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Unknotting Operations of Polygonal Type

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1. Introduction.

In this paper, we consider about knots and links in the 3-sphere S^3.

A 3-gon move is a local move on a link diagram as indicated in Figure 1.1. In [3], Y. Nakanishi showed that a Δ-unknotting operation can be realized by a finite sequence of 3-gon moves. A Δ-unknotting operation is a local move on a diagram as shown in Figure 1.2, and it is a kind of unknotting operation ([2]). Hence a 3-gon move is a kind of unknotting operation. We generalize the notion of 3-gon moves to n-gon moves as shown in Figure 1.3. In section 2, we obtain some results about n-gon moves which are similar to those about H(n)-moves in [1]. For example, we will show that for given any knot K, there exists an integer n such that K can be transformed into a trivial knot by one n-gon move.

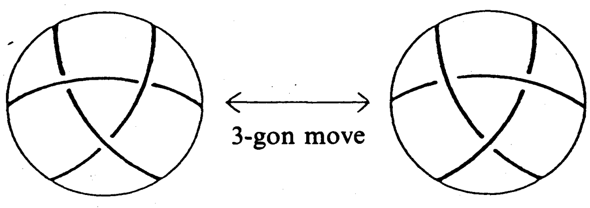

FIGURE 1.1.

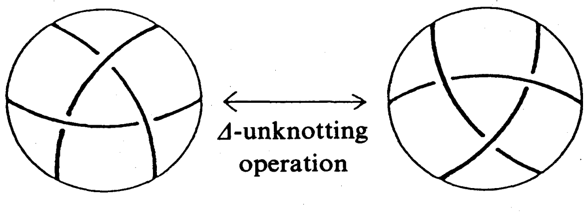

FIGURE 1.2.

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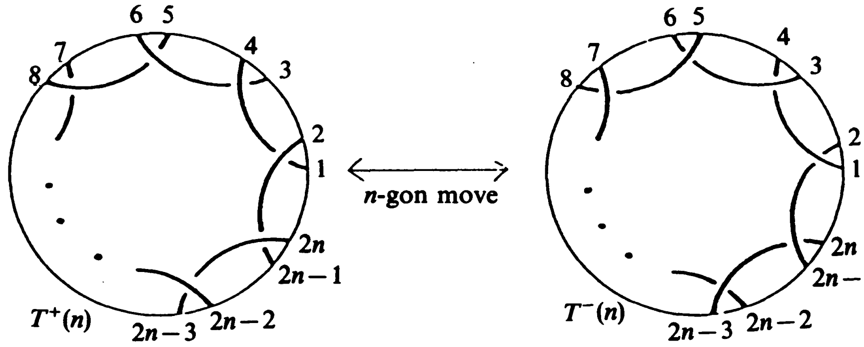


FIGURE 1.3.

In section 3, we will apply n-gon moves to links and will consider the number of the equivalence classes generated by n-gon moves.

2. n-gon moves on knots.

DEFINITION. For any integer n (≥ 3), an *n-gon move on a link diagram* is a local move on the diagram as indicated in Figure 1.3. And more precisely, under the notations $T^+(n)$, $T^-(n)$ in Figure 1.3, an *n^+-gon move on a diagram* is a local move that replace a tangle $T^+(n)$ with a tangle $T^-(n)$ and an *n^--gon move on a diagram* is the inverse move of an n^+-gon move.

REMARK. In fact, for any integer n (≥ 3), n^+-gon and n^--gon moves are equivalent moves, i.e. each move can be realized by a finite sequence of the other moves.

DEFINITION. For any integer n (≥ 3), an *n-gon move on a link* is a deformation of the link which corresponds to an n-gon move on a diagram of the link. Similarly, *n^+-gon and n^--gon moves on a link* are defined.

