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Inclusions of Type III Factors Arising from Finite Group Actions

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Abstract. We consider inclusions of type III factors arising from finite group actions. We show the relation between actions and the corresponding subfactors, and compute conjugacy invariants for the subfactors.

§0. Introduction.

In the previous paper [18], we considered the inclusions of type III factors:

$$M=P\otimes M_2(C)\supset N=\left\{\begin{pmatrix} x & \\ & \alpha(x)\end{pmatrix}\,\middle|\, x\in P\right\},$$

where P is a type III factor and α is an automorphism of P. We showed "equivalence" between classification of automorphisms up to outer conjugacy and that of the corresponding subfactors up to conjugacy, namely, we gave the relationship between Z-actions and the subfactors. We also computed conjugacy invariants for the subfactors. In the present paper, we shall consider inclusions of type III factors arising from finite group actions, more precisely, we consider the following:

$$M=P\otimes M_n(C)\supset N=\left\{\sum_{g\in G}\alpha_g(x)e_{gg}\,\middle|\, x\in P\right\},$$

where α is an action of a finite group G with order n on a type III factor P. Our main purposes are to give the relation between actions and the corresponding subfactors, and to compute conjugacy invariants for the subfactors: the tower of the relative commutants, the mirroring in the sense of Ocneanu [14] and the dual action arising from the associated type II_∞ inclusion. We also remark the known facts on the crossed product case in order to compare them.

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§1. Preliminaries.

1.1. Invariants for actions. We recall the cocycle conjugacy invariant for actions on a type III factor from Sutherland-Takesaki [19].

Let $\alpha: G \to \operatorname{Aut} M$ be an action of a discrete group G on a type III factor M. Then the cocycle conjugacy invariant of α is given by $(N(\alpha), \mathrm{mod}, \chi, \nu)$. Each of them is defined as follows: Let φ be a dominant weight on M. Remark that if we denote the continuous decomposition of M by $M = M_\varphi \rtimes_\theta \boldsymbol{R}$, then the flow of weights of M is given by $(Z(M_\varphi), \{\theta_t\}_{t \in \boldsymbol{R}})$ (see Connes-Takesaki [2]). For each automorphism α of M, the module $\mathrm{mod}\,\alpha$ in $\operatorname{Aut} Z(M_\varphi)$ which commutes with θ is given by

$$\mathrm{mod}\,\alpha = \operatorname{Ad} u \circ \alpha|_{Z(M_\varphi)},$$

where u is a unitary in M satisfying $\varphi \circ \alpha^{-1} = \varphi \circ \operatorname{Ad} u$. If we set

$$N(\alpha) = \alpha^{-1}(\{\operatorname{Ad} u \circ \bar{\sigma}_c^\varphi \mid u \in U(M),\ c \in Z_\theta^1(\boldsymbol{R}, U(Z(M_\varphi)))\}),$$

it is a normal subgroup of G. Denoting $\alpha_h = \operatorname{Ad} u_h \circ \bar{\sigma}_{c(h)}^\varphi$, $h \in N(\alpha)$, the characteristic invariant $\chi = [\lambda, \mu]$ in $\Lambda(G, N(\alpha), U(Z(M_\varphi)))$ is defined by the relations

$$u_h \bar{\sigma}_{c(h)}^\varphi(u_k) = u_{hk} \mu(h, k), \qquad h, k \in N(\alpha),$$

$$\alpha_g(u_{g^{-1}hg})(D\varphi \circ \alpha_g^{-1} : D\varphi)_{(\mathrm{mod}\,\alpha_g)(c(g^{-1}hg))} = u_h \lambda(g, h), \qquad g \in G,\ h \in N(\alpha),$$

and the modular invariant ν which is a homomorphism from $N(\alpha)$ into the first cohomology group of the flow of weights is defined by

$$\nu(h) = [c(h)], \qquad h \in N(\alpha).$$

Furthermore, these invariants satisfy the following relations:

$$c(h)c(k) = (\partial \mu(h, k))c(hk), \tag{1.1.1}$$

$$(\mathrm{mod}\,\alpha_g)(c(g^{-1}hg)) = (\partial \lambda(g, h))c(h). \tag{1.1.2}$$

Here, for a unitary u in $Z(M_\varphi)$, ∂u means the coboundary defined by

$$(\partial u)(s) = u^* \theta_s(u), \qquad s \in \boldsymbol{R}.$$

These invariants are independent of the choice of φ and depend only on the cocycle conjugacy class of α. For details, see [19].

