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Simplifying Certain Mappings from Simply Connected 4-Manifolds into the Plane

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1. Introduction and summary.

In the study of C^∞-manifolds by means of mappings, the following is a primary and deep problem: To what extent can we simplify mappings between manifolds? We study this problem in the present paper for a certain class of stable mappings from closed simply connected 4-manifolds into the plane. It is the class such that the associated quotient mapping q_f from M onto the closed 2-disc D^2 of each member $f: M \to \boldsymbol{R}^2$ has only tori and spheres as regular map-fibres. For example, S^4, $\boldsymbol{C}^2P$, $\boldsymbol{C}^2P \sharp \overline{\boldsymbol{C}^2P}$, $S^2 \times S^2$ and their finite connected sums admit such stable mappings (see Examples in Appendix 1).

Let f be a stable mapping from M into $\boldsymbol{R}^2$. We call a point p in M a singular point of f if df_p is not of maximum rank. The set of the singular points of f is denoted by $S(f)$, which is a 1-dimensional closed submanifold of M consisting of fold points and a finite number of cusp points ([4]). For x and y in M, we define $x \sim y$ by the conditions that $f(x) = f(y)$ $(=a)$ and that x and y are in the same connected component of $f^{-1}(a)$. The quotient space of M by this equivalence relation is called the *quotient space associated with* f and is denoted by W_f. The quotient mapping is denoted by q_f. Let $\bar{f}: W_f \to \boldsymbol{R}^2$ be the mapping which satisfies $\bar{f} \circ q_f = f$. Then q_f is a local homeomorphism when it is restricted to $S(f)$ and $\bar{f}$ is a local homeomorphism outside $q_f(S(f))$ ([5], [6]). For points in W_f, the topological types of their neighborhoods in W_f are listed in [6].

We call a stable mapping $f: M^4 \to \boldsymbol{R}^2$ *simple*, if (i) f has at most one cusp, (ii) W_f is homeomorphic to D^2, and (iii) q_f is an embedding when it is restricted to $S(f) \setminus \{\text{cusps}\}$. If M is oriented, all the regular fibres of q_f are oriented closed surfaces. Let the maximum genus of all the regular q_f-fibres be denoted by q_f. Let R be a connected component of $W_f \setminus q_f(S(f))$. We say R is a 0-*region* if the regular fibre over a point in R is a sphere,

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and a 1-*region* if it is a torus.

Now we state our theorems.

THEOREM A. *Let $f: M^4 \to \boldsymbol{R}^2$ be a simple mapping with $g_f \leq 1$ and $\pi_1(M)=1$. Then*

$$\# S(f) \geq \frac{1}{2} b_2(M)+3 \qquad (\textit{if } b_2(M) \textit{ is even and non-zero}),$$

$$\# S(f) \geq \frac{1}{2}(b_2(M)+5) \qquad (\textit{if } b_2(M) \textit{ is odd}),$$

where $\# S(f)$ denotes the number of connected components of $S(f)$ and $b_2(M)$ the second Betti number of M.

THEOREM B. *Let $g: M^4 \to \boldsymbol{R}^2$ be a simple mapping with $g_g \leq 1$ and $\pi_1(M)=1$. Then, by a finite iteration of the S- and C-operations, which are defined in section 6, we can change the pair (M, g) to (N, f) such that N is homeomorphic to M, and that $f: N \to \boldsymbol{R}^2$ is a simple mapping with $g_f \leq 1$ which satisfies the following conditions.*

(1)

$$\# S(f)=1 \qquad (\textit{if } b_2(M)=0),$$

$$\# S(f) \leq \frac{3}{2} b_2(M)+1 \qquad (\textit{if } b_2(M) \textit{ is even and non-zero}),$$

$$\# S(f) \leq \frac{3}{2}(b_2(M)+1) \qquad (\textit{if } b_2(M) \textit{ is odd}).$$

(2) *The pair (N, f) has a decomposition*

$$(N, f)=(N_1, f_1) \natural (N_2, f_2) \natural \cdots \natural (N_k, f_k)$$

such that $f_i: N_i \to \boldsymbol{R}^2$ is a simple mapping with $g_{f_i} \leq 1$ and has at most one 1-region.

The notation $(M, f)=(M_1, f_1) \natural (M_2, f_2)$ means that $M=M_1 \# M_2$ and that q_f is right-left equivalent to $q_{f_1 \# f_2}$, where the connected sum of simple mappings is defined precisely in section 6. If a pair (M, f) has a decomposition as in the second condition in Theorem B, we say (M, f) is *configuration trivial.*

THEOREM C. *Let $g: N \to \boldsymbol{R}^2$ be a simple mapping with $g_g \leq 1$ and $\pi_1(N)=1$. Then for $M=N \# S^2 \times S^2$ and $N \# \boldsymbol{C}^2 P \# \overline{\boldsymbol{C}^2 P}$, there exists a simple mapping $f: M \to \boldsymbol{R}^2$ with $g_f \leq 1$ which satisfies the following conditions.*

(1)

$$\# S(f)=4 \qquad (\textit{if } b_2(M)=2),$$

$$\# S(f) \leq \frac{3}{2} b_2(M)-1 \qquad (\textit{if } b_2(M) \textit{ is even and not equal to } 0,\ 2),$$

$$\# S(f) \leq \frac{1}{2}(3b_2(M)-1) \qquad (\textit{if } b_2(M) \textit{ is odd}).$$

(2) *The pair* (M, f) *is configuration trivial.*

The basic tools used in this paper have been prepared in [3], [4], [5]. Local properties of the quotient spaces have been studied in [2], [6].

