TOKYO J. MATH.
VOL. 14, NO. 2, 1991

On the Generalized Hilbert Transforms in R^2 and the Generalized Harmonic Analysis*

Katsuo MATSUOKA

Keio Shiki High School
(Communicated by Y. Ito)

Dedicated to Professor Tatsuo Kawata on his eightieth birthday

Introduction.

In [17, 18], N. Wiener established the generalized harmonic analysis (GHA) in order to give an account of phenomena which cannot be described by the classical harmonic analysis. Especially, his GHA was motivated by even earlier investigations in the theory of Brownian motion, in order to study the functions with continuous spectra. And, he studied the uniformly almost periodic functions in the sense of H. Bohr from the point of view of the GHA (cf. S. Koizumi [7, 8] and P. Masani [9–11]).

S. Koizumi [4] introduced the generalized Hilbert transform (GHT) of $g\in L_c^2(\boldsymbol{R})$, i.e. g such that $g(x)/(x+i)\in L^2(\boldsymbol{R})$, by

$$\tilde{g}(x)=\lim_{\varepsilon\to 0}\frac{x+i}{\pi}\int_{0<\varepsilon\leq|x-t|}\frac{g(t)}{t+i}\frac{dt}{x-t}.$$

And, in [5, 6], he constructed the theory of the spectral analysis of the GHT by using Wiener's generalized Fourier transform (GFT)

$$s(u;g)=\underset{A\to\infty}{\text{l.i.m.}}\frac{1}{\sqrt{2\pi}}\left[\int_1^A+\int_{-A}^{-1}\right]g(t)\frac{e^{-iut}}{-it}dt+\frac{1}{\sqrt{2\pi}}\int_{-1}^{1}g(t)\frac{e^{-iut}-1}{-it}dt\qquad(g\in L_c^2(\boldsymbol{R})),$$

where "l.i.m." means the limit in $L^2(\boldsymbol{R})$. Then he proved the following fundamental Wiener's GFT relation between any $g\in L_c^2(\boldsymbol{R})$ and its GHT $\tilde{g}$:

Received December 7, 1990
Revised July 9, 1991

* Part of this paper was reported in [15].

$$s(u+\varepsilon;\tilde{g})-s(u-\varepsilon;\tilde{g})=\begin{cases}(-i\,\mathrm{sgn}\,u)\{s(u+\varepsilon;g)-s(u-\varepsilon;g)\} & (|u|>\varepsilon)\\ i\{s(u+\varepsilon;g)-s(u-\varepsilon;g)\}+2r_1^g(u+\varepsilon)+2r_2^g(u+\varepsilon) & (|u|\le\varepsilon)\,,\end{cases}$$

where

$$r_1^g(u)=\underset{B\to\infty}{\mathrm{l.i.m.}}\frac{1}{\sqrt{2\pi}}\int_{-B}^{B}\frac{g(s)}{s+i}\frac{e^{-ius}-1}{-is}ds$$

and

$$r_2^g(u)=\underset{B\to\infty}{\mathrm{l.i.m.}}\frac{1}{\sqrt{2\pi}}\int_{-B}^{B}\frac{g(s)}{s+i}e^{-ius}ds\,.$$

Along the same line, the author ([13, 16]) extended Koizumi's theory of the spectral analysis of the GHT to the $\boldsymbol{R}^2$ case. Then, since the spectra of the GHT appear at the origin and along the coordinate axes, we were compelled to treat the complicated calculation.

On the other hand, P. Masani [10] pointed out that Wiener's GHA can be assimilated in a widened form of functional analysis (Masani's GHA) in which he defined the GFT

$$\alpha(u;g)=\underset{A\to\infty}{\mathrm{l.i.m.}}\frac{1}{\sqrt{2\pi}}\int_{-A}^{A}g(t)l_0(t)e^{-iut}dt\qquad(g\in L_c^2(\boldsymbol{R}))\,,$$

where

$$l_0(t)=\frac{i}{\sqrt{\pi}}\frac{1}{t+i}\,.$$

Here, the above two GFT's have the same contents essentially. Recently, we have known that Masani's GFT matches well the GHT (cf. K. Matsuoka [14, 15]).

In the $\boldsymbol{R}^2$ case, another remarkable circumstance occurs. This is the manner of limit. For some reasons, in the previous paper (K. Anzai, S. Koizumi and K. Matsuoka [1]), we studied the restricted limit as follows: The limit

$$\lim_{S,T\to\infty}\frac{1}{4ST}\int_{-T}^{T}\int_{-S}^{S}|f(s,t)|^2dsdt$$

exists and has the same limit whenever S and T tend to infinity in such a way that $S=CT$ for every positive constant C. As for the spherical limit and the independent limit, see N. Wiener–A. C. Berry [17] and T. Kawata [2] respectively.