1.2. Type III index theory. Let $M \supset N$ be a pair of type III factors and $E: M \to N$ a faithful normal conditional expectation with finite minimal index. Choosing a faithful state φ on N, we set $\psi = \varphi \circ E \in M_*^+$. Then we get the following inclusion of type II_∞ von Neumann algebras:

$$\tilde{M} = M \rtimes_{\sigma^\psi} \boldsymbol{R} \supset \tilde{N} = N \rtimes_{\sigma^\varphi} \boldsymbol{R}.$$

Moreover, there exists a faithful normal conditional expectation $\tilde{E}$ from $\tilde{M}$ onto $\tilde{N}$ such that $\tilde{E}|_M = E$ and

$$\tilde{E} \circ \theta_t = \theta_t \circ \tilde{E}, \qquad t \in \boldsymbol{R}.$$

Here θ means the dual action of σ^{ψ} on $\tilde{M}$. Thanks to Connes' result on Radon-Nikodym cocycle, the conjugacy class of $\tilde{M} \supset \tilde{N}$ with $\tilde{E}$ is independent of the choice of φ. Therefore we canonically get the two towers consisting of type III factors

$$N \subset M \subset M_1 \subset M_2 \subset \cdots$$

and type II_∞ von Neumann algebras

$$\tilde{N} \subset \tilde{M} \subset \tilde{M}_1 \subset \tilde{M}_2 \subset \cdots$$

by iterating the basic extension. We remark that $\tilde{M}_k$ is the crossed product of M_k by the modular action because the basic extension is compatible with taking the crossed product with respect to the modular action. For details, see [4], [5], [7], [9], [10], [11] and [13].

1.3. Construction of an inclusion of type III factors. We recall the way to construct an inclusion of factors from Popa [15] (see also Popa [16]).

Let $\alpha: G \to \mathrm{Aut}P$ be an action of a finite group G on a type III factor P. We denote by $\{e_{gh}\}_{g,h \in G}$ the usual matrix units parametrized by G in $M_n(\boldsymbol{C})$, where n is the order of G. Then we define a factor M and a subfactor N by

$$M = P \otimes M_n(\boldsymbol{C}),$$

$$N = \left\{ \sum_{g \in G} \alpha_g(x) e_{gg} \,\middle|\, x \in P \right\},$$

and define a faithful normal conditional expectation E by

$$E\left(\sum_{g,h \in G} x_{gh} e_{gh} \right) = \sum_{g \in G} \alpha_g(x) e_{gg}, \qquad \text{where} \quad x = \frac{1}{n} \sum_{g \in G} \alpha_g{}^{-1}(x_{gg}).$$

Since the following lemma follows from the standard argument, we leave its proof to the reader (see [7] and [10]).

LEMMA 1.3. *With the above notations, we have*

(i) *M and N are isomorphic to P,*

(ii) $\mathrm{Index}\, E = n^2$ *and E is the minimal conditional expectation for $M \supset N$.*

§2. Relation between actions and subfactors.

Let P be a type III factor and let α, β be actions of a finite group G ($|G| = n$) on P. We denote by N_α, N_β the corresponding subfactors as in 1.3 respectively.

PROPOSITION 2.1. *The following conditions are equivalent*:

(i) $M \supset N_\alpha$ *is conjugate to* $M \supset N_\beta$,

(ii) *There exist an automorphism* θ *of* P *and a unit preserving bijection* φ *of* G *such that*

$$\theta \circ \alpha_g \circ \theta^{-1} = \beta_{\varphi(g)} \quad in \quad \operatorname{Out} P = \operatorname{Aut} P / \operatorname{Int} P .$$

PROOF. (i)→(ii): Suppose that there exists an automorphism Φ of M such that $\Phi(N_\alpha) = N_\beta$. Then there exists an automorphism θ of P such that

$$\Phi\Big(\sum_{g \in G} \alpha_g(x) e_{gg}\Big) = \sum_{g \in G} \beta_g(\theta(x)) e_{gg} , \qquad x \in P .$$

Since $\Phi(N'_\alpha \cap M) = N'_\beta \cap M$, after a suitable perturbation we may assume that there exists a bijection ψ of G such that

$$\Phi(e_{gg}) = e_{\psi(g)\psi(g)} , \qquad g \in G .$$

We compute, for any $x \in P$,

$$\begin{aligned}\Phi(\alpha_g(x) e_{gg}) &= \Phi\Big(e_{gg} \cdot \Big(\sum_{h \in G} \alpha_h(x) e_{hh} \Big) \cdot e_{gg} \Big) \\ &= e_{\psi(g)\psi(g)} \cdot \Big(\sum_{h \in G} \beta_h(\theta(x)) e_{hh} \Big) \cdot e_{\psi(g)\psi(g)} \\ &= \beta_{\psi(g)}(\theta(x)) e_{\psi(g)\psi(g)} .\end{aligned}$$

Since there exists a unitary u_g in P such that

$$\Phi(e_{ge}) = u_g e_{\psi(g)\psi(e)} ,$$

we also compute

$$\begin{aligned}\Phi(\alpha_g(x) e_{gg}) &= \Phi(e_{ge} \cdot \alpha_g(x) e_{ee} \cdot e_{eg}) \\ &= u_g e_{\psi(g)\psi(e)} \cdot \beta_{\psi(e)}(\theta(\alpha_g(x))) e_{\psi(e)\psi(e)} \cdot u_g{}^* e_{\psi(e)\psi(g)} \\ &= u_g \beta_{\psi(e)}(\theta(\alpha_g(x))) u_g{}^* e_{\psi(g)\psi(g)} ,\end{aligned}$$

where $e \in G$ is the unit of G. Hence we get

$$\beta_{\psi(g)}(\theta(x)) = u_g \beta_{\psi(e)}(\theta(\alpha_g(x))) u_g{}^* , \qquad x \in P .$$

This means that $\theta \circ \alpha_g \circ \theta^{-1} = \beta_{\psi(e)^{-1}\psi(g)}$ in $\operatorname{Out} P$. If we set $\varphi(g) = \psi(e)^{-1}\psi(g)$, $g \in G$, we obtain the conclusion.