PROPOSITION 2.1. *For any integer n (≥ 3), every knot can be transformed into a trivial knot by a finite sequence of n-gon moves, i.e. an n-gon move is a kind of unknotting operation.*

PROOF. This follows from the following lemma:

LEMMA 1. (1) *Any knot can be transformed into a trivial knot by a finite sequence of 3-gon moves.*

(2) *For any integer n (≥ 3), an n-gon move can be realized by an $(n+1)$-gon move.*

PROOF. (1) In [3], Y. Nakanishi showed that a Δ-unknotting operation can be realized by a finite sequence of 3-gon moves. So the proof is completed.

(2) Figure 2.1 illustrates how this can be accomplished. □

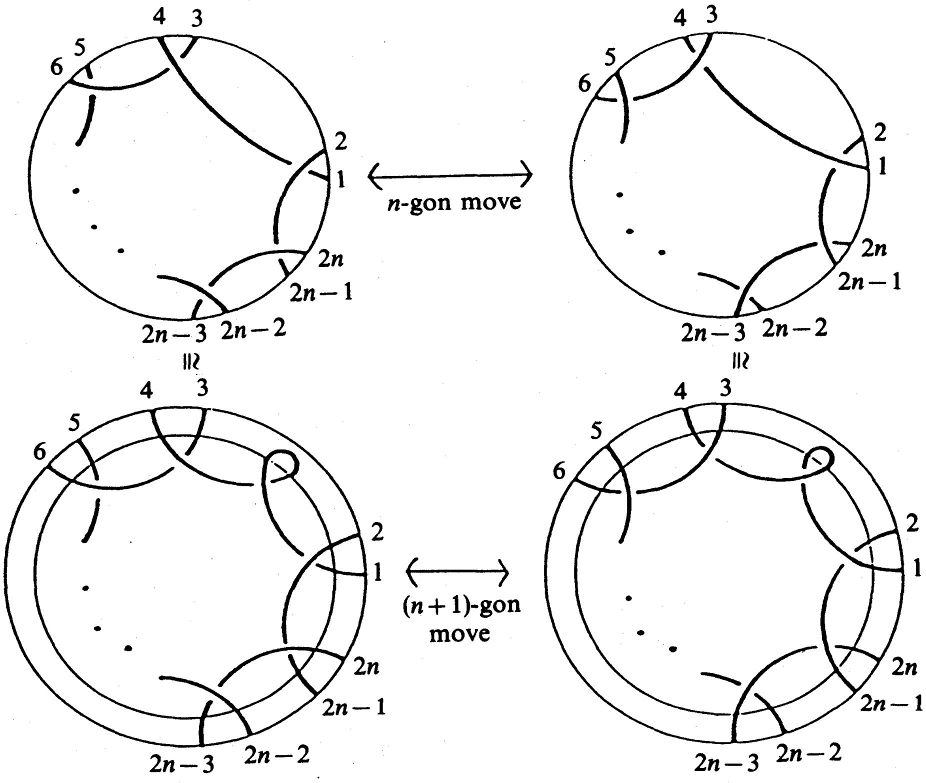


FIGURE 2.1.

PROPOSITION 2.2. *For any integer n (≥3), every knot can be transformed into a trivial knot by a finite sequence of n^+-gon moves, i.e. an n^+-gon move is a kind of unknotting operation.*

PROOF. This follows from the following lemma:

LEMMA 2. (1) *Any knot can be transformed into a trivial knot by a finite sequence of 3^+-gon moves.*

(2) *For any integer n (≥3), an n^+-gon move can be realized by an $(n+1)^+$-gon move.*

PROOF. (1) Figure 2.2 shows that a Δ-unknotting operation can be realized by a finite sequence of 3^+-gon moves.

