity and $S_\chi \in \boldsymbol{Z}[\zeta_q]$ for all $\chi \in H_1$. Furthermore, since

$$\sum_{t=1}^{p-1} t(t-1) = \frac{p(p-1)(p-2)}{3},$$

$q \mid \sum_{t=1}^{p-1} t(t-1)$ if $q \geq 5$ and $3 \mid \sum_{t=1}^{p-1} t(t-1)$ if $q=3$. Now consider congruences modulo q in $\boldsymbol{Z}[\zeta_q]$. Fix $\chi \in H_1$, $\chi \neq 1$. Since $p \equiv 1 \pmod{q}$,

$$\begin{aligned} S_\chi &\equiv \sum_{t=1}^{p-1} \chi(t)t(t-1) \\ &\equiv \sum_{t=1}^{p-1} \chi(t)t(t-1) - \sum_{t=1}^{p-1} t(t-1) \\ &= \sum_{t=1}^{p-1} (\chi(t)-1)t(t-1) \pmod{q}\,. \end{aligned}$$

Now $(\zeta_q-1)\,|\,(\chi(t)-1)$, $1\le t\le p-1$, and $(\zeta_q-1)\,|\,q$, hence $(\zeta_q-1)\,|\,S_\chi$ for all $\chi\in H_1$, $\chi\neq 1$. Therefore,

$$(\zeta_q-1)^{q-1}\,\Big|\,\prod_{\substack{\chi\in H_1\\ \chi\neq 1}} S_\chi\,.$$

Since $(\zeta_q-1)^{q-1}=(q)$ as ideals in $\boldsymbol{Z}[\zeta_q]$ and since $\prod_{\substack{\chi\in H_1\\ \chi\neq 1}} S_\chi\in\boldsymbol{Z}$, we obtain

$$q\,\Big|\,\prod_{\substack{\chi\in H_1\\ \chi\neq 1}} S_\chi\,.$$

Next we shall prove (4-$(r+1)$) assuming

(4-r)
$$q^r\,\Big|\,\prod_{\substack{\chi\in H_r\\ \chi\neq 1}} S_\chi\,.$$

Since the order of H_{r+1} is q^{r+1}, all $\chi(t)$ are q^{r+1}-th roots of unity and $S_\chi\in\boldsymbol{Z}[\zeta_{q^{r+1}}]$ for all $\chi\in H_{r+1}$. Similarly to case (4-1), it is easy to see that $(\zeta_{q^{r+1}}-1)\,|\,(\chi(t)-1)$, $1\le t\le p-1$, $(\zeta_{q^{r+1}}-1)\,|\,q$, and $(\zeta_{q^{r+1}}-1)\,|\,S_\chi$ for all $\chi\in H_{r+1}$, $\chi\neq 1$. Now consider the product

$$\prod_{\substack{\chi\in H_{r+1}\\ \chi\neq 1}} S_\chi=\prod_{\substack{\chi\in H_r\\ \chi\neq 1}} S_\chi\times\prod_{\chi\in H_{r+1}\backslash H_r} S_\chi\,.$$

From the above, we see

$$(\zeta_{q^{r+1}}-1)^{q^{r+1}-q^r}\,\Big|\,\prod_{\chi\in H_{r+1}\backslash H_r} S_\chi\,.$$

Since $(\zeta_{q^{r+1}}-1)^{q^{r+1}-q^r}=(q)$ as ideals in $\boldsymbol{Z}[\zeta_{q^{r+1}}]$ and since

$$\prod_{\chi\in H_{r+1}\backslash H_r} S_\chi=\prod_{\substack{\chi\in H_{r+1}\\ \chi\neq 1}} S_\chi\Big/\prod_{\substack{\chi\in H_r\\ \chi\neq 1}} S_\chi\in\boldsymbol{Z}\,,$$

we obtain

$$q\,\Big|\,\prod_{\chi\in H_{r+1}\backslash H_r} S_\chi\,,$$

hence

$$q^{r+1}\,\Big|\,\prod_{\substack{\chi\in H_{r+1}\\ \chi\neq 1}} S_\chi\,.$$

COROLLARY. *For any $n\in\boldsymbol{N}$ there exist infinitely many totally real abelian fields F with the property that n divides $\#K_2\mathcal{O}_F$.*

PROOF. For $n=2^e m$, $2\nmid m$, we put

$$P_n:=\{p \text{ prime} \mid p\equiv 1 \pmod{3m} \text{ and } p\geq 2e+1\}\,.$$

By Dirichlet's theorem on arithmetic progressions, P_n has infinitely many elements. For $p\in P_n$ we consider F which is a subfield of $\boldsymbol{Q}(\zeta_p)^+$ such that m divides $[F:\boldsymbol{Q}]$ and $e\leq[F:\boldsymbol{Q}]$, for example $F:=\boldsymbol{Q}(\zeta_p)^+$. By the Main Theorem, m divides $\sharp K_2\mathcal{O}_F$. Furthermore, by Tate's 2-rank formula (cf. [6]), which implies that $2^{[F:\boldsymbol{Q}]}$ *divides* $\sharp K_2\mathcal{O}_F$ *for any totally real number field* F, we have

$$2^e \mid \sharp K_2\mathcal{O}_F\,,$$

hence

$$n=2^e m \mid \sharp K_2\mathcal{O}_F\,. \qquad \blacksquare$$

REMARK. In case $e=0$, we may consider F with

$$[F:\boldsymbol{Q}]=m=n\,.$$

In case $e>0$, we put $d:=(m, e)$ and

$$P'_n:=\left\{p \text{ prime} \;\middle|\; p\equiv 1\left(\bmod \frac{6me}{d}\right)\right\}.$$

For $p\in P'_n$, we may consider F with

$$[F:\boldsymbol{Q}]=\frac{me}{d}<n\,.$$

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