(ii)→(i): It is sufficient to show the assertion in the following three cases.

(1) The case of $\beta_g = \theta \circ \alpha_g \circ \theta^{-1}$ for some automorphism θ of P.

If we set $\Phi=\theta\otimes\mathrm{id.}$, then we have

$$\Phi\left(\sum_{g\in G}\alpha_g(x)e_{gg}\right)=\sum_{g\in G}\theta(\alpha_g(x))e_{gg}$$

$$=\sum_{g\in G}\beta_g(\theta(x))e_{gg}\,,\qquad x\in P\,.$$

(2) The case of $\beta_g=\mathrm{Ad}\,u_g\circ\alpha_g$ for some unitary u_g in P, $g\in G$. If we set $\Phi=\mathrm{Ad}(\sum_{g\in G}u_ge_{gg})$, then

$$\Phi\left(\sum_{g\in G}\alpha_g(x)e_{gg}\right)=\sum_{g\in G}u_g\alpha_g(x)u_g{}^*e_{gg}$$

$$=\sum_{g\in G}\beta_g(x)e_{gg}\,,\qquad x\in P\,.$$

(3) The case of $\beta_g=\alpha_{\varphi(g)}$ for some bijection φ of G. If we put $\Phi=\mathrm{Ad}(\sum_{g\in G}e_{g\varphi(g)})$, then

$$\Phi\left(\sum_{g\in G}\alpha(x)e_{gg}\right)=\sum_{g\in G}\alpha_{\varphi(g)}(x)e_{gg}$$

$$=\sum_{g\in G}\beta_g(x)e_{gg}\,,\qquad x\in P\,.$$

q.e.d.

REMARK 2.2. Let us assume that the conditions in Proposition 2.1 are satisfied. If α is outer (hence so is β), then φ is an automorphism of G. In fact, since it follows that, for $g, h\in G$,

$$\beta_{\varphi(g)\varphi(h)}=\beta_{\varphi(gh)}\qquad\text{in}\quad\mathrm{Out}\,P\,,$$

we have $\varphi(g)\varphi(h)=\varphi(gh)$.

REMARK 2.3. For outer actions α, β of a finite group G on a type III factor P, the following three conditions are equivalent.

(i) $P\rtimes_\alpha G\supset P$ is conjugate to $P\rtimes_\beta G\supset P$,

(ii) $P\supset P^\alpha$ is conjugate to $P\supset P^\beta$, where P^α, P^β mean the fixed point algebras,

(iii) There exists an automorphism φ of G such that α is cocycle conjugate to $\varphi^*\beta$, where $\varphi^*\beta$ is the action defined by $(\varphi^*\beta)_g=\beta_{\varphi(g)}$, $g\in G$.

Indeed, the equivalence of (i) and (ii) follows from the fact that $P\rtimes_\alpha G\supset P$ is the basic extension of $P\supset P^\alpha$.

We assume that the condition (i) is satisfied, that is, there exists an isomorphism Φ from $P\rtimes_\alpha G$ onto $P\rtimes_\beta G$ such that $\Phi(P)=P$. Since it preserves the Jones projections, it can be extended to the isomorphism between the towers of the basic extension. In particular, it preseves the towers of the relative commutants. Since it intertwines the modular conjugation operators, it induces an automorphism φ of G.

The cocycle conjugacy of two actions follows from a direct calculation. The converse is trivial.

§3. Conjugacy invariants for subfactors.

For the subfactors defined in 1.3, we shall compute conjugacy invariants. Since the informations of the tower of the relative commutants and the mirrorings in the sense of Ocneanu [14] arising from type III inclusion are included in those arising from the associated type II_∞ inclusion and the dual action, it is sufficient to compute the latter (see Kosaki-Longo [12] and [18; Remark 2.8].)

Let $\alpha: G \to \mathrm{Aut}\, P$ be an action of a finite group G with order n on a type III factor P: Let $M \supset N$ be as in 1.3 and $\tilde{M} \supset \tilde{N}$ the canonical inclusion of type II_∞ von Neumann algebras with the dual action θ^M (see 1.2). We set, for a non-negative integer k, $P_k = M_n(\boldsymbol{C})$ with matrix units $\{e^k_{gh}\}_{g,h\in G}$ and put

$$\tilde{M}_k = \tilde{P} \otimes P_0 \otimes P_1 \otimes \cdots \otimes P_k\,, \qquad k \geq 0\,,$$
$$\tilde{M}_{-1} = \left\{ \sum_{g\in G} \tilde{\alpha}_g(x) e_{gg} \;\middle|\; x \in \tilde{P} \right\}.$$