(2) Figure 2.1 illustrates how this can be done. □

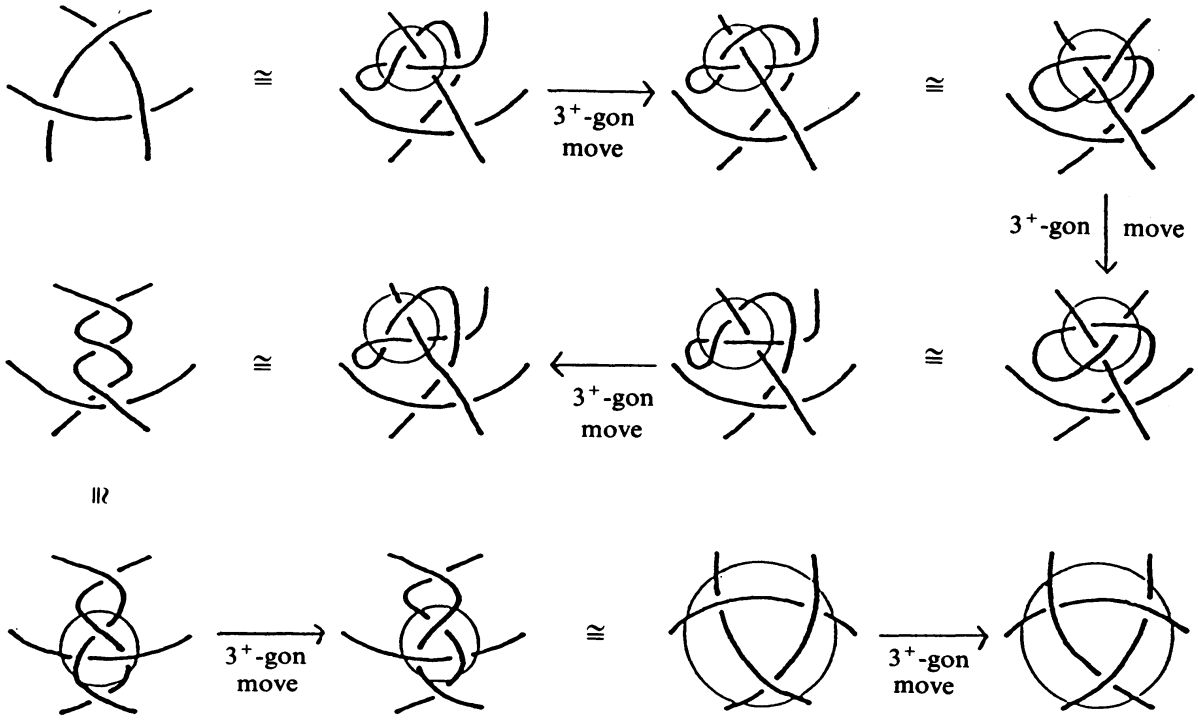


FIGURE 2.2.

REMARK. Similarly, we can also prove that an n^--gon move is a kind of unknotting operation.

DEFINITION. For a knot K and an integer n (≥ 3), $u_n(K)$ is defined to be the minimum number of n-gon moves which are necessary to transform K into a trivial knot.

For a knot K and an integer r (≥ 2), let $mg(K, r)$ denote the minimum number of generators of the first integral homology group of the r-fold cyclic branched covering space of S^3 branched along K.

PROPOSITION 2.3. *For a knot K and integers n (≥ 3) and r (≥ 2), we have*

$$u_n(K) \geq mg(K, r)/(n-1)(r-1)\,.$$

PROOF. By replacing an H(n)-move by an n-gon move in the proof of Theorem 4 in [1], we can obtain Proposition 2.3. □

PROPOSITION 2.4. *For any knot K and any integer n (≥ 3), we have*

$$u_n(K) \geq u_{n+1}(K)\,.$$

PROOF. This is obvious from Lemma 1 (2). □

THEOREM 2.5. *For any knot K, there exists an integer n such that $u_n(K) = 1$.*

PROOF. First of all, we show the following lemma:

LEMMA 3. *There exists a diagram k of the knot K satisfying the following condition*:

(*) *k includes disjoint tangles* $T_1^+(3)$, $T_2^+(3)$, $\cdots$, $T_l^+(3)$ *for some l, so that, k is transformed into a diagram of a trivial knot by* 3^+*-gon moves on* $T_1^+(3)$, $T_2^+(3)$, $\cdots$, $T_l^+(3)$.