A REMARK ON CONFIGURATION TRIVIALITY. If a pair (M, f) is configuration trivial, then the location of $q_f(S(f))$ is very simple: The region adjacent to the boundary of W_f is a 0-region, since f has the normal form $(u, x, y, z) \to (u, x^2+y^2+z^2)$ near the points in $q_f^{-1}(\partial W_f) \cap S(f)$. Therefore if (M, f) is configuration trivial, then the location of $q_f(S(f))$ in W_f is simple as illustrated in Figure 1.1.

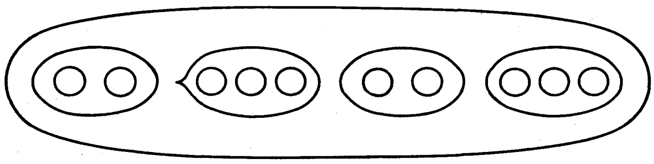

FIGURE 1.1

Throughout this paper, the symbol $\cong$ between two manifolds means that the manifolds are diffeomorphic.

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2. Preliminaries.

A stable mapping f: $M^4 \to \boldsymbol{R}^2$ is characterized by the following local and global conditions ([4]).

Local condition: For a given point $p \in S(f)$, there are local coordinate systems centred at p and $f(p)$ such that in a neighbourhood of p, f has the normal form (L_1) or (L_2);

(L_1) $f: (u, z_1, z_2, z_3) \mapsto (u, Q(z))$, where $Q(z) = \sum \varepsilon_i z_i^2$, $|\varepsilon_i| = 1$,

or

(L_2) $f: (u, x, z_1, z_2) \mapsto (u, Q(z) + aux + bx^3)$,
where $Q(z) = \sum \varepsilon_i z_i^2$, $|\varepsilon_i| = |a| = |b| = 1$.

We call p a *fold point*, or simply a *fold* if it is of type (L_1), and a *cusp point* or a *cusp* if it is of type (L_2).

Global conditions:

(G_1) if $p \in S(f)$ is a cusp point, then $f^{-1}(f(p)) \cap S(f) = \{p\}$,

and

(G_2) $f|_{S(f)\setminus\{\text{cusps}\}}$ is an immersion with normal-crossings.

For a fold point p, we can choose local coordinates so that the index of $Q(z)$ is even, which we call the index of the fold point p. Thus the index of a fold point is either 0 or 2. A fold point is called *definite* if its index is 0, and *indefinite* if it is 2.

For the quotient spaces associated with stable mappings, one should refer to [5] and [6]. We give only two remarks here.

REMARK (1). For a stable mapping $f: M \to \boldsymbol{R}^2$, assume that the quotient space W_f is a topological manifold possibly with boundary. Then one can give a C^∞-structure to W_f with respect to which $\bar{f}$: $W_f \to \boldsymbol{R}^2$ is an immersion, since the mapping $\bar{f}$ is a local homeomorphism (see Fig. 2, [6]). With respect to this C^∞-structure, q_f is C^∞.

REMARK (2). For the same f as in Remark (1), fix the C^∞-structure of W_f given above. Then for any generic immersion h: $W_f \to \boldsymbol{R}^2$, the composed mapping $g = h \circ q_f$ is a stable mapping such that $W_f = W_g$ and $q_f = q_g$.

3. Basic tools.

Let $f: M^4 \to \boldsymbol{R}^2$ be a simple mapping with $g_f \leq 1$, and let S_i be a connected component of $S(f)$ consisting of indefinite folds. Then $q_f(S_i)$ separates W_f into two regions. We say S_i is *positive* (resp. *negative*) if the inside region of $q_f(S_i)$ is a 0-region (resp. 1-region).

NOTATION. $S_\pm(f) = \{S_i \mid S_i$ is a positive (resp. negative) connected component of $S(f)$ consisting of indefinite fold points$\}$.

DEFINITION (type of a simple mapping). Let f: $M^4 \to \boldsymbol{R}^2$ be a simple mapping with a cusp and $g_f \leq 1$. Let S be the connected component of $S(f)$ with the unique cusp. Then $q_f(S)$ separates W_f into two regions. We say f is of *type A* (resp. *type B*) if the inside region of $q_f(S)$ is a 1-region (resp. 0-region).

LEMMA 3.1. *For a simple mapping f: $M^4 \to \boldsymbol{R}^2$ with $g_f \leq 1$, we have*

$$\begin{aligned}
\chi(M) &= 2(\# S_+(f) - \# S_-(f)) + 2 && \textit{if } \chi(M) \textit{ is even}, \\
&= 2(\# S_+(f) - \# S_-(f)) + 1 && \textit{if } \chi(M) \textit{ is odd} \quad (\textit{type } A), \\
&= 2(\# S_+(f) - \# S_-(f)) + 5 && \textit{if } \chi(M) \textit{ is odd} \quad (\textit{type } B),
\end{aligned}$$

where $\chi(M)$ is the Euler number of M.