In this paper, therefore, under the restricted limit, we will reconstruct the theory of the spectral analysis of the GHT by using Masani's GFT instead of Wiener's GFT. As a result, we can easily deal with the spectra of the GHT at the origin and along the coordinate axes.

§1. Masani's generalized harmonic analysis (GHA).

Throughout this paper, all functions we consider will be complex valued and measurable on $\boldsymbol{R}$ or $\boldsymbol{R}^2$.

First, we list the notation, which will be used in what follows (see K. Matsuoka [12–16]):

(a) $$L_c^2(\boldsymbol{R}^2)=\left\{f\in L_{\mathrm{loc}}^2(\boldsymbol{R}^2):\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|f(x_1,x_2)|^2dc(x_1,x_2)=\frac{1}{\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{|f(x_1,x_2)|^2}{(1+x_1^2)(1+x_2^2)}dx_1dx_2<\infty\right\};$$

(b) [Masani's generalized Fourier transform (GFT)]

$$\alpha(u,v;f)=\underset{A\to\infty}{\mathrm{l.i.m.}}\frac{1}{2\pi}\int_{-A}^{A}\int_{-A}^{A}f(s,t)l_0(s)l_0(t)e^{-i(us+vt)}dsdt\qquad(f\in L_c^2(\boldsymbol{R}^2)),$$

where

$$l_0(x)=\frac{i}{\sqrt{\pi}}\frac{1}{x+i}\qquad(x\in\boldsymbol{R})$$

and the notation "l.i.m." means the limit in $L^2(\boldsymbol{R}^2)$;

(c) $\Delta_\varepsilon\alpha(u;g)=\alpha(u+\varepsilon;g)-\alpha(u-\varepsilon;g)$,

$\Delta_\varepsilon^+\alpha(u;g)=\alpha(u+\varepsilon;g)+\alpha(u-\varepsilon;g)$,

$\Delta_\varepsilon^{(1)}\alpha(u,v;f)=\alpha(u+\varepsilon,v;f)-\alpha(u-\varepsilon,v;f)$,

$\Delta_\varepsilon^{(1)+}\alpha(u,v;f)=\alpha(u+\varepsilon,v;f)+\alpha(u-\varepsilon,v;f)$,

$\Delta_\eta^{(2)}\alpha(u,v;f)=\alpha(u,v+\eta;f)-\alpha(u,v-\eta;f)$,

$\Delta_\eta^{(2)+}\alpha(u,v;f)=\alpha(u,v+\eta;f)+\alpha(u,v-\eta;f)$,

$\Delta_{\varepsilon,\eta}=\Delta_{\varepsilon,\eta}^{--}=\Delta_\varepsilon^{(1)}\Delta_\eta^{(2)}$,

$\Delta_{\varepsilon,\eta}^{+-}=\Delta_\varepsilon^{(1)+}\Delta_\eta^{(2)}$,

$\Delta_{\varepsilon,\eta}^{-+}=\Delta_\varepsilon^{(1)}\Delta_\eta^{(2)+}$,

$\Delta_{\varepsilon,\eta}^{++}=\Delta_\varepsilon^{(1)+}\Delta_\eta^{(2)+}$;

(d) The notations "$\mathscr{R}_1$-$\lim_{S,T\to\infty}$" and "$\mathscr{R}_2$-$\lim_{\varepsilon,\eta\to+0}$" mean that in each of them a limit exists and has the same limit whenever S and T tend to infinity or ε and η tend to zero in such a way that $S=CT$ or $\eta=C\varepsilon$ for every positive constant C, respectively;

(e) $W^2(\boldsymbol{R}^2)=\left\{f\in L^2_{\mathrm{loc}}(\boldsymbol{R}^2)\ :\ \sup_{S,T>0}\frac{1}{4ST}\int_{-T}^{T}\int_{-S}^{S}|f(s,t)|^2dsdt<\infty\right\}$;

(f) $\mathscr{W}^2(\boldsymbol{R}^2)=\left\{f\in W^2(\boldsymbol{R}^2)\ :\ \mathscr{R}_1\text{-}\lim_{S,T\to\infty}\frac{1}{4ST}\int_{-T}^{T}\int_{-S}^{S}|f(s,t)|^2dsdt\ \text{exists}\right\}$;

(g) $S(\boldsymbol{R}^2)=\{f\in W^2(\boldsymbol{R}^2)\ :\ \phi(x_1,x_2;f)\ \text{exists for all}\ (x_1,x_2)\in\boldsymbol{R}^2\}$,

where

$$\phi(x_1,x_2;f)=\mathscr{R}_1\text{-}\lim_{S,T\to\infty}\frac{1}{4ST}\int_{-T}^{T}\int_{-S}^{S}f(x_1+s,x_2+t)\overline{f(s,t)}dsdt,$$

which is called the covariance function of f;

(h) $S'(\boldsymbol{R}^2)=\{f\in S(\boldsymbol{R}^2)\ :\ \phi(x_1,x_2;f)\ \text{is continuous on}\ \boldsymbol{R}^2\}$.