PROOF OF LEMMA 3. By Proposition 2.2, K can be transformed into a trivial knot by a finite sequence of 3^+-gon moves. For each 3^+-gon move in this sequence, attach a disk to K as in Figure 2.3 such that these attaching disks are disjoint from each other. Thus K is deformed into a trivial knot K_0 with attaching disks. Let l be the number of these disks. Moreover, by contracting these disks, we may draw a diagram k_0 of K_0 with attaching disks, as follows:

k_0 contains disjoint 3-tangles TS_1, TS_2, $\cdots$, TS_l with attaching disks as in Figure 2.4, and the remainder part of k_0 is a diagram of disjoint arcs in S^3 (for example, see Figure 2.5).

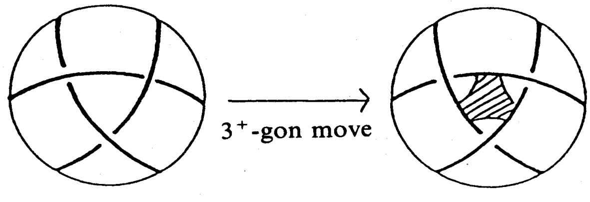
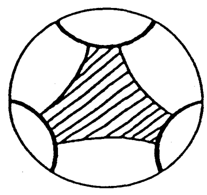


FIGURE 2.3.



TS_j

FIGURE 2.4.

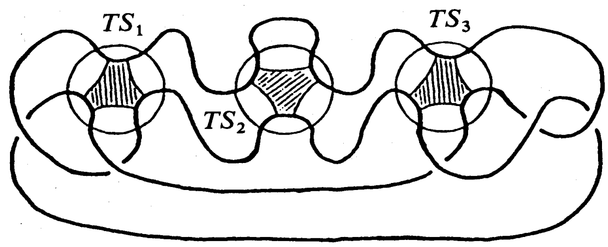


FIGURE 2.5.

Finally we construct a diagram of K in the following way. For each tangle TS_j ($j=1, 2, \cdots, l$) in k_0, we remove the attaching disk and apply a 3^--gon move as shown in Figure 2.6. Denote $T_j^+(3)$ be the resulting "small" 3-tangle in Figure 2.6. Then we obtain a diagram of K which satisfies the condition (*). □

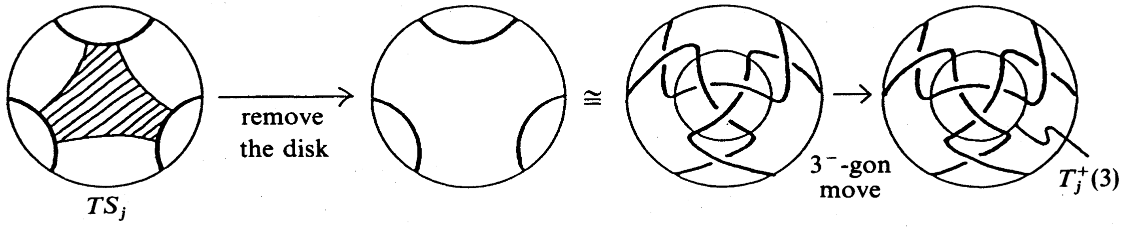


FIGURE 2.6.

Let k be a diagram of K satisfying (*). More generally, we consider the following condition on a knot diagram D:

$(**)_l$ D includes l disjoint tangles $T_1^+(n_1)$, $T_2^+(n_2)$, $\cdots$, $T_l^+(n_l)$ for some l (where n_1, n_2, $\cdots$, $n_l \geq 3$), so that, by an n_i^+-gon move on $T_i^+(n_i)$ for all i $(1 \leq i \leq l)$, D is transformed into a diagram of a trivial knot.

Since k satisfies $(**)_l$, so, by the induction on l, we can show Theorem 2.5 using the following lemma:

LEMMA 4. *Assume $l \geq 2$. Let k be a diagram of K satisfying $(**)_l$. Then there exists a diagram of K satisfying $(**)_{l-1}$.*

PROOF OF LEMMA 4. Let G be the diagram which is obtained from k by collapsing the tangles $T_1^+(n_1)$, $T_2^+(n_2)$, $\cdots$, $T_l^+(n_l)$ to vertices v_1, v_2, $\cdots$, v_l respectively. Let v_i, v_j be vertices which are connected by an edge e in G.