PROOF. By Remark (2) in section 2, there exists a stable mapping g: $M \to \boldsymbol{R}^2$ such that $W_f = W_g$, $q_f = q_g$, and that $\bar{g}$: $W_g \to \boldsymbol{R}^2$ is an embedding. For such g, the lemma is immediately seen by Theorem 1 of [3]. We obtain the required equalities for f, since $\# S_\pm(f) = \# S_\pm(g)$. q.e.d.

Let $f: M^4 \to \boldsymbol{R}^2$ be a simple mapping, C a connected component of $q_f(S(f))$ which is adjacent to both a 0-region and a 1-region, and a a point in the 1-region. Let J be an embedded closed arc in W_f which connects a and a point in the 0-region, such that it meets $q_f(S(f))$ transversely at a single point in C. We see that the restriction of q_f to $q_f^{-1}(J)$ is a Morse function onto J with a single saddle critical point, in the same way as in Proposition 4, §1.3 of [5]. It turns out that $q_f^{-1}(J)$ is a solid torus with an open 3-disc removed.

DEFINITION (meridian and longitude with respect to (C, J)). (Notations a, C, J are as above.) An essential simple closed curve m in $q_f^{-1}(a)$ is called a *meridian of* $q_f^{-1}(a)$ *with respect to* (C, J) if it is the boundary of a closed 2-disc embedded in the solid torus $q_f^{-1}(J) \cup D^3$. A simple closed curve l in $q_f^{-1}(a)$ is called a *longitude of* $q_f^{-1}(a)$ *with respect to* (C, J) if l and m meet transversely at a single point.

Note that the isotopy class or the homology class of m is unique up to sign.

More generally, we consider the following situation. Set $J=[-1, 1]$ and $\tilde{J}=(S^1 \times D^2) \setminus \operatorname{Int} D^3$. Let $g: \tilde{J} \to J$ be a C^∞-function such that $g^{-1}(-1)$ is the sphere component of $\partial\tilde{J}$, that $g^{-1}(1)$ is the torus component of $\partial\tilde{J}$, and that $g|\operatorname{Int}\tilde{J}$ is a Morse function with a single critical point of Morse-index 1. Then we can define a longitude and a meridian of $g^{-1}(1)$ as before.

NOTATION. *Let* φ be a diffeomorphism on $\tilde{J}=(S^1 \times D^2) \setminus \operatorname{Int} D^3$ and T the torus component of $\partial\tilde{J}$. We denote by $[\varphi]$ the isomorphism $(\varphi|_T)_*$ on $H_1(T, \boldsymbol{Z})$ induced by φ.

NOTATION. For an integer α, let T_α denote the matrix defined by

$$T_\alpha = \begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix}.$$

PROPOSITION 3.2. *(g, J, $\tilde{J}$ are as above.) Let (l, m) be a longitude and a meridian of $g^{-1}(1)$. If an orientation preserving diffeomorphism φ on $\tilde{J}$ satisfies $g = g \circ \varphi$, then* $[\varphi]$ *has*

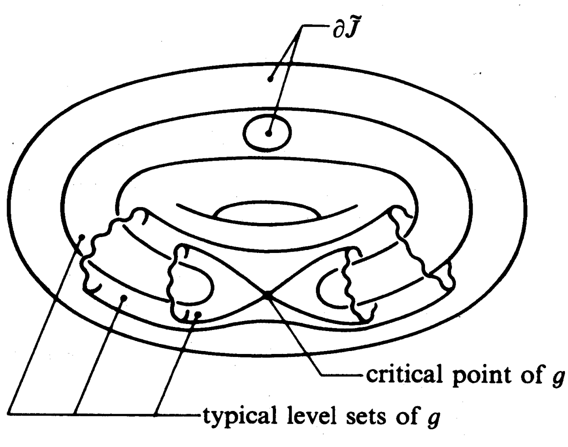


FIGURE 3.1

a matrix representation of the form $[\varphi]=\pm T_\alpha$, $\alpha\in \boldsymbol{Z}$ *with respect to the basis* $\langle [l], [m]\rangle$ *of* $H_1(g^{-1}(1), \boldsymbol{Z})$. *Conversely, for* $A=\pm T_\alpha$, *there is a diffeomorphism* φ *on* $\tilde{J}$ *such that* $[\varphi]=A$ *and that* $g=g\circ\varphi$.

PROOF. Both assertions are obvious since g has the level sets illustrated in Figure 3.1. q.e.d.

4. Proof of Theorem A.

In the case $b_2(M)$ is even, $\# S(f)=\# S_+(f)+\# S_-(f)+1$. Using Lemma 3.1, one obtains

$$\# S(f)=2\# S_-(f)+\frac{1}{2}b_2(M)+1 .$$

It is enough to show $\# S_-(f)\geq 1$. Suppose that $\# S_-(f)=0$, then W_f has no 1-region, since the region adjacent to ∂W_f is a 0-region. This implies that $\# S(f)=1$, and hence that $b_2(M)=0$, which is excluded. Therefore $\# S_-(f)\geq 1$ for M with $b_2(M)>0$.