Similarly, the $\boldsymbol{R}$ case is defined.

Note that

$$L^2_{\mathrm{loc}}\supset L^2_c\supset W^2\supset\mathscr{W}^2\supset S\supset S' \tag{1.1}$$

on $\boldsymbol{R}$ or $\boldsymbol{R}^2$ (see N. Wiener [18] and K. Matsuoka [12]). Also,

$$\Delta_{\varepsilon,\eta}\alpha(u,v;f)=\underset{A\to\infty}{\text{l.i.m.}}\ \frac{1}{2\pi}\int_{-A}^{A}\int_{-A}^{A}f(s,t)l_0(s)l_0(t) \cdot(-2i\sin\varepsilon s)(-2i\sin\eta t)e^{-i(us+vt)}dsdt \qquad (f\in L^2_c(\boldsymbol{R}^2)). \tag{1.2}$$

Next, we state the $\boldsymbol{R}^2$ case of Wiener's identity which is fundamental in Masani's GHA (as for the $\boldsymbol{R}$ case, see P. Masani [10]).

THEOREM 1.1 (K. Matsuoka [15]). *If $f\in W^2(\boldsymbol{R}^2)$, then*

$$\mathscr{R}_1\text{-}\lim_{S,T\to\infty}\frac{1}{4ST}\int_{-T}^{T}\int_{-S}^{S}|f(s,t)|^2dsdt =\mathscr{R}_2\text{-}\lim_{\varepsilon,\eta\to+0}\frac{1}{16\varepsilon\eta}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|\Delta_{\varepsilon,\eta}\alpha(u,v;f)|^2dudv, \tag{1.3}$$

in the sense that if either side of (1.3) *exists, the other side exists and assumes the same value.*

PROOF. It follows from (1.2), the Plancherel theorem and (3.10), (3.11) of K. Matsuoka [12] that

$$\mathscr{R}_2\text{-}\lim_{\varepsilon,\eta\to+0}\frac{1}{16\varepsilon\eta}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|\Delta_{\varepsilon,\eta}\alpha(u,v;f)|^2dudv =\mathscr{R}_2\text{-}\lim_{\varepsilon,\eta\to+0}\frac{1}{\pi^2\varepsilon\eta}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|f(s,t)|^2\frac{\sin^2\varepsilon s}{1+s^2}\frac{\sin^2\eta t}{1+t^2}dsdt \tag{1.4}$$

$$=\mathscr{R}_2\text{-}\lim_{\varepsilon,\eta\to+0}\frac{1}{\pi^2\varepsilon\eta}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|f(s,t)|^2\frac{\sin^2\varepsilon s}{s^2}\frac{\sin^2\eta t}{t^2}dsdt$$

$$-\mathscr{R}_2\text{-}\lim_{\varepsilon,\eta\to+0}\frac{\varepsilon}{\pi^2\eta}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{|f(s,t)|^2}{1+s^2}\left(\frac{\sin\varepsilon s}{\varepsilon s}\right)^2\frac{\sin^2\eta t}{t^2}dsdt$$

$$-\mathscr{R}_2\text{-}\lim_{\varepsilon,\eta\to+0}\frac{\eta}{\pi^2\varepsilon}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{|f(s,t)|^2}{1+t^2}\frac{\sin^2\varepsilon s}{s^2}\left(\frac{\sin\eta t}{\eta t}\right)^2dsdt$$

$$+\mathscr{R}_2\text{-}\lim_{\varepsilon,\eta\to+0}\frac{\varepsilon\eta}{\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{|f(s,t)|^2}{(1+s^2)(1+t^2)}\left(\frac{\sin\varepsilon s}{\varepsilon s}\right)^2\left(\frac{\sin\eta t}{\eta t}\right)^2dsdt$$

$$=\mathscr{R}_2\text{-}\lim_{\varepsilon,\eta\to+0}\frac{1}{\pi^2\varepsilon\eta}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|f(s,t)|^2\frac{\sin^2\varepsilon s}{s^2}\frac{\sin^2\eta t}{t^2}dsdt\,.$$

Thus, applying Theorem 4 of K. Anzai, S. Koizumi and K. Matsuoka [1] or Theorem 2′ of K. Matsuoka [12] (or Theorem 6 of T. Kawata [3]), we have (1.3). ■

In the remains of this section, we shall give the following theorem connecting the covariance function to Masani's GFT (as for the $\boldsymbol{R}$ case, see P. Masani [10]).

