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A Note on the Rational Approximations to e

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Introduction.

We have some consequences on the rational approximations to e. P. Bundschuh [2] proved the following inequality.

BUNDSCHUH'S THEOREM. *For all integers p, q such that $q>0$,*

$$\left|e-\frac{p}{q}\right|>\frac{\log\log 4q}{18q^2\log 4q}.$$

On the other hand, C. S. Davis [3] proved the following theorem.

DAVIS' THEOREM. *For any $\varepsilon>0$ there is an infinity of solutions of the inequality*

$$\left|e-\frac{p}{q}\right|<\left(\frac{1}{2}+\varepsilon\right)\frac{\log\log q}{q^2\log q}$$

in integers p, q. Further, there exists a number $q'=q'(\varepsilon)$ such that

$$\left|e-\frac{p}{q}\right|>\left(\frac{1}{2}-\varepsilon\right)\frac{\log\log q}{q^2\log q}$$

for all integers p, q with $q\geqq q'$.

The last inequality suggests the possibility of replacing the constant $1/18$ in Bundschuh's theorem by a larger one; which will be done in this note.

THEOREM. *Let p, q be positive integers such that $q\geqq 2$. Then*

$$\left|e-\frac{p}{q}\right|>\frac{\log\log q}{3q^2\log q}.$$

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§1. Lemma.

LEMMA. *Let p, q be positive integers. Let p_n/q_n be the n-th convergent of e. Then*

$$\left|e-\frac{p}{q}\right|>\frac{\log\log q}{\gamma_N q^2\log q}$$

for all integers p, q such that $q\geqq q_{3N+1}$ ($N\geqq 6$), where γ_N is any constant such that

$$\gamma_N>\left(2+\frac{3}{N+1/2}\right)\left(1+\frac{\log\log((4N+7)/e)}{\log(N+7/4)}\right).$$

PROOF. If p/q is not a convergent of e, then

$$\left|e-\frac{p}{q}\right|>\frac{1}{2q^2}.$$

Therefore, the lemma is proved in this case. We must consider the case that p/q is a convergent of e. The continued fraction of e is

$$e=[a_0, a_1, a_2, a_3, \cdots]=[2, 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, \cdots].$$

In other words, $a_0=2$, and for $m\geqq 1$,

$$a_{3m}=a_{3m-2}=1 \quad\text{and}\quad a_{3m-1}=2m.$$

Case 1. Let $n=3m$ ($m>N$). Since $q_{3m+1}=a_{3m+1}q_{3m}+q_{3m-1}=q_{3m}+q_{3m-1}$, we have

$$\left|e-\frac{p_{3m}}{q_{3m}}\right|>\frac{1}{q_{3m}(q_{3m+1}+q_{3m})}>\frac{1}{3q_{3m}^2}.$$

As we can see that $\log\log x/\log x$ ($x\geqq 16$) is a strictly decreasing function, we have

$$\frac{\log\log q_{3m}}{\log q_{3m}}\leqq\frac{\log\log q_{18}}{\log q_{18}}=0.1768\cdots<1/3,$$

therefore

$$\left|e-\frac{p_{3m}}{q_{3m}}\right|>\frac{\log\log q_{3m}}{q_{3m}^2\log q_{3m}}.$$

Case 2. Let $n=3m+1$ ($m\geqq N$). Since $q_{3m+2}=a_{3m+2}q_{3m+1}+q_{3m}=2(m+1)q_{3m+1}+q_{3m}$, we have

$$\left|e-\frac{p_{3m+1}}{q_{3m+1}}\right|>\frac{1}{q_{3m+1}(q_{3m+2}+q_{3m+1})}>\frac{1}{2(m+2)q_{3m+1}^2}.$$

Now we must estimate q_{3m+1}. Since $q_{3m+1}\geqq 2(2m+1)q_{3m-2}\geqq\cdots\geqq 2^m(2m+1)(2m-1)\cdots$