Here $\tilde{P}$ means the crossed product $P \rtimes_{\sigma^\varphi} \boldsymbol{R}$ for some weight φ on P and $\tilde{\alpha}$ is the canonical extension of α in the sense of Haagerup-Størmer [3]. We denote the dual action on $\tilde{P}$ by θ^P. We embed $\tilde{M}_k$ into $\tilde{M}_{k+1}$ by

$$\sum x_{g_0h_0\cdots g_kh_k} e^0_{g_0h_0} \cdots e^k_{g_kh_k} \in \tilde{M}_k \longrightarrow$$
$$\sum \tilde{\alpha}_{g_{k+1}}(x_{g_0h_0\cdots g_kh_k}) e^0_{g_0h_0} \cdots e^k_{g_kh_k} e^{k+1}_{g_{k+1}g_{k+1}} \in \tilde{M}_{k+1}\,,$$

and define a faithful normal conditional expectation $\tilde{E}_k$ from $\tilde{M}_{k+1}$ onto $\tilde{M}_k$ by

$$\tilde{E}_k\big(\sum x_{g_0h_0\cdots g_{k+1}h_{k+1}} e^0_{g_0h_0} \cdots e^{k+1}_{g_{k+1}h_{k+1}}\big)$$
$$= \frac{1}{n} \sum \tilde{\alpha}_{g_{k+1}}\left(\sum_{l\in G} \tilde{\alpha}_l^{-1}(x_{g_0h_0\cdots g_kh_kll}) \right) e^0_{g_0h_0} \cdots e^k_{g_kh_k} e^{k+1}_{g_{k+1}g_{k+1}}\,.$$

LEMMA 3.1. *With the above notations, we have*

(i) *There exists an isomorphism Φ from $\tilde{M}$ onto $\tilde{M}_0$ such that*

$$\Phi(\tilde{N}) = \tilde{M}_{-1} \quad and \quad \Phi \circ \theta^M_t = (\theta^P_t \otimes \mathrm{id.}) \circ \Phi\,, \qquad t \in \boldsymbol{R}\,,$$

(ii) *$\{\tilde{M}_k\}_{k=-1}^\infty$ is the canonical tower of type II_∞ von Neumann algebras arising from $M \supset N$ and $\{\tilde{E}_k\}_{k=-1}^\infty$ is the sequence of the canonical conditional expectations,*

(iii) *The dual action θ^k on $\tilde{M}_k$ is given by*

$$\theta^k = \theta^P \otimes \mathrm{id.} \otimes \mathrm{id.} \otimes \cdots \otimes \mathrm{id.}$$

PROOF. (i) Taking a state (or a weight) τ on P such that $\tau \circ \alpha_g = \tau$, $g \in G$, we set

$\tilde{P}=P\rtimes_{\sigma^\tau}G$. We define a state φ on N by

$$\varphi\left(\sum_{g\in G}\alpha_g(x)e_{gg}\right)=\tau(x)\,, \qquad x\in P\,,$$

and set $\psi=\varphi\circ E\in M_*^+$. We then have $\psi=\tau\otimes(1/n)\mathrm{Tr}$, where Tr is the usual trace on $M_n(\boldsymbol{C})$. Since $\tilde{M}=M\rtimes_{\sigma^\psi}\boldsymbol{R}$ is generated by

$$\begin{aligned} &x\otimes 1\otimes 1\,, && x\in P\,,\\ &1\otimes e_{gh}\otimes 1\,, && g,h\in G\,,\\ &\Delta_\tau^{it}\otimes 1\otimes\lambda_t\,, && t\in\boldsymbol{R}\,,\end{aligned}$$

and $\tilde{M}_0=\tilde{P}\otimes M_n(\boldsymbol{C})$ is generated by

$$\begin{aligned} &x\otimes 1\otimes 1\,, && x\in P\,,\\ &\Delta_\tau^{it}\otimes\lambda_t\otimes 1\,, && t\in\boldsymbol{R}\,,\\ &1\otimes 1\otimes e_{gh}\,, && g,h\in G\,,\end{aligned}$$

we get a natural isomorphism Φ from $\tilde{M}$ onto $\tilde{M}_0$ which has the desired properties.

(ii) These follow from the characterization of the basic extension by Hamachi-Kosaki [6] (see also Bisch [1]).

(iii) This is a direct consequence of (i) and (ii). q.e.d.

For the indices $g_0, h_0, \cdots, g_k, h_k\in G$, we set

$$I(g_0, h_0, \cdots, g_k, h_k)=g_k\cdots g_0h_0{}^{-1}\cdots h_k{}^{-1}\,.$$

LEMMA 3.2 (Bisch [1]). *The relative commutant $\tilde{N}'\cap\tilde{M}_k$, $k\geq -1$, is given as follows*:

$$\tilde{N}'\cap\tilde{M}_k=\left\{\sum_{I(g_0,h_0,\cdots,g_k,h_k)\in N(\alpha)} c_{g_0h_0\cdots g_kh_k}\,\tilde{u}_{I(g_0,h_0,\cdots,g_k,h_k)}e^0_{g_0h_0}\cdots e^k_{g_kh_k}\;\middle|\; c_{g_0h_0\cdots g_kh_k}\in Z(\tilde{P})\right\},$$

where $\tilde{u}_{I(g_0,h_0,\cdots,g_k,h_k)}$ means an implementing unitary of $\tilde{\alpha}_{I(g_0,h_0,\cdots,g_k,h_k)}$ in $\tilde{P}$.