THEOREM 1.2. *If* $f\in S(\boldsymbol{R}^2)$, *then*

(1.5) $\phi(x_1,x_2;f)$

$$=\mathscr{R}_2\text{-}\lim_{\varepsilon,\eta\to+0}\frac{1}{16\varepsilon\eta}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{i(ux_1+vx_2)}|\Delta_{\varepsilon,\eta}\alpha(u,v;f)|^2dudv\qquad((x_1,x_2)\in\boldsymbol{R}^2)\,.$$

PROOF. By (1.4),

$$\mathscr{R}_2\text{-}\lim_{\varepsilon,\eta\to+0}\frac{1}{\pi^2\varepsilon\eta}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|f(s,t)|^2\frac{\sin^2\varepsilon s}{1+s^2}\frac{\sin^2\eta t}{1+t^2}dsdt \tag{1.6}$$

$$=\mathscr{R}_2\text{-}\lim_{\varepsilon,\eta\to+0}\frac{1}{\pi^2\varepsilon\eta}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|f(s,t)|^2\frac{\sin^2\varepsilon s}{s^2}\frac{\sin^2\eta t}{t^2}dsdt\,.$$

Therefore, it follows from Lemma 5 of K. Matsuoka [12] that

$$\mathscr{R}_2\text{-}\lim_{\varepsilon,\eta\to+0}\frac{1}{16\varepsilon\eta}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|\Delta_{\varepsilon,\eta}\alpha_{x_1,x_2}(u,v;f) \tag{1.7}$$

$$-e^{i(ux_1+vs_2)}\Delta_{\varepsilon,\eta}\alpha(u,v;f)|^2dudv=0\,,$$

where

$$\alpha_{x_1,x_2}(u,v;f)=\underset{A\to\infty}{\text{l.i.m.}}\frac{1}{2\pi}\int_{-A}^{A}\int_{-A}^{A}f(x_1+s,x_2+t)l_0(s)l_0(t)e^{-i(us+vt)}dsdt\,.$$

Thus, applying the same argument as in the proof of Theorem 6 of K. Matsuoka [12], we obtain (1.5) by Theorem 1.1 and (1.7). ■

§2. The spectral analysis of the generalized Hilbert transform (GHT).

In [5, 6], S. Koizumi established the spectral relation between a given function on $\boldsymbol{R}$ and its GHT by using Wiener's GHA. Also, the $\boldsymbol{R}^2$ case was investigated by K. Matsuoka [14, 17].

In this section, using Masani's GHA, we shall prove the main theorems in [13, 16] (cf. K. Matsuoka [14, 15]).

In a way similar to [4], we introduce the GHT defined on $\boldsymbol{R}^2$ as follows:

$$(2.1)\quad (H^{(1)}f)(x_1, x_2)=\lim_{\varepsilon_1\to 0}\frac{x_1+i}{\pi}\int_{0<\varepsilon_1\leq|x_1-s|}\frac{f(s, x_2)}{s+i}\frac{ds}{x_1-s}\qquad (f\in L_c^2(\boldsymbol{R}^2)),$$

$$(2.2)\quad (H^{(2)}f)(x_1, x_2)=\lim_{\varepsilon_2\to 0}\frac{x_2+i}{\pi}\int_{0<\varepsilon_2\leq|x_2-t|}\frac{f(x_1, t)}{t+i}\frac{dt}{x_2-t}\qquad (f\in L_c^2(\boldsymbol{R}^2)),$$

$$(2.3)\quad (Hf)(x_1, x_2)=(H^{(2)}H^{(1)}f)(x_1, x_2)=(H^{(1)}H^{(2)}f)(x_1, x_2)$$
$$=\lim_{\varepsilon_1,\varepsilon_2\to 0}\frac{(x_1+i)(x_2+i)}{\pi^2}\iint_{\substack{0<\varepsilon_1\leq|x_1-s|\\ 0<\varepsilon_2\leq|x_2-t|}}\frac{f(s, t)}{(s+i)(t+i)}\frac{dsdt}{(x_1-s)(x_2-t)}\qquad (f\in L_c^2(\boldsymbol{R}^2)).$$

Now, we remark that whenever $f\in W^2(\boldsymbol{R}^2)$, by (1.1), the GHT's $H^{(1)}f$, $H^{(2)}f$, Hf are defined, and the GFT's of these are also defined. On the other hand, it is to be noted that there exists $f\in W^2(\boldsymbol{R}^2)$ such that $Hf\notin W^2(\boldsymbol{R}^2)$. For example, letting $f(x_1, x_2)=\chi_{(0,\infty)}(x_1)\chi_{(0,\infty)}(x_2)$, where $\chi_{(0,\infty)}$ is the characteristic function of the interval $(0, \infty)$, we get the required function

$$(Hf)(x_1, x_2)=\left(\frac{1}{\pi}\log|x_1|-\frac{i}{2}\right)\left(\frac{1}{\pi}\log|x_2|-\frac{i}{2}\right).$$