5·3·1, we have

$$\log q_{3m+1} \geqq m\log 2 + \sum_{k=1}^{m} \log(2k+1) \geqq m\log 2 + \int_0^m \log(2x+1)dx$$
$$= m\log 2 + (m+1/2)\log(2m+1) - m \geqq (m+1/2)\log((4m+2)/e)\,.$$

Conversely,

$$q_{3m+1} \leqq (4m+3)q_{3m-2}\,.$$

Hence,

$$q_{3m+1} \leqq \prod_{\mu=1}^{m} (4\mu+3)\,.$$

Therefore,

$$\log q_{3m+1} \leqq \sum_{\mu=1}^{m} \log(4\mu+3) \leqq \int_1^{m+1} \log(4x+3)dx$$
$$= (m+7/4)\log(4m+7) - m - 7\log 7/4$$
$$\leqq (m+7/4)\log((4m+7)/e)\,,$$
$$\log\log q_{3m+1} \leqq \log(m+7/4) + \log\log((4m+7)/e)\,.$$

As we can see that

$$l(x) = \frac{\log\log((4x+7)/e)}{\log(x+7/4)} \qquad (x \geqq 6)$$

is a strictly decreasing function, we have

$$\log\log q_{3m+1} \leqq (1+l(N))\log(m+7/4) \leqq (1+l(N))\log((4m+2)/e)\,.$$

From these consequences, we find

$$\frac{\log\log q_{3m+1}}{\log q_{3m+1}} \leqq \frac{1+l(N)}{m+1/2}$$
$$\leqq (2+3/(N+1/2))\left(1+\frac{\log\log((4N+7)/e)}{\log(N+7/4)}\right)\frac{1}{2(m+2)}$$
$$< \frac{\gamma_N}{2(m+2)}\,.$$

Therefore,

$$\left| e - \frac{p_{3m+1}}{q_{3m+1}} \right| > \frac{\log\log q_{3m+1}}{\gamma_N q_{3m+1}^2 \log q_{3m+1}}\,.$$

Case 3. Let $n=3m+2$ $(m\geqq N)$. We can prove the lemma in this case similarly in the case 1.

This completes the proof.

§2. Proof of the theorem.

It suffices only to consider that p/q is a $(3m+1)$-th convergent of e. If $N=22$, then

$$(2+3/(N+1/2))\left(1+\frac{\log\log((4N+7)/e)}{\log(N+7/4)}\right)=2.9873\cdots.$$

Hence we can choose γ_{22} so that $\gamma_{22}=3$. From Lemma, for all positive integers p, q such that $q\geqq q_{3m+1}$ $(m\geqq 22)$,

$$\left|e-\frac{p}{q}\right|>\frac{\log\log q}{3q^2\log q}.$$

We deffine δ_m as follows:

$$\left|e-\frac{p_{3m+1}}{q_{3m+1}}\right|>\frac{1}{2(m+2)q_{3m+1}^2}=\frac{\log\log q_{3m+1}}{\delta_m q_{3m+1}^2\log q_{3m+1}},$$

i.e.

$$\delta_m=\frac{2(m+2)\log\log q_{3m+1}}{\log q_{3m+1}}.$$

We show that $\delta_m\leqq 3$ for $m\leqq 21$.

$\delta_1=2.0527\cdots$, $\delta_2=2.7211\cdots$, $\delta_3=2.7975\cdots$, $\delta_4=2.7942\cdots$, $\delta_5=2.7757\cdots$,
$\delta_6=2.7553\cdots$, $\delta_7=2.7364\cdots$, $\delta_8=2.7195\cdots$, $\delta_9=2.7047\cdots$, $\delta_{10}=2.6916\cdots$,
$\delta_{11}=2.6801\cdots$, $\delta_{12}=2.6699\cdots$, $\delta_{13}=2.6607\cdots$, $\delta_{14}=2.6525\cdots$, $\delta_{15}=2.6451\cdots$,
$\delta_{16}=2.6384\cdots$, $\delta_{17}=2.6322\cdots$, $\delta_{18}=2.6266\cdots$, $\delta_{19}=2.6214\cdots$, $\delta_{20}=2.6166\cdots$,
$\delta_{21}=2.6121\cdots$:

This completes the proof of the theorem.

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