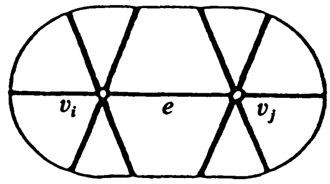


FIGURE 2.7.

For simplicity, we consider the case $n_i = n_j = 3$. We can assume that there is a neighborhood U of v_i, v_j and e in S^2 such that $U \cap G$ looks like Figure 2.7. If it is not, move v_j along e by an ambient isotopy of S^3, and then we can obtain a diagram as indicated in Figure 2.7. Therefore, we can assume that the corresponding part of k looks like (a) or (b) in Figure 2.8. In the case (b), deform the diagram and choose the other tangle $T'^+(3)$ as shown in Figure 2.9. Then 3^+-gon moves on $T_i^+(3)$ and $T_j^+(3)$ are equivalent to that on $T'^+(3)$. So we can reduce to the case Figure 2.8 (a).

(a) (b)

FIGURE 2.8.

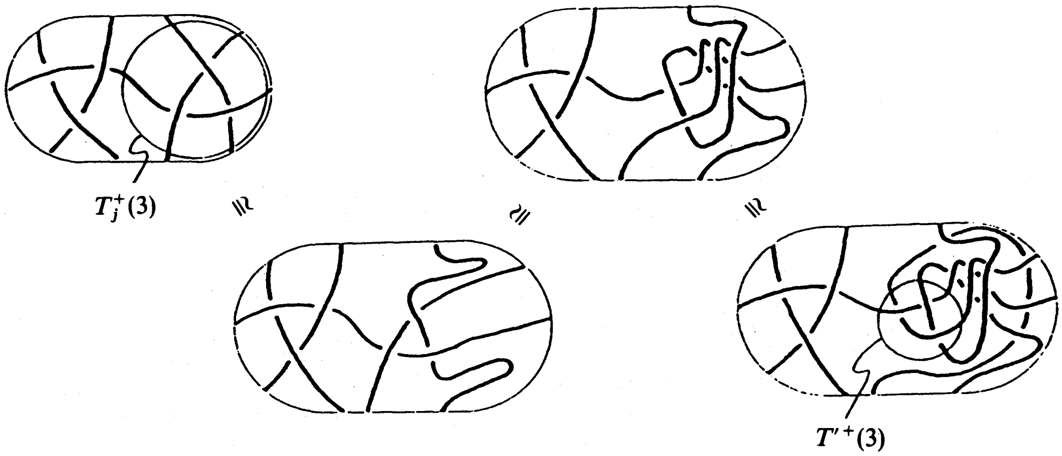


FIGURE 2.9.

Next deform the diagram $U \cap k$ and choose a tangle $T^+(3+3+2)=T^+(8)$ as shown in Figure 2.10. Then the 3^+-gon moves on $T_i^+(3)$ and $T_j^+(3)$ in k can be realized by an 8^+-gon move on $T^+(8)$ in the new diagram. Hence the proof is completed in the case of $n_i=n_j=3$.

In the general case, by the same method, we can show that:

Two polygonal moves in k (that is, the n_i^+-gon move on $T_i^+(n_i)$ and the n_j^+-gon move on $T_j^+(n_j)$) can be realized by a single (n_i+n_j+2)-gon move on $T^+(n_i+n_j+2)$ in another diagram of K. □

COROLLARY 2.6. *For any knot K, we have*

$$\lim_{n\to\infty} u_n(K)=1 .$$

PROOF. This follows from Proposition 2.4 and Theorem 2.5. □

DEFINITION. For a knot K, $p(K)$ is defined to be the minimum integer n satisfying $u_n(K)=1$.