Therefore, in order to apply Theorem 1.1 to GHT, moreover, we introduce the following subclasses of $W^2(\boldsymbol{R}^2)$:

$$(2.4)\quad W_{H^{(1)}}^2(\boldsymbol{R}^2)=\{f\in W^2(\boldsymbol{R}^2) : H^{(1)}f\in W^2(\boldsymbol{R}^2)\},$$

$$(2.5)\quad W_{H^{(2)}}^2(\boldsymbol{R}^2)=\{f\in W^2(\boldsymbol{R}^2) : H^{(2)}f\in W^2(\boldsymbol{R}^2)\},$$

$$(2.6)\quad W_H^2(\boldsymbol{R}^2)=\{f\in W^2(\boldsymbol{R}^2) : H^{(1)}f, H^{(2)}f, Hf\in W^2(\boldsymbol{R}^2)\}.$$

First, we state the theorem concerning the mean total power of the GHT.

THEOREM 2.1. *Suppose* $f\in \mathscr{W}^2(\boldsymbol{R}^2)$, $Hf\in W^2(\boldsymbol{R}^2)$, *and there exist the constants* K_1^{--}, K_1^{+-}, K_2^{--}, K_2^{-+}, K_3^{--}, K_3^{+-}, K_3^{-+}, K_3^{++} *such that*

(2.7) $$K_1^{\pm -} = \mathscr{R}_2\text{-}\lim_{\varepsilon,\eta \to +0} \frac{1}{16\varepsilon\eta} \int_{-\infty}^{\infty} \int_{-\varepsilon}^{\varepsilon} |\Delta_{\varepsilon,\eta}^{\pm -} \alpha(u, v; f)|^2 dudv ,$$

(2.8) $$K_2^{-\pm} = \mathscr{R}_2\text{-}\lim_{\varepsilon,\eta \to +0} \frac{1}{16\varepsilon\eta} \int_{-\eta}^{\eta} \int_{-\infty}^{\infty} |\Delta_{\varepsilon,\eta}^{-\pm} \alpha(u, v; f)|^2 dudv ,$$

(2.9) $$K_3^{\pm\pm} = \mathscr{R}_2\text{-}\lim_{\varepsilon,\eta \to +0} \frac{1}{16\varepsilon\eta} \int_{-\eta}^{\eta} \int_{-\varepsilon}^{\varepsilon} |\Delta_{\varepsilon,\eta}^{\pm\pm} \alpha(u, v; f)|^2 dudv .$$

Then $Hf \in \mathscr{W}^2(\boldsymbol{R}^2)$ *and*

(2.10) $$\begin{aligned} &\mathscr{R}_1\text{-}\lim_{S,T \to \infty} \frac{1}{4ST} \int_{-T}^{T} \int_{-S}^{S} |(Hf)(s, t)|^2 dsdt \\ &= \mathscr{R}_1\text{-}\lim_{S,T \to \infty} \frac{1}{4ST} \int_{-T}^{T} \int_{-S}^{S} |f(s, t)|^2 dsdt \\ &\quad -(K_1^{--} + K_2^{--} - K_3^{--}) + (K_1^{+-} - K_3^{+-}) + (K_2^{-+} - K_3^{-+}) + K_3^{++} . \end{aligned}$$

Before proving Theorem 2.1, we mention the following: By the multiplier property of the ordinal Hilbert transform, we have Masani's GFT relations between any $f \in L_c^2(\boldsymbol{R}^2)$ and its GHT's:

(2.11) $$\alpha(u, v; H^{(1)}f) = (-i \operatorname{sgn} u)\alpha(u, v; f) ,$$

(2.12) $$\alpha(u, v; H^{(2)}f) = (-i \operatorname{sgn} v)\alpha(u, v; f)$$

and

(2.13) $$\alpha(u, v; Hf) = (-i \operatorname{sgn} u)(-i \operatorname{sgn} v)\alpha(u, v; f) .$$

Therefore, it follows from (2.11)–(2.13) that for any $\varepsilon, \eta > 0$,

(2.14) $$\Delta_{\varepsilon,\eta}\alpha(u, v; H^{(1)}f) = \begin{cases} (-i \operatorname{sgn} u)\Delta_{\varepsilon,\eta}\alpha(u, v; f) & (|u| > \varepsilon) \\ (-i)\Delta_{\varepsilon,\eta}^{+-} \alpha(u, v; f) & (|u| \le \varepsilon) , \end{cases}$$