PROOF. Since the canonical extension $\tilde{\alpha}$ is free in the sense of [8] or inner ([3] or [17]), the conclusion follows from a direct computation. q.e.d.

THEOREM 3.3. *Let $(\tilde{P}, \tilde{H}, \tilde{J}, \tilde{P}^\natural)$ be a standard form of $\tilde{P}$ and $\{\tilde{v}_g\}_{g\in G}$ the canonical implementation of $\tilde{\alpha}$. We choose and fix a cyclic and separating vector $\tilde{\xi}$ for $\tilde{P}$ in $\tilde{P}^\natural$.*

We represent $\tilde{M}_k$ in $B(\tilde{H}\otimes l^2(G)\otimes\cdots\otimes l^2(G))$ by

$$\sum x_{g_0h_0\cdots g_kh_k}e^0_{g_0h_0}\cdots e^k_{g_kh_k}$$
$$\longrightarrow \sum \tilde{\alpha}_{g_{2k+1}}\tilde{\alpha}_{g_{2k}}\cdots\tilde{\alpha}_{g_{k+1}}(x_{g_0h_0\cdots g_kh_k})$$
$$e^0_{g_0h_0}\cdots e^k_{g_kh_k}e^{k+1}_{g_{k+1}g_{k+1}}\cdots e^{2k+1}_{g_{2k+1}g_{2k+1}},$$

and define a vector $\tilde{\xi}_k$ *in the Hilbert space* $\tilde{H}\otimes l^2(G)\otimes\cdots\otimes l^2(G)=l^2(G\times\cdots\times G,\tilde{H})$ *by*

$$\tilde{\xi}(g_0,g_1,\cdots,g_{2k+1})=\delta_{g_0,g_{2k+1}^{-1}}\delta_{g_1,g_{2k}^{-1}}\cdots\delta_{g_k,g_{k+1}^{-1}}\tilde{\xi}.$$

Here $\delta_{g,h}$ *means the Kronecker symbol.*

Then $\tilde{\xi}_k$ *is a cyclic and separating vector for* $\tilde{M}_k$ *and the modular conjugation operator* $\tilde{J}_k$ *arising from* $\tilde{\xi}_k$ *is given by*

$$\tilde{J}_k=\sum\tilde{v}_{g_{2k+1}}\tilde{v}_{g_{2k}}\cdots\tilde{v}_{g_1}\tilde{v}_{g_0}\tilde{J}e^0_{g_0g_{2k+1}^{-1}}e^1_{g_1g_{2k}^{-1}}\cdots e^{2k+1}_{g_{2k+1}g_0^{-1}}.$$

PROOF. For $X=\sum\tilde{\alpha}_{g_{2k+1}}\tilde{\alpha}_{g_{2k}}\cdots\tilde{\alpha}_{g_{k+1}}(x_{g_0h_0\cdots g_0h_k})e^0_{g_0h_0}\cdots e^k_{g_kh_k}e^{k+1}_{g_{k+1}g_{k+1}}\cdots e^{2k+1}_{g_{2k+1}g_{2k+1}}$ $\in\tilde{M}_k$, we have

$$(X\tilde{\xi}_k)(g_0,g_1,\cdots,g_{2k+1})$$
$$=\sum_{h_0,\cdots,h_k}\tilde{\alpha}_{g_{2k+1}}\tilde{\alpha}_{g_{2k}}\cdots\tilde{\alpha}_{g_{k+1}}(x_{g_0h_0\cdots g_kh_k})\tilde{\xi}_k(h_0,\cdots,h_k,g_{k+1},\cdots,g_{2k+1})$$
$$=\tilde{\alpha}_{g_{2k+1}}\tilde{\alpha}_{g_{2k}}\cdots\tilde{\alpha}_{g_{k+1}}(x_{g_0g_{2k+1}^{-1}g_1g_{2k}^{-1}\cdots g_kg_{k+1}^{-1}})\tilde{\xi}.$$

Hence $\tilde{\xi}_k$ is a cyclic and separating vector for $\tilde{M}_k$.

From the definition, $S_k(=S_{\tilde{\xi}_k})$ is the closure of the operator $S^0_k: X\tilde{\xi}_k\to X^*\tilde{\xi}_k$, $X\in\tilde{M}_k$. Since we compute

$$(X^*\tilde{\xi}_k)(h_0,h_1,\cdots,h_{2k+1})$$
$$=\sum_{g_0,\cdots,g_k}\tilde{\alpha}_{h_{2k+1}}\tilde{\alpha}_{h_{2k}}\cdots\tilde{\alpha}_{h_{k+1}}(x_{g_0h_0\ldots g_kh_k})^*\tilde{\xi}_k(g_0,\cdots,g_k,h_{k+1},\cdots,h_{2k+1})$$
$$=\tilde{\alpha}_{h_{2k+1}}\tilde{\alpha}_{h_{2k}}\cdots\tilde{\alpha}_{h_{k+1}}(x_{h_{2k+1}^{-1}h_0h_{2k}^{-1}h_1\cdots h_{k+1}^{-1}h_k})^*\tilde{\xi},$$

we get

$$S^0_k=\sum\tilde{v}_{g_{2k+1}}\tilde{v}_{g_{2k}}\cdots\tilde{v}_{g_{k+1}}S^0_{\tilde{v}_{g_{k+1}}^*\cdots\tilde{v}_{g_{2k}}^*\tilde{v}_{g_{2k+1}}^*\tilde{\xi},\tilde{v}_{g_k}\cdots\tilde{v}_{g_1}\tilde{v}_{g_0}\tilde{\xi}}$$
$$\tilde{v}_{g_k}\cdots\tilde{v}_{g_1}\tilde{v}_{g_0}e^0_{g_0g_{2k+1}^{-1}}e^1_{g_1g_{2k}^{-1}}\cdots e^{2k+1}_{g_{2k+1}g_0^{-1}}.$$