PROPOSITION 2.7. *For a knot K and integers n* (≥ 3) *and r* (≥ 2), *we have*

$$p(K) \geq mg(K, r)/(r-1)+1 .$$

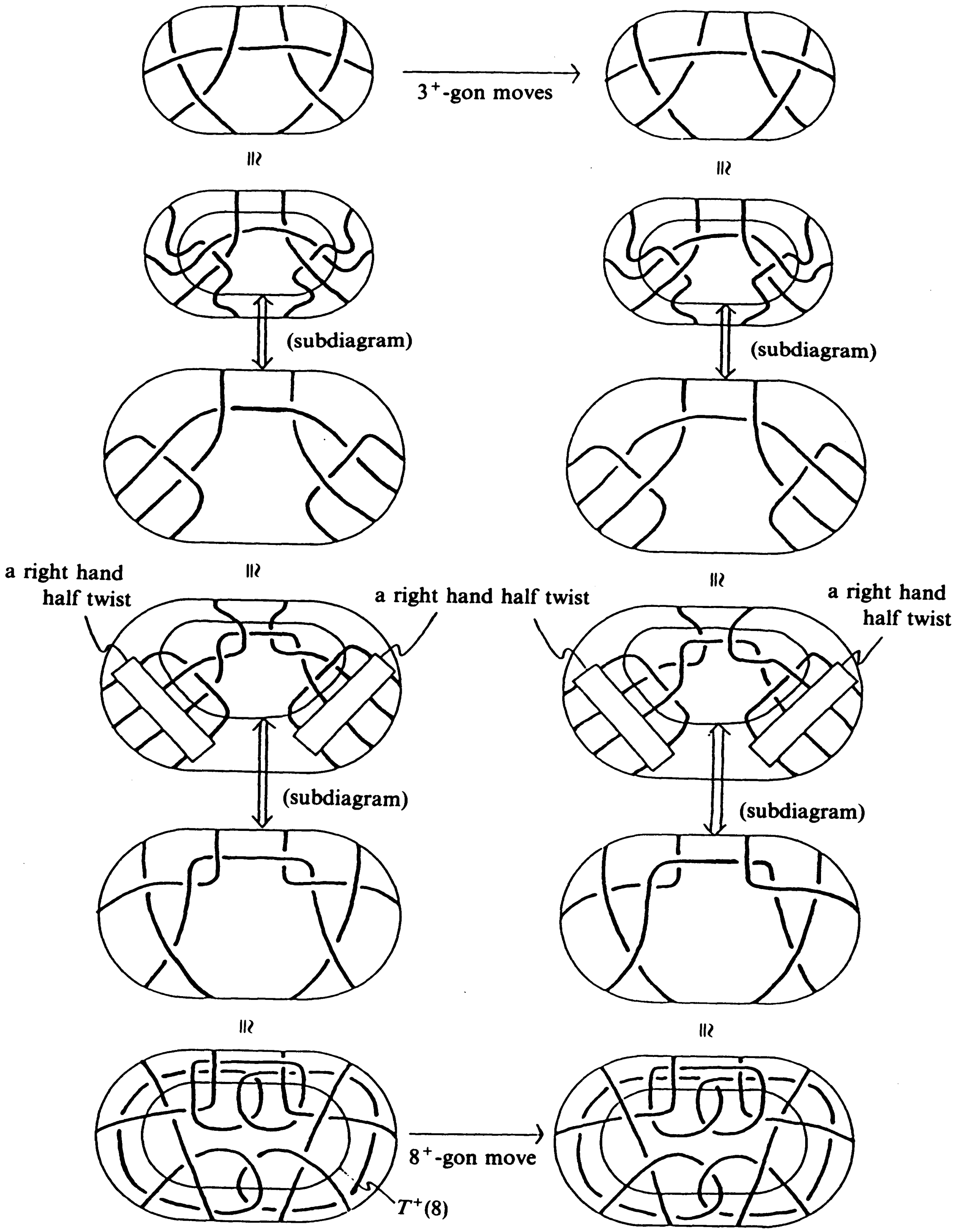
PROOF. This follows by letting $n=p(K)$ in Proposition 2.3. □

EXAMPLE (Y. Nakanishi). Let n be an integer which satisfies $n \geq 4$ and $n \not\equiv 0$ (mod 3). Then the knot K_n as indicated in Figure 2.11 has $p(K_n)=n$ (where $n=3k\pm 1$).