In the case $b_2(M)$ is odd, $\# S(f)=\# S_+(f)+\# S_-(f)+2$, and using Lemma 3.1 one obtains

$$\# S(f)=2\# S_-(f)+\frac{1}{2}b_2(M)+\frac{5}{2} \qquad \text{if } f \text{ is of type A, and}$$

$$\# S(f)=2\# S_-(f)+\frac{1}{2}b_2(M)+\frac{1}{2} \qquad \text{if } f \text{ is of type B} .$$

If f is of type A, then the required inequality is obvious ($\# S_-(f)$ can be 0. In fact $\boldsymbol{C}^2\boldsymbol{P}$ has such a simple mapping. See Figure 9.1). If f is of type B, then an element of $S_-(f)$ must be located outside the connected component of $q_f(S(f))$ which contains the image of the cusp. Therefore $\# S_-(f)\geq 1$. The same required inequality follows immediately. q.e.d.

5. Transversal trees.

For a simple mapping f with $g_f\leq 1$, we define a graph Λ_f embedded in W_f as follows:

(1) Take a point p_i ($i\geq 1$) in each connected component R_i of $W_f\setminus q_f(S(f))$ and a point p_0 in ∂W_f, which are the vertices of Λ_f.

(2) If R_i and R_j are separated by a connected component, say C_{ij} ($i>j\geq 1$), of $q_f(S(f))\setminus\partial W_f$ (such C_{ij} is unique for each (i,j), if it exists), then connect p_i and p_j by a path σ_{ij} so that σ_{ij} meets $q_f(S(f))$ transversely at a single point in C_{ij}.

(3) Let p_1 be the vertex chosen from the 0-region adjacent to ∂W_f. Then connect p_0 and p_1 by a path σ_{10} so that σ_{10} is normal to ∂W_f and $\sigma_{10}\cap q_f(S(f))=\{p_0\}$; σ_{ij}

$(i>j\geq 1)$ and σ_{10} are the edges of Λ_f.

Note that Λ_f is a tree. Suppose that Λ_f has a cycle γ and let C be a connected component of $q_f(S(f))$ which meets γ. Let D be the closed 2-disc in W_f which bounds C. Since γ meets C transversely at a single point, say p, $\gamma\setminus p$ is divided into two open sets $\gamma\cap \operatorname{Int} D$ and $\gamma\cap(W_f\setminus D)$, both of which are non-empty. This is a contradiction.

DEFINITION (transversal tree). For a simple mapping $f: M^4\to \boldsymbol{R}^2$, we call the tree Λ_f in W_f thus obtained a *transversal tree of* f. We give Λ_f the orientation towards the inside: Let σ be an edge of Λ_f and C the connected component of $q_f(S(f))$ which meets σ. We give σ the orientation, from the vertex of σ lying outside of C towards the vertex lying inside of C. To σ_{10} we give the orientation from p_0 towards p_1.

We say a vertex p of a graph is of *degree* k if the number of edges which contain p as a boundary point is k.

LEMMA 5.1. *Let* $f: M\to \boldsymbol{R}^2$ *be a simple mapping with* $g_f\leq 1$ *and* $\pi_1(M)=1$. *Then the degree one vertices of a transversal tree* Λ_f *are in* 0*-regions except for* p_0.

The proof will be given at the end of this section.

DEFINITION (elementary tree of Λ_f). Let $\{p_1, \cdots, p_k\}$ be the vertices of Λ_f contained in the 0-regions and of degree greater than one. We call the closure of each connected component of $\Lambda_f\setminus\{p_1, \cdots, p_k\}$ an *elementary tree of* Λ_f.

To each elementary tree Λ_i, we give an orientation which is induced from Λ_f. We say a degree one vertex p of Λ_i is an *initial* (resp. a *terminal*) *point of* Λ_i if p is the initial (resp. the terminal) point of the unique edge of Λ_i which contains p as a boundary point. We call the unique elementary tree which meets ∂W_f the *initial tree*.

PROOF OF LEMMA 5.1. If a connected component of $W_f\setminus q_f(S(f))$ contains a vertex of Λ_f of degree one, then it is diffeomorphic to the open disc. Therefore we have only to show that a region R is a 0-region if it is diffeomorphic to the open disc. Suppose that such R is a 1-region and set $C=\partial\bar{\mathrm{R}}$.

Case 1 *where* C *does not contain* $q_f(\{cusp\})$, *the image of the unique cusp.* One can show that there is a tubular neighbourhood $N(C)$ of C such that $q_f^{-1}(J)\to q_f^{-1}(N(C))\to C$ is a local trivial fibration where J is a fibre of the canonical projection $N(C)\to C$, by using the same argument as in Proposition, §1.6 of [5]. The fibre $q_f^{-1}(J)$ is diffeomorphic to $(S^1\times D^2)\setminus \operatorname{Int} D^3$ (see section 3). Let C' be the outside boundary of $N(C)$ and D_c' the closed neighbourhood of R enclosed by C'. Set $E=W_f\setminus \operatorname{Int} D_c'$. Let $i: q_f^{-1}(C')\to q_f^{-1}(D_c')$ and $j: q_f^{-1}(C')\to q_f^{-1}(E)$ be the inclusions.