Taking the closure, we have

$$S_k=\sum\tilde{v}_{g_{2k+1}}\tilde{v}_{g_{2k}}\cdots\tilde{v}_{g_{k+1}}S_{\tilde{v}_{g_{k+1}}^*\cdots\tilde{v}_{g_{2k}}^*\tilde{v}_{g_{2k+1}}^*\tilde{\xi},\tilde{v}_{g_k}\cdots\tilde{v}_{g_1}\tilde{v}_{g_0}\tilde{\xi}}$$
$$\tilde{v}_{g_k}\cdots\tilde{v}_{g_1}\tilde{v}_{g_0}e^0_{g_0g_{2k+1}^{-1}}e^1_{g_1g_{2k}^{-1}}\cdots e^{2k+1}_{g_{2k+1}g_0^{-1}}.$$

Thus we obtain

$$\Delta_k = S_k{}^* S_k = \sum \tilde{v}_{g_0}{}^* \tilde{v}_{g_1}{}^* \cdots \tilde{v}_{g_k}{}^* \Delta_{\tilde{v}_{g_{k+1}}^* \cdots \tilde{v}_{g_{2k}}^* \tilde{v}_{g_{2k+1}}^* \tilde{\xi},\ \tilde{v}_{g_k} \cdots \tilde{v}_{g_1} \tilde{v}_{g_0} \tilde{\xi}}$$
$$\tilde{v}_{g_k} \cdots \tilde{v}_{g_1} \tilde{v}_{g_0} e^0_{g_{2k+1}^{-1} g_{2k+1}^{-1}} e^1_{g_{2k}^{-1} g_{2k}^{-1}} \cdots e^{2k+1}_{g_0^{-1} g_0^{-1}},$$

and

$$\tilde{J}_k = S_k \Delta_k^{-1/2} = \sum \tilde{v}_{g_{2k+1}} \tilde{v}_{g_{2k}} \cdots \tilde{v}_{g_1} \tilde{v}_{g_0} \tilde{J} e^0_{g_0 g_{2k+1}^{-1}} e^1_{g_1 g_{2k}^{-1}} \cdots e^{2k+1}_{g_{2k+1} g_0^{-1}}.$$

q.e.d.

By Sutherland-Takesaki [19; Lemma 5.11], there exists a dominant weight φ on P such that $\varphi \circ \alpha_g = \varphi$, $g \in G$, and for the continuous decomposition $P = P_\varphi \rtimes_\theta \boldsymbol{R}$,

$$\hat{\alpha}_g \circ \theta_t = \theta_t \circ \hat{\alpha}_g, \qquad g \in G, \quad t \in \boldsymbol{R},$$

$$\begin{cases} \alpha_g(x) = \hat{\alpha}_g(x), & x \in P_\varphi, \\ \alpha_g(\lambda(t)) = \lambda(t), & t \in \boldsymbol{R}, \end{cases}$$

where $\hat{\alpha}$ is the action on P_φ induced by α. Then each of the invariants of α is calculated as follows: The module is given by

$$\operatorname{mod} \alpha_g = \hat{\alpha}_g|_{Z(P_\varphi)}, \qquad g \in G.$$

Writing $\alpha_h = \operatorname{Ad} u_h \circ \bar{\sigma}^\varphi_{c(h)}$, $h \in N(\alpha)$,

$$u_h u_k = \mu(h, k) u_{hk}, \qquad h, k \in N(\alpha),$$
$$\alpha_g(u_{g^{-1}hg}) = \lambda(g, h) u_h, \qquad g \in G, \quad h \in N(\alpha),$$
$$\theta_t(u_h) = c(h, t) u_h, \qquad t \in \boldsymbol{R}, \quad h \in N(\alpha).$$

COROLLARY 3.4. *With the above notations, we have*

(i) *The canonical tower* $\{\tilde{M}_k\}_{k=-1}^{\infty}$ *of type* II_∞ *von Neumann algebras arising from* $M \supset N$ *is given by*

$$\tilde{M}_k = P_\varphi \otimes P_0 \otimes P_1 \otimes \cdots \otimes P_k, \qquad k \geq 0,$$
$$\tilde{M}_{-1} = \tilde{N} = \left\{ \sum_{g \in G} \hat{\alpha}_g(x) e_{gg} \,\middle|\, x \in P_\varphi \right\},$$

and the embedding is given by

$$\sum x_{g_0 h_0 \cdots g_k h_k} e^0_{g_0 h_0} \cdots e^k_{g_k h_k} \in \tilde{M}_k \longrightarrow$$
$$\sum \hat{\alpha}_{g_{k+1}}(x_{g_0 h_0 \cdots g_k h_k}) e^0_{g_0 h_0} \cdots e^k_{g_k h_k} e^{k+1}_{g_{k+1} g_{k+1}} \in \tilde{M}_{k+1},$$