(2.15) $$\Delta_{\varepsilon,\eta}\alpha(u, v; H^{(2)}f) = \begin{cases} (-i \operatorname{sgn} v)\Delta_{\varepsilon,\eta}\alpha(u, v; f) & (|v| > \eta) \\ (-i)\Delta_{\varepsilon,\eta}^{-+} \alpha(u, v; f) & (|v| \le \eta) \end{cases}$$

and

(2.16) $$\Delta_{\varepsilon,\eta}\alpha(u, v; Hf) = \begin{cases} (-i \operatorname{sgn} u)(-i \operatorname{sgn} v)\Delta_{\varepsilon,\eta}\alpha(u, v; f) & (|u| > \varepsilon, |v| > \eta) \\ (-i)(-i \operatorname{sgn} v)\Delta_{\varepsilon,\eta}^{+-} \alpha(u, v; f) & (|u| \le \varepsilon, |v| > \eta) \\ (-i \operatorname{sgn} u)(-i)\Delta_{\varepsilon,\eta}^{-+} \alpha(u, v; f) & (|u| > \varepsilon, |v| \le \eta) \\ (-i)^2 \Delta_{\varepsilon,\eta}^{++} \alpha(u, v; f) & (|u| \le \varepsilon, |v| \le \eta) , \end{cases}$$

respectively.

PROOF OF THEOREM 2.1. Using (2.16), we have

$$\mathscr{R}_2\text{-}\lim_{\varepsilon,\eta\to+0}\frac{1}{16\varepsilon\eta}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|\Delta_{\varepsilon,\eta}\alpha(u,v;Hf)|^2dudv$$

$$=\mathscr{R}_2\text{-}\lim_{\varepsilon,\eta\to+0}\frac{1}{16\varepsilon\eta}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|\Delta_{\varepsilon,\eta}\alpha(u,v;f)|^2dudv$$

$$-(K_1^{--}+K_2^{--}-K_3^{--})+(K_1^{+-}-K_3^{+-})+(K_2^{-+}-K_3^{-+})+K_3^{++}.$$

Thus, by Theorem 1.1, (2.10) is proved. ■

From Theorem 2.1, in result, we obtain the following corollaries.

COROLLARY 2.2 (K. Matsuoka [15]). *Suppose $f\in W^2_{H^{(1)}}(\boldsymbol{R}^2)\cap\mathscr{W}^2(\boldsymbol{R}^2)$, and it satisfies that*

$$(\mathrm{M}_1)\qquad \mathscr{R}_2\text{-}\lim_{\varepsilon,\eta\to+0}\frac{1}{8\varepsilon\eta}\int_{-\infty}^{\infty}\int_{-\varepsilon}^{\varepsilon}|\Delta_{\varepsilon,\eta}\alpha(u,v;f)|^2dudv=0$$

and that

$(\mathrm{M}_2)_{\mathscr{W}^2}$ *there exists a function $k_1^f(x_2)\in\mathscr{W}^2(\boldsymbol{R}^2)$ such that*

$$(2.17)\qquad \mathscr{R}_2\text{-}\lim_{\varepsilon,\eta\to+0}\frac{1}{8\varepsilon\eta}\int_{-\infty}^{\infty}\int_{-\varepsilon}^{\varepsilon}\left|\frac{1}{i}\Delta_{\varepsilon,\eta}^{+-}\alpha(u,v:f)-\sqrt{2}\,\Delta_\eta\alpha(v;k_1^f)\right|^2dudv=0.$$

Then $H^{(1)}f\in\mathscr{W}^2(\boldsymbol{R}^2)$ and

$$(2.18)\qquad \mathscr{R}_1\text{-}\lim_{S,T\to\infty}\frac{1}{4ST}\int_{-T}^{T}\int_{-S}^{S}|(H^{(1)}f)(s,t)|^2dsdt$$

$$=\mathscr{R}_1\text{-}\lim_{S,T\to\infty}\frac{1}{4ST}\int_{-T}^{T}\int_{-S}^{S}|f(s,t)|^2dsdt+\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|k_1^f(t)|^2dt.$$

COROLLARY 2.2′ (K. Matsuoka [15]). *Suppose $f\in W^2_{H^{(2)}}(\boldsymbol{R}^2)\cap\mathscr{W}^2(\boldsymbol{R}^2)$, and it satisfies that*

$$(\mathrm{M}_3)\qquad \mathscr{R}_2\text{-}\lim_{\varepsilon,\eta\to+0}\frac{1}{8\varepsilon\eta}\int_{-\eta}^{\eta}\int_{-\infty}^{\infty}|\Delta_{\varepsilon,\eta}\alpha(u,v;f)|^2dudv=0$$

and that

$(\mathrm{M}_4)_{\mathscr{W}^2}$ *there exists a function $k_2^f(x_1)\in\mathscr{W}^2(\boldsymbol{R})$ such that*

$$(2.19)\qquad \mathscr{R}_2\text{-}\lim_{\varepsilon,\eta\to+0}\frac{1}{8\varepsilon\eta}\int_{-\eta}^{\eta}\int_{-\infty}^{\infty}\left|\frac{1}{i}\Delta_{\varepsilon,\eta}^{-+}\alpha(u,v;f)-\sqrt{2}\,\Delta_\varepsilon\alpha(u;k_2^f)\right|^2dudv=0.$$