We show that $\operatorname{rank} H_1(q_f^{-1}(E), \boldsymbol{Z})\geq 1$. One can take a free generator s' of $H_1(q_f^{-1}(C'), \boldsymbol{Z})$, since $q_f^{-1}(C')$ is diffeomorphic to $S^2\times S^1$. Then $j_*(s')$ is of infinite order in $H_1(q_f^{-1}(E), \boldsymbol{Z})$, since $(q_f|q_f^{-1}(E))_*\circ j_*(s')$ is a free generator of $H_1(E, \boldsymbol{Z})\cong \boldsymbol{Z}$.

Next we show that $\operatorname{rank} H_1(q_f^{-1}(D_c'), \boldsymbol{Z})=1$, by using a Mayer-Vietoris exact sequence. Let C'' be the inside boundary of $N(C)$ and D_c'' the closure of $D_c' \setminus N(C)$. We have the following exact sequence;

$$H_1(q_f^{-1}(C''), \mathbf{Z}) \longrightarrow H_1(q_f^{-1}(D_c''), \mathbf{Z}) \oplus H_1(q_f^{-1}(N(C)), \mathbf{Z}) \longrightarrow H_1(q_f^{-1}(D_c'), \mathbf{Z}) \longrightarrow 0 .$$

Since $q_f | q_f^{-1}(C'') : q_f^{-1}(C'') \to C''$ is a trivial T^2 fibration, one can see that the fibration $q_f^{-1}(N(C)) \to C$ is trivial. It is shown that $H_1(q_f^{-1}(C''), \mathbf{Z})$, $H_1(q_f^{-1}(D_c''), \mathbf{Z})$ and $H_1(q_f^{-1}(N(C)), \mathbf{Z})$ are torsion free and of rank 3, 2 and 2 respectively, by the Künneth formula in the last case. Let (l, m) be a longitude and a meridian with respect to (C, J), and set $s = S(f) \cap q_f^{-1}(C)$. Let s'' be a cross-section of q_f over C''. We may assume that the above three homology groups are generated by l, m, and s'', by l and m, and by l and s, respectively. Therefore it is easy to see that the first homomorphism of the exact sequence is injective. Hence we have $H_1(q_f^{-1}(D_c'), \mathbf{Z}) \cong \mathbf{Z}\langle l \rangle$.

On the other hand, we have the following Mayer-Vietoris exact sequence;

$$H_1(q_f^{-1}(C'), \mathbf{Z}) \longrightarrow H_1(q_f^{-1}(D_c'), \mathbf{Z}) \oplus H_1(q_f^{-1}(E), \mathbf{Z}) \longrightarrow H_1(M, \mathbf{Z}) = 0 .$$

Since j_* is an injection, the first homomorphism is an isomorphism. This is a contradiction. In fact the left side of the isomorphism has rank 1, and the right side has rank greater than or equal to 2.

Case 2 *where* C *contains* $q_f(\{\text{cusp}\})$. Let $g: M \to \mathbf{R}^2$ be a simple mapping such that $W_f = W_g$, $q_f = q_g$, (thus $g_g \leq 1$) and that $\bar{g}$ is an embedding (see Remark (2) in section 2). We will prove the claim for g. As R is supposed to be a 1-region, g must be of type A. Let $\gamma: \mathbf{R}^2 \to \mathbf{R}$ be a linear function such that $\gamma \circ g$ is a Morse function (for the existence of such γ, see 1.3 of [3]). After we compose g with some isotopy in $\mathbf{R}^2$ if necessary,

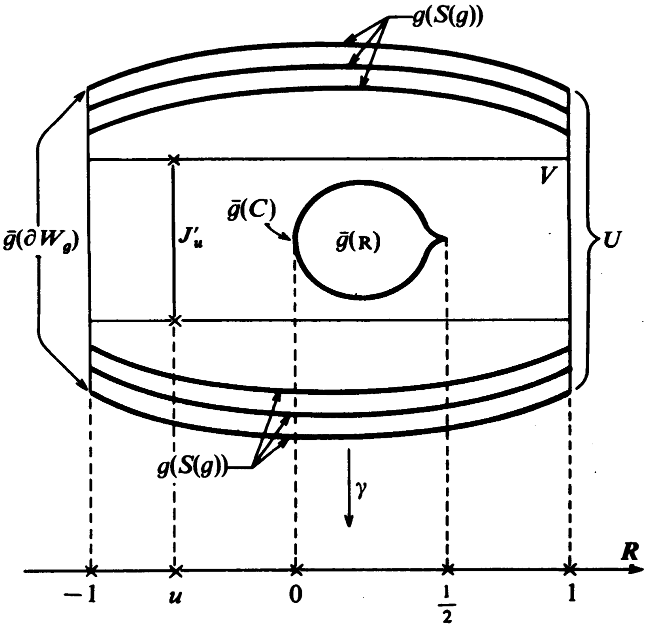


FIGURE 5.1

we can take closed neighbourhoods U and V of $\bar{g}(\mathrm{R})$ which satisfy the following conditions, by [3] (see Figure 5.1):

a) $\bar{g}(\bar{\mathrm{R}}) \subset V \subset U$,

b) there is a diffeomorphism $\varphi: I \times J \to U$,

c) $\varphi(u \times J) = g(M) \cap \gamma^{-1}(u)$ for $u \in I$,

d) $V \cap g(S(g)) = \bar{g}(C)$,

e) set $V \cap \gamma^{-1}(u) = J'_u$, then $(\gamma \circ g)^{-1}(u) \setminus g^{-1}(J'_u)$, $u \in I$, are all diffeomorphic, where $I, J \cong [-1, 1]$.