This follows from Proposition 2.7 using $mg(K_n, 2)=n-1$.

3. An equivalence relation of links.

Let L^μ be the set of all μ-component links. We define the equivalence relation $\underset{n}{\sim}$



3⁺-gon moves
(subdiagram)
(subdiagram)
a right hand half twist
a right hand half twist
a right hand half twist
(subdiagram)
(subdiagram)
8⁺-gon move
T⁺(8)

FIGURE 2.10.

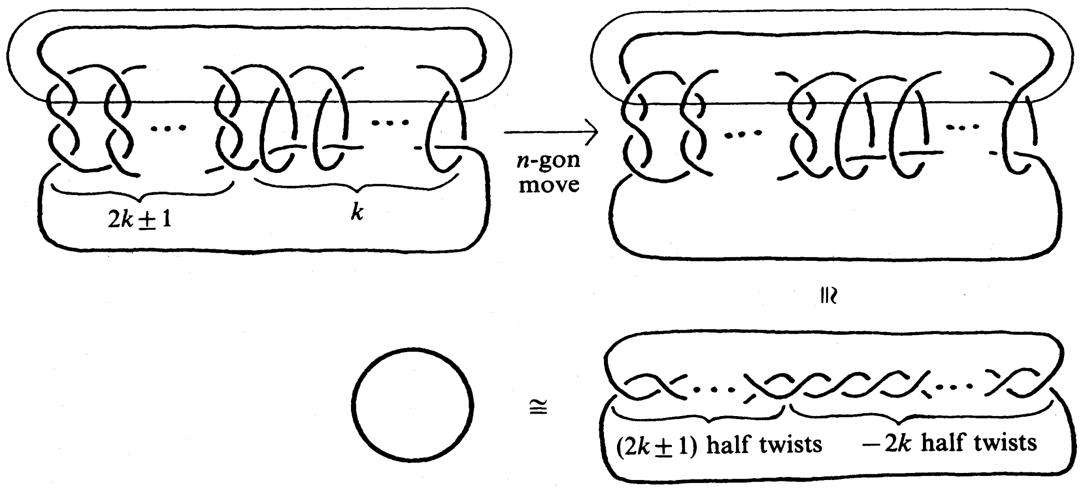


FIGURE 2.11.

for links $L,\ L' \in L^{\mu}$ as follows:

$$L \underset{n}{\sim} L' \underset{\text{def}}{\Longleftrightarrow} L \text{ is obtained from } L' \text{ by a finite sequence of } n\text{-gon moves}.$$

THEOREM 3.1. *Let* $L = K_1 \cup K_2 \cup \cdots \cup K_{\mu}$ *and* $L' = K'_1 \cup K'_2 \cup \cdots \cup K'_{\mu}$ *be two μ-component links. For any integer n (≥ 3), $L \underset{n}{\sim} L'$ if and only if, after suitably oriented and/or ordered if necessary, L and L' satisfy the following condition with respect to their linking numbers*:

$$\sum_{\substack{j=1\\ j\neq i}}^{\mu} lk(K_i, K_j) \equiv \sum_{\substack{j=1\\ j\neq i}}^{\mu} lk(K'_i, K'_j) \pmod 2 \qquad \textit{for all } i = 1, 2, \cdots, \mu .$$

PROOF. At first, we prove the following lemma:

LEMMA 5. *For any integer n (≥ 4), an n-gon move can be realized by a finite sequence of $(n-1)$-gon moves.*

PROOF. This follows from Lemma 1 (2) and Figure 3.1. □

By Lemma 1 (2) and 5, for any integers m and n (≥ 3), if $L \underset{m}{\sim} L'$ then $L \underset{n}{\sim} L'$. On the other hand, in [3], it was shown the case of $n = 3$. So the proof is completed. □

$L^{\mu}/\underset{n}{\sim}$ denotes the set of equivalence classes of L^{μ} by $\underset{n}{\sim}$.