Then $H^{(2)}f\in\mathscr{W}^2(\boldsymbol{R}^2)$ and

$$(2.20)\qquad \mathscr{R}_1\text{-}\lim_{S,T\to\infty}\frac{1}{4ST}\int_{-T}^{T}\int_{-S}^{S}|(H^{(2)}f)(s,t)|^2dsdt$$
$$=\mathscr{R}_1\text{-}\lim_{S,T\to\infty}\frac{1}{4ST}\int_{-T}^{T}\int_{-S}^{S}|f(s,t)|^2dsdt+\lim_{S\to\infty}\frac{1}{2S}\int_{-S}^{S}|k_2^f(s)|^2ds\,.$$

COROLLARY 2.3 (K. Matsuoka [15]). *Suppose* $f\in W_H^2(\boldsymbol{R}^2)\cap\mathscr{W}^2(\boldsymbol{R}^2)$, *and it satisfies* ($M_1$), $(M_2)_{\mathscr{W}^2}$, (M_3), $(M_4)_{\mathscr{W}^2}$ *and, in addition, that*

$$(M_5)\qquad \mathscr{R}_2\text{-}\lim_{\varepsilon,\eta\to+0}\frac{1}{4\varepsilon\eta}\int_{-\eta}^{\eta}\int_{-\varepsilon}^{\varepsilon}|\Delta_{\varepsilon,\eta}^{+-}\alpha(u,v;f)|^2dudv=0\,,$$

$$(M_6)\qquad \mathscr{R}_2\text{-}\lim_{\varepsilon,\eta\to+0}\frac{1}{4\varepsilon\eta}\int_{-\eta}^{\eta}\int_{-\varepsilon}^{\varepsilon}|\Delta_{\varepsilon,\eta}^{-+}\alpha(u,v;f)|^2dudv=0$$

and

(M_7) *there exists a constant* k_3^f *such that*

$$\mathscr{R}_2\text{-}\lim_{\varepsilon,\eta\to+0}\frac{1}{4\varepsilon\eta}\int_{-\eta}^{\eta}\int_{-\varepsilon}^{\varepsilon}\left|\left(\frac{1}{i}\right)^2\Delta_{\varepsilon,\eta}^{++}\alpha(u,v;f)-2k_3^f\right|^2dudv=0\,.$$

Then $Hf\in\mathscr{W}^2(\boldsymbol{R}^2)$ *and*

$$(2.21)\qquad \mathscr{R}_1\text{-}\lim_{S,T\to\infty}\frac{1}{4ST}\int_{-T}^{T}\int_{-S}^{S}|(Hf)(s,t)|^2dsdt$$
$$=\mathscr{R}_1\text{-}\lim_{S,T\to\infty}\frac{1}{4ST}\int_{-T}^{T}\int_{-S}^{S}|f(s,t)|^2dsdt$$
$$+\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|k_1^f(t)|^2dt+\lim_{S\to\infty}\frac{1}{2S}\int_{-S}^{S}|k_2^f(s)|^2ds+|k_3^f|^2\,.$$

Next, we show the theorems concerning the covariance function of the GHT.

THEOREM 2.4. *Suppose* $f\in W_{H^{(1)}}^2(\boldsymbol{R}^2)\cap S(\boldsymbol{R}^2)$, *and it satisfies* ($M_1$) *and that*

$(M_2)_S$ *there exists a function* $k_1^f(x_2)\in S(\boldsymbol{R})$ *such that* (2.17) *holds.*

Then $H^{(1)}f\in S(\boldsymbol{R}^2)$ *and*

$$(2.22)\qquad \phi(x_1,x_2;H^{(1)}f)=\phi(x_1,x_2;f)+\phi(x_2;k_1^f)\,.$$

THEOREM 2.4′. *Suppose* $f\in W_{H^{(2)}}^2(\boldsymbol{R}^2)\cap S(\boldsymbol{R}^2)$, *and it satisfies* ($M_3$) *and that*

$(M_4)_S$ *there exists a function* $k_2^f(x_1)\in S(\boldsymbol{R})$ *such that* (2.19) *holds.*

Then $H^{(2)}f\in S(\boldsymbol{R}^2)$ *and*

$$(2.23)\qquad \phi(x_1,x_2;H^{(2)}f)=\phi(x_1,x_2;f)+\phi(x_1;k_2^f)\,.$$

THEOREM 2.5. *Suppose* $f \in W_H^2(\boldsymbol{R}^2) \cap S(\boldsymbol{R}^2)$, *and it satisfies* (M_1), $(\mathrm{M}_2)_S$, (M_3), $(\mathrm{M}_4)_S$ *and* (M_5)–(M_7). *Then* $Hf \in S(\boldsymbol{R}^2)$ *and*

$$\phi(x_1, x_2; Hf) = \phi(x_1, x_2; f) + \phi(x_2; k_1^f) + \phi(x_1; k_2^f) + |k_3^f|^2 . \tag{2.24}$$

Applying the same argument as in the proof of Theorem 2.1, we prove Theorems 2.4–2.5 by Theorem 1.2.