In fact, one can choose $I \subset (\gamma \circ g)(M)$ so that I contains exactly one critical value of $\gamma \circ g$. We may assume that this critical value is 0 and that the value of the cusp is 1/2. Then by the argument of [3] we see that $(\gamma \circ g)^{-1}(-\infty, 1/4]$ can be obtained from $(\gamma \circ g)^{-1}(-\infty, -1/2]$ by a 1-handle attaching. From d), one has $g^{-1}(J'_{-1/2}) \cong S^2 \times J$, hence by e), one has $g^{-1}(J'_{1/4}) \cong S^2 \times S^1 \setminus 2\,\mathrm{Int}\,D^3$. On the other hand, $g^{-1}(J'_{1/4}) \cong g^{-1}(J'_{3/4}) \cong S^2 \times J$, where the former diffeomorphism follows from the fact that $[1/4, 3/4]$ contains no critical values of $\gamma \circ g$ and the latter follows from d). This is a contradiction. q.e.d.

6. *S*- and *C*-operations.

Let $f: M^4 \to \boldsymbol{R}^2$ be a simple mapping and suppose $\pi_1(M) = 1$. We construct a C^∞-stable mapping $\tilde{f}$ from $M \# S^2 \times S^2$ onto S^2 as follows.

CONSTRUCTION OF $\tilde{f}$: Let c be the unique connected component of $S(f)$ such that $q_f(c) = \partial W_f$. Take a collar neighbourhood $v(c)$ of ∂W_f so that $v(c) \cap q_f(S(f)) = c$. Then $q_f^{-1}(v(c))$ is a tubular neighbourhood of c in M and the restriction of q_f to its boundary is right-left equivalent to the projection $S^2 \times S^1 \to S^1$. One can glue $\overline{M \setminus q_f^{-1}(v(c))}$ and $S^2 \times D^2$ and at the same time $\overline{W_f \setminus v(c)}$ and D^2 along their boundaries by diffeomorphisms φ and ψ which satisfy the commutative diagram (Figure 6.1), where $p: S^2 \times D^2 \to D^2$ is the projection.

$$\begin{array}{ccc}
\partial(\overline{M \setminus q_f^{-1}(v(c))}) & \xrightarrow{\varphi} & \partial(S^2 \times D^2) \\
{\scriptstyle q_f|\partial(\overline{M \setminus q_f^{-1}(v(c))})} \big\downarrow & & \big\downarrow {\scriptstyle p|\partial(S^2 \times D^2)} \\
\partial(\overline{W_f \setminus v(c)}) & \xrightarrow{\psi} & \partial D^2
\end{array}$$

FIGURE 6.1

By this gluing, we also glue q_f and q_p and obtain a smooth mapping $\tilde{f}$ from $\overline{M \setminus q_f^{-1}(v(c))} \cup_\varphi D^2 \times S^2$ onto $\overline{W_f \setminus v(c)} \cup_\psi D^2$.

We can choose the attaching diffeomorphism φ so that the source manifold is diffeomorphic to $M \# S^2 \times S^2$ ([9]). Thus we have constructed a smooth mapping

$\tilde{f}: M \# S^2 \times S^2 \to S^2$, which is obviously stable. Note that the right-left equivalence classes of $\tilde{f}$'s are not unique according to the various isotopy types of φ. We say that $\tilde{f}$ is a mapping onto S^2 associated with f.

DEFINITION (S-equivalence). Let $f: M^4 \to \boldsymbol{R}^2$ and $g: N^4 \to \boldsymbol{R}^2$ be simple mappings with $\pi_1(M) = \pi_1(N) = 1$. The pairs (M, f) and (N, g) are said to be *S-equivalent* if $\tilde{f}: M \# S^2 \times S^2 \to S^2$ and $\tilde{g}: N \# S^2 \times S^2 \to S^2$ are right-left equivalent, where $\tilde{f}$ and $\tilde{g}$ are mappings onto S^2 associated with f and g respectively.

Let M_i be an oriented manifold and $f_i: M_i \to \boldsymbol{R}^2$ a simple mapping for $i = 1, 2$. We can construct a smooth stable mapping $f \# g$ from $M_1 \# M_2$ into $\boldsymbol{R}^2$ in the following way.

Let a_i be a point in ∂W_{f_i} for $i = 1, 2$. Take a tubular neighbourhood U_i of a_i in W_{f_i} so that $U_i \cap q_{f_i}(S(f_i)) \subset \partial W_{f_i}$ for $i = 1, 2$ (see Figure 6.2). Let λ_i be the closure of $\partial U_i \setminus \partial W_{f_i}$ for $i = 1, 2$. We may assume that λ_i is transverse to ∂W_{f_i}. Then we see that $q_f^{-1}(U_i)$ is diffeomorphic to D^4 and $q_f^{-1}(\lambda_i) = \partial q_f^{-1}(U_i)$, $i = 1, 2$, by applying Levine's argument in [3]. We give an orientation to each W_{f_i} so that $\bar{f}_i$ is an orientation preserving immersion, and to each λ_i as a subset of the boundary of $\overline{W_{f_i} \setminus U_i}$.