COROLLARY 3.2. *For integers μ (≥ 1) and n (≥ 3), we have*

$$\#(L^{\mu}/\underset{n}{\sim}) = 2^{\mu-1},$$

where $\#A$ denotes the number of the elements contained in A.

PROOF. By Lemma 1 (2) and 5, the number of the equivalence classes generated by n-gon moves is independent of n. On the other hand, in [3], it was shown that

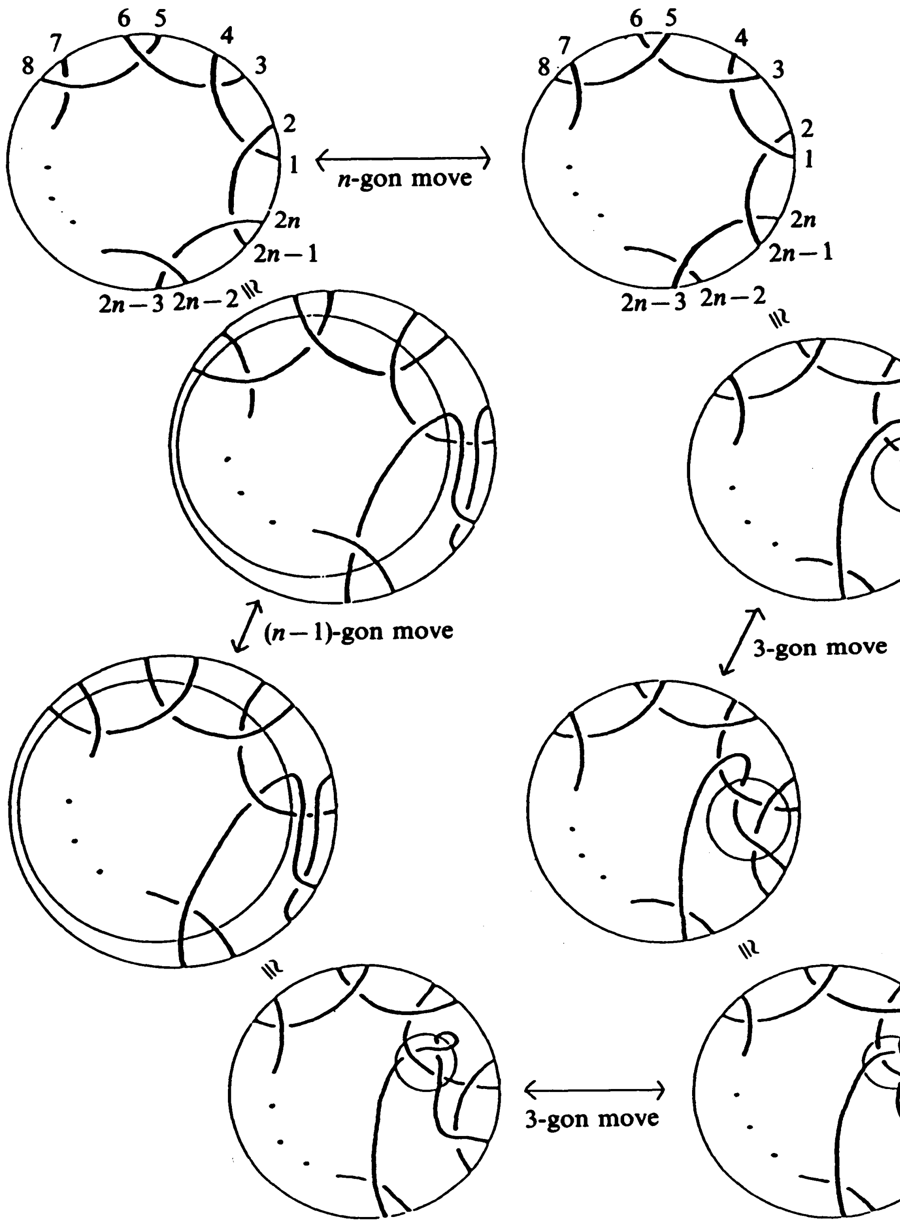


FIGURE 3.1.

$\#(L^{\mu}/\underset{3}{\sim})=2^{\mu-1}$. □

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