Furthermore, by Theorems 2.4–2.5, the results concerning the GHT of functions in the class $S'(\boldsymbol{R}^2)$ follow.

THEOREM 2.6. *Suppose* $f \in W_{H^{(1)}}^2(\boldsymbol{R}^2) \cap S'(\boldsymbol{R}^2)$, *and it satisfies* (M_1) *and that*

$(\mathrm{M}_2)_{S'}$ *there exists a function* $k_1^f(x_2) \in S'(\boldsymbol{R})$ *such that* (2.17) *holds.*

Then $H^{(1)}f \in S'(\boldsymbol{R}^2)$ *and* (2.22) *holds.*

THEOREM 2.6′. *Suppose* $f \in W_{H^{(2)}}^2(\boldsymbol{R}^2) \cap S'(\boldsymbol{R}^2)$, *and it satisfies* (M_3) *and that*

$(\mathrm{M}_4)_{S'}$ *there exists a function* $k_2^f(x_1) \in S'(\boldsymbol{R})$ *such that* (2.19) *holds.*

Then $H^{(2)}f \in S'(\boldsymbol{R}^2)$ *and* (2.23) *holds.*

THEOREM 2.7. *Suppose* $f \in W_H^2(\boldsymbol{R}^2) \cap S'(\boldsymbol{R}^2)$, *and it satisfies* (M_1), $(\mathrm{M}_2)_{S'}$, (M_3), $(\mathrm{M}_4)_{S'}$ *and* (M_5)–(M_7). *Then* $Hf \in S'(\boldsymbol{R}^2)$ *and* (2.24) *holds.*

ACKNOWLEDGEMENT. I am very grateful to Professor Sumiyuki Koizumi of Keio University for several useful discussions with me and for his guidance and encouragement. I would also like to thank the referee for some helpful comments.

References

[1] K. ANZAI, S. KOIZUMI and K. MATSUOKA, On the Wiener formula of functions of two variables, Tokyo J. Math., **3** (1980), 249–270.

[2] T. KAWATA, Fourier analysis of nonstationary stochastic processes, Trans. Amer. Math. Soc., **118** (1965), 276–302.

[3] T. KAWATA, *On Generalized Harmonic Analysis in* $\boldsymbol{R}^n$, Keio Univ. Res. Rep., **5** (1986).

[4] S. KOIZUMI, On the singular integrals V, Proc. Japan Acad., **35** (1959), 1–6.

[5] S. KOIZUMI, On the singular integrals VI, Proc. Japan Acad., **35** (1959), 323–328.

[6] S. KOIZUMI, On the Hilbert transform I, J. Fac. Sci. Hokkaido Univ. Ser. I, **14** (1959), 153–224.

[7] S. KOIZUMI, Comments on [26a,b], [28a], 181–182 in Nobert Wiener: *Collected Works II*, ed. by P. Masani, MIT Press, 1979.

[8] S. KOIZUMI, Commentary on the Memoire [30a] on generalized harmonic analysis, 325–332 in Nobert Wiener: *Collected Works II*, ed. by P. Masani, MIT Press, 1979.

[9] P. MASANI, Wiener's contributions to generalized harmonic analysis, prediction theory and filter theory, Bull. Amer. Math. Soc., **72** (1966), 73–125.

[10] P. MASANI, Commentary on the Memoire [30a] on generalized harmonic analysis, 333–379 in Nobert Wiener: *Collected Works II*, ed. by P. Masani, MIT Press, 1979.

[11] P. MASANI, *Nobert Wiener*, Birkhäuser, 1990.
[12] K. MATSUOKA, Generalized harmonic analysis of functions of two variables, Keio Engrg. Rep., **33** (1980), 67–81.
[13] K. MATSUOKA, On the generalized Hilbert transforms of functions of two variables, Keio Engrg. Rep., **33** (1980), 97–116.
[14] K. MATSUOKA, On the mean total power of the generalized Hilbert transform, Annual Rep. Takamatsu Nat. Coll. Tech., **21** (1985), 137–139.
[15] K. MATSUOKA, On the mean total power of the generalized Hilbert transforms in $\boldsymbol{R}^2$, Mem. Keio Shiki High School, **18** (1988), 72–78.
[16] K. MATSUOKA, On the generalized Hilbert transforms of functions of two variables II, Keio Sci. Tech. Rep., **43** (1990), 21–27.
[17] N. WIENER, Generalized harmonic analysis, Acta Math., **55** (1930), 117–258.
[18] N. WIENER, *The Fourier Integral and Certain of its Applications*, Cambridge Univ. Press, 1933.

Present Address:
HONCHO, SHIKISHI, SAITAMA-KEN 353, JAPAN