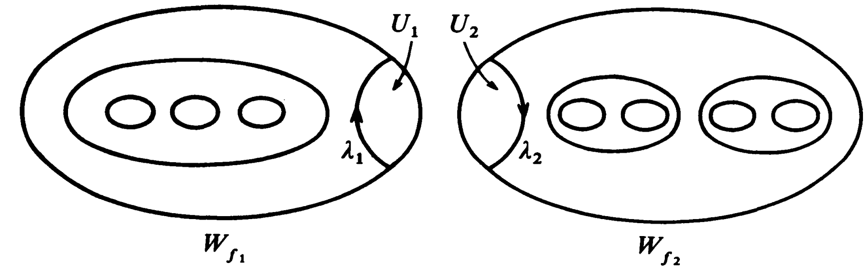


FIGURE 6.2

Let $g_i: q_{f_i}^{-1}(\lambda_i) \to \lambda_i$ be the restrictions of q_{f_i} to $q_{f_i}^{-1}(\lambda_i)$, $i = 1, 2$. They are Morse functions by the definition of definite fold points and are right-left equivalent; in fact both are the simplest Morse functions on S^3. Let $\varphi: q_{f_1}^{-1}(\lambda_1) \to q_{f_2}^{-1}(\lambda_2)$ and $\psi: \lambda_1 \to \lambda_2$ be diffeomorphisms satisfying $\psi \circ g_1 = g_2 \circ \varphi$. We take ψ as orientation reversing. We may assume that φ is orientation reversing with respect to the orientations of $q_{f_i}^{-1}(\lambda_i)$ as the boundaries of $\overline{M_i \setminus q_{f_i}^{-1}(U_i)}$, $i = 1, 2$. In fact if φ is orientation preserving, let $k: q_{f_2}^{-1}(\lambda_2) \to q_{f_2}^{-1}(\lambda_2)$ be an orientation reversing diffeomorphism which satisfies $g_2 = g_2 \circ k$ and which reverses the orientation of each level set of g_2. Then the diffeomorphism $k \circ \varphi$ is orientation reversing and $\psi \circ g_1 = g_2 \circ k \circ \varphi$.

Let M_i' be the closure of $M_i \setminus q_{f_i}^{-1}(U_i)$ and W_i' the closure of $W_{f_i} \setminus U_i$, for $i = 1, 2$. We can glue the triads $(M_1', q_{f_1}|_{M_1'}, W_1')$ and $(M_2', q_{f_2}|_{M_2'}, W_2')$ by φ and ψ, and obtain $(M_1 \# M_2, \tilde{q}, W_{f_1} \natural W_{f_2})$, where $\tilde{q}$ is a C^0-mapping satisfying $\tilde{q} = q_{f_i}$ on M_i', $i = 1, 2$. After a slight perturbation, $\tilde{q}$ becomes C^∞. We denote the C^∞-mapping by $q_{f_1} \# q_{f_2}$. Now change the immersion $\bar{f}_2$ left equivalently so that $f_2(M_2')$ attaches to $f_1(M_1')$. Then we can glue $\bar{f}_1$ and $\bar{f}_2$ and obtain an immersion h from $W_{f_1} \natural W_{f_2}$ into $\boldsymbol{R}^2$. After a slight perturbation of h, we obtain a C^∞-stable mapping $h \circ q_{f_1} \# q_{f_2}$ from $M_1 \# M_2$ into $\boldsymbol{R}^2$. Let $f_1 \# f_2$ denote the mapping thus obtained.

Note that $W_{f_1 \# f_2} = W_{f_1} \natural W_{f_2}$ and that $q_{f_1 \# f_2}$ is right-left equivalent to $q_{f_1} \# q_{f_2}$. The mapping $q_{f_1} \# q_{f_2}$, (thus also $q_{f_1 \# f_2}$) is determined uniquely by f_1 and f_2 up to right-left equivalence.

NOTATION. For a stable mapping $f: M \to \boldsymbol{R}^2$ and simple mappings $f_i: M_i \to \boldsymbol{R}^2$, $i = 1, 2$, we shall write $(M, f) = (M_1, f_1) \natural (M_2, f_2)$ if q_f is right-left equivalent to $q_{f_1} \# q_{f_2}$.

REMARK. It is easily checked that $(M_1, f_1) \natural (M_2, f_2) = (M_2, f_2) \natural (M_1, f_1)$ and $((M_1, f_1) \natural (M_2, f_2)) \natural (M_3, f_3) = (M_1, f_1) \natural ((M_2, f_2) \natural (M_3, f_3))$ where the two equalities mean that the manifolds of both sides are diffeomorphic and the quotient mappings of the stable mappings of the both sides are right-left equivalent.

DEFINITION (S-operation). Let $f: M^4 \to \boldsymbol{R}^2$ be a simple mapping with $\pi_1(M^4) = 1$. Assume that (M, f) has the following decomposition:

$$(M, f) = (M_1, f_1) \natural (M_2, f_2) \natural \cdots \natural (M_k, f_k) .$$

Then, an *S-operation* is to replace an (M_i, f_i) with a pair which is S-equivalent to (M_i, f_i).

REMARK. (1) If (M, f) and (N, g) are S-equivalent, then M and N are homeomorphic, by [1].