(ii) *The tower of the relative commutants* $\{\tilde{N}' \cap \tilde{M}_k\}_{k=-1}^{\infty}$ *is calculated by*

$$\tilde{N}'\cap\tilde{M}_k=\left\{\sum_{I(g_0,h_0,\cdots,g_k,h_k)\in N(\alpha)} c_{g_0h_0\cdots g_kh_k}\, u_{(g_0,h_0,\cdots,g_k,h_k)}e^0_{g_0h_0}\cdots e^k_{g_kh_k}\;\middle|\; c_{g_0h_0\cdots h_kh_k}\in Z(P_\varphi)\right\},$$

(iii) *The mirroring $\tilde{\gamma}_k$ of $\tilde{N}'\cap\tilde{M}_{2k+1}$ is calculated by*

$$\begin{aligned}&\tilde{\gamma}_k(\sum c_{g_0h_0\cdots g_{2k+1}h_{2k+1}}u_{I(g_0,h_0,\cdots,g_{2k+1},h_{2k+1})}e^0_{g_0h_0}\cdots e^{2k+1}_{g_{2k+1}h_{2k+1}})\\ &=\sum(\operatorname{mod}\alpha_{h_0^{-1}h_1^{-1}\cdots h_{2k+1}^{-1}})(c_{g_0h_0\cdots g_{2k+1}h_{2k+1}})\\ &\quad\lambda(h_0^{-1}h_1^{-1}\cdots h_{2k+1}^{-1}, I(h_{2k+1}^{-1}, g_{2k+1}^{-1},\cdots,h_0^{-1},g_0^{-1}))\\ &\quad u_{I(h_{2k+1}^{-1},g_{2k+1}^{-1},\cdots,h_0^{-1},g_0^{-1})}e^0_{h_{2k+1}^{-1}g_{2k+1}^{-1}}\cdots e^{2k+1}_{h_0^{-1}g_0^{-1}},\end{aligned}$$

and the canonical shift $\tilde{\Gamma}$ in the sense of Ocneanu [14] *is given by*

$$\begin{aligned}&\tilde{\Gamma}(\sum c_{g_0h_0\cdots g_{2k+1}h_{2k+1}}u_{I(g_0,h_0,\cdots,g_{2k+1},h_{2k+1})}e^0_{g_0h_0}\cdots e^{2k+1}_{g_{2k+1}h_{2k+1}})\\ &=\sum c_{g_0h_0\cdots g_{2k+1}h_{2k+1}}u_{I(g_0,h_0,\cdots,g_{2k+1},h_{2k+1})}e^2_{g_0h_0}e^3_{g_1h_1}\cdots e^{2k+3}_{g_{2k+1}h_{2k+1}},\end{aligned}$$

(iv) *The restriction of the dual action θ^k to the relative commutant $\tilde{N}'\cap\tilde{M}_k$ is given by*

$$\begin{aligned}&\theta^k_t(\sum c_{g_0h_0\cdots g_kh_k}u_{I(g_0,h_0,\cdots,g_k,h_k)}e^0_{g_0h_0}\cdots e^k_{g_kh_k})\\ &=\sum\theta_t(c_{g_0h_0\cdots g_kh_k})c(I(g_0,h_0,\cdots g_kh_k),t)\\ &\qquad u_{I(g_0,h_0,\cdots,g_k,h_k)}e^0_{g_0h_0}\cdots e^k_{g_kh_k},\qquad t\in\boldsymbol{R}.\end{aligned}$$

PROOF. By [17], $\tilde{P}$ is identified with P_φ. Then assertions (i) and (ii) follow from Lemma 3.1 and Lemma 3.2.

(iii) We get, from Theorem 3.3,

$$\begin{aligned}&\tilde{\gamma}_k(\sum c_{g_0h_0\cdots g_{2k+1}h_{2k+1}}u_{I(g_0,h_0,\cdots,g_{2k+1},h_{2k+1})}e^0_{g_0h_0}\cdots e^{2k+1}_{g_{2k+1}h_{2k+1}})\\ &=\sum\tilde{v}^*_{h_0}\tilde{v}_{h_1}{}^*\cdots\tilde{v}^*_{h_{2k}}\tilde{v}_{h_{2k+1}}{}^*\tilde{J}c_{g_0h_0\cdots g_{2k+1}h_{2k+1}}{}^*\\ &\quad u_{I(g_0,h_0,\cdots,g_{2k+1},h_{2k+1})}{}^*\tilde{v}_{g_{2k+1}}\tilde{v}_{g_{2k}}\cdots\tilde{v}_{g_1}\tilde{v}_{g_0}\tilde{J}\\ &\quad e^0_{g_{2k+1}^{-1}h_{2k+1}^{-1}}e^1_{g_{2k}^{-1}h_{2k}^{-1}}\cdots e^{2k+1}_{g_0^{-1}h_0^{-1}}\\ &=\sum\tilde{v}^*_{h_0}\tilde{v}^*_{h_1}\cdots\tilde{v}^*_{h_{2k}}\tilde{v}^*_{h_{2k+1}}c_{g_0h_0\cdots g_{2k+1}h_{2k+1}}\\ &\quad u_{I(g_0,h_0,\cdots,g_{2k+1},h_{2k+1})}\tilde{v}_{h_{2k+1}}\tilde{v}_{h_{2k}}\cdots\tilde{v}_{h_1}\tilde{v}_{h_0}\\ &\quad e^0_{h_{2k+1}^{-1}g_{2k+1}^{-1}}e^1_{h_{2k}^{-1}g_{2k}^{-1}}\cdots e^{2k+1}_{h_0^{-1}g_0^{-1}}\\ &=\sum(\operatorname{mod}\alpha_{h_0^{-1}h_1^{-1}\cdots h_{2k+1}^{-1}})(c_{g_0h_0\cdots g_{2k+1}h_{2k+1}})\\ &\quad\lambda(h_0^{-1}h_1^{-1}\cdots h_{2k+1}^{-1}, I(h_{2k+1}^{-1}, g_{2k+1}^{-1},\cdots,h_0^{-1},g_0^{-1}))\\ &\quad u_{I(h_{2k+1}^{-1},g_{2k+1}^{-1},\cdots,h_0^{-1},g_0^{-1})}e^0_{h_{2k+1}^{-1}g_{2k+1}^{-1}}\cdots e^{2k+1}_{h_0^{-1}g_0^{-1}}.\end{aligned}$$

Furthermore, we have

$$\tilde{\gamma}_{k+1}\circ\tilde{\gamma}_k(\sum c_{g_0h_0\cdots g_{2k+1}h_{2k+1}}u_{I(g_0,h_0,\cdots,g_{2k+1},h_{2k+1})}e^0_{g_0h_0}\cdots e^{2k+1}_{g_{2k+1}h_{2k+1}})$$
$$=\sum c_{g_0h_0\cdots g_{2k+1}h_{2k+1}}u_{I(g_0,h_0,\cdots,g_{2k+1},h_{2k+1})}e^2_{g_0h_0}\cdots e^{2k+3}_{g_{2k+1}h_{2k+1}}).$$

(iv) follows from Lemma 3.1. q.e.d.

REMARK 3.5. We compare the above example with the crossed product. Let $\alpha: G\to \mathrm{Aut}\,P$ be an outer action of a finite group G on a type III factor P. Let $M=P\rtimes_\alpha G\supset N=P$. Then the canonical inclusion $\tilde{M}\supset\tilde{N}$ is conjugate to $\tilde{P}\rtimes_{\tilde{\alpha}}G\supset\tilde{P}$ ([3] or [17]) and the basic extension $\tilde{M}_1$ is given by $\tilde{P}\otimes M_n(\boldsymbol{C})$, $n=|G|$.

Let $(\tilde{P}, \tilde{H}, \tilde{J}, P^\natural)$ be a standard form and $\{\tilde{v}_g\}_{g\in G}$ the canonical implementation of $\tilde{\alpha}$. Then the standard representation of $\tilde{M}$ is the usual one, namely,

$$\sum_{g\in G} x_g\lambda_g \longrightarrow \sum_{g,h\in G}\tilde{\alpha}_{gh}{}^{-1}(x_g)e_{gh,h}\,.$$

If we take a cyclic and separating vector $\tilde{\xi}$ for $\tilde{P}$ in $\tilde{P}^\natural$ and define a cyclic and separating vector $\tilde{\xi}_0$ for $\tilde{M}$ in $\tilde{H}\otimes l^2(G)$ by

$$\tilde{\xi}_0(g)=\delta_{g,e}\tilde{\xi}\,,\qquad g\in G\,,$$

then the modular conjugation operator $\tilde{J}_0$ coming from $\tilde{\xi}_0$ is calculated by

$$\tilde{J}_0=\sum_{g\in G}\tilde{v}_g\tilde{J}e_{g^{-1},g}\,.$$

Moreover, the following are valid:

$$\tilde{N}'\cap\tilde{M}_1=\left\{\sum_{g^{-1}h\in N(\alpha)} c_{g,h}u_{g^{-1}h}e_{g,h}\;\middle|\; c_{g,h}\in Z(P_\varphi)\right\},$$

$$\tilde{\gamma}_0(\sum c_{g,h}u_{g^{-1}h}e_{g,h})=\sum\lambda(h, hg^{-1})(\mathrm{mod}\,\alpha_h)(c_{g,h})u_{hg^{-1}}e_{h^{-1},g^{-1}}\,,$$

$$\theta^1_t(\sum c_{g,h}u_{g^{-1}h}e_{g,h})=\sum\theta_t(c_{g,h})c(g^{-1}h,t)u_{g^{-1}h}e_{g,h}\,,\qquad t\in\boldsymbol{R}\,.$$

REMARK 3.6. For both examples, a simple calculation shows that the relation (1.1.2) corresponds exactly to the commutativity of the mirroring and the dual action.

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