(2) If (M, f) is changed to (N, g) by finitely iterated S-operations, then $M \# S^2 \times S^2$ is diffeomorphic to $N \# S^2 \times S^2$.

DEFINITION (C-operation). Let $f: M^4 \to \boldsymbol{R}^2$ be a simple mapping with $g_f \leq 1$ and $\pi_1(M^4) = 1$. Assume that (M, f) has a decomposition $(M, f) = (M_1, f_1) \natural (M_2, f_2)$ such that (i) f_2 has no cusp, (ii) W_{f_2} has a unique 1-region which is diffeomorphic to an open annulus, and (iii) $q_{f_2}(S(f_2))$ consists of three simple closed curves, one of which is the boundary of W_{f_2} and the others bound the 1-region. Then, a *C-operation* is to replace (M, f) with (M_1, f_1).

Note that if (M, f) is replaced with (M_1, f_1) by a C-operation, then M_1 is diffeomorphic to M, by the following proposition.

PROPOSITION 6.1. (*Notations are as above*). *M_2 is diffeomorphic to S^4.*

PROOF. Note that M_2 is a homotopy 4-sphere. In fact, M_2 is simply connected because so is M, and the Euler characteristic of M_2 is 2 by [3].

Let a be a point chosen from the 0-region of W_{f_2} enclosed by the inside boundary of the unique 1-region, D a 2-disc in the 0-region which contains a in its interior, and C one of the two connected components of $q_{f_2}(S(f_2))$ which bound the 1-region. There exists a trivial fibration π from $W_{f_2} \setminus \mathrm{Int} D$ onto C, and the composition

$$\pi \circ q_{f_2}: \quad q_{f_2}^{-1}(W_{f_2} \setminus \mathrm{Int} D) \longrightarrow W_{f_2} \setminus \mathrm{Int} D \longrightarrow C$$

is a locally trivial fibration, which follows from the same argument as in Proposition, §1.6 of [5]. Fibres of this fibration are punctured lens spaces whose boundaries are $q_{f_2}^{-1}(r)$, $r \in \partial D$. We see this easily by the definition of fold points (see the paragraph following Lemma 3.1). Since $q_{f_2}^{-1}(D)$, which is diffeomorphic to $S^2 \times D^2$, is a tubular neighbourhood of $q_{f_2}^{-1}(a)$, it follows that the exterior of the 2-knot $q_{f_2}^{-1}(a)$ is a punctured lens space bundle over C. Note that the boundary of each fibre is isotopic to $q_{f_2}^{-1}(a)$. It turns out that $q_{f_2}^{-1}(a)$ is a fibred 2-knot in a homology 4-sphere, fibred by punctured lens spaces. Such a homology sphere must be diffeomorphic to S^4: In fact $q_{f_2}^{-1}(a)$ is Zeeman's 2-twist-spun knot of a two bridge knot in S^3 [7, 8, 10]. q.e.d.

7. Proof of Theorem B.

We shall prove the theorem in two steps. In the first step, performing S-operations, we change the pair (M, g) to a configuration trivial pair $(N, f')=(N_1, f_1)\natural \cdots \natural(N_s, f_s)$. In the second step, performing C-operations, we change (N, f') to a configuration trivial pair (N, f) which satisfies the inequality (1).

First step, simplifying configurations. Let Λ_g be a transversal tree of g and p_1 the terminal point of the initial elementary tree Λ_0. Suppose that the degree of p_1 equals one. It turns out that Λ_g consists of only the initial elementary tree, and hence that (M, g) is already a configuration trivial pair. Therefore we may assume that the set S of the vertices in the 0-regions with degree greater than one is non-empty. Divide Λ_g into elementary trees Λ_i's at the points in S.

To show that the pair (M, g) can be changed to a configuration trivial pair, we have only to show it in the case where the degree of p_1 is two: If the degree of p_1 is greater than two, then there is a natural decomposition

$$(M, g)=(L_1, g_1)\natural(L_2, g_2)\natural \cdots \natural(L_m, g_m)$$

such that for the closures $\Lambda_{11}, \Lambda_{12}, \cdots, \Lambda_{1m}$ of the connected components of $\Lambda_f \setminus \Lambda_0$, each $\Lambda_{1i} \cup \Lambda_0$ is naturally identified with a transversal tree of g_i for $i=1, \cdots, m$. If each (L_i, g_i) can be changed to a configuration trivial pair (L_i', g_i'), then (M, g) can be changed to the connected sum $(N, g')=(L_1', g_1')\natural \cdots \natural(L_m', g_m')$ which is also configuration trivial. Therefore this case is reduced to the degree two case.

Assuming that the degree of p_1 is two, let p_2 be one of the terminal points of the elementary tree of Λ_g whose initial point is p_1. In the case that $p_2 \in S$, to simplify the configuration of $q_g(S(g))$ at p_2, we shall perform an S-operation centred at p_2 to the pair (M, g), that is, an S-operation which replaces a tubular neighbourhood of $q_f^{-1}(p_2)$ with $S^1 \times D^3$ and a collar neighbourhood of ∂W_g with $D^2 \times S^2$. The resulting pair (L, h) satisfies the following conditions: Let D_2 be a small 2-disc centred at p_2 in the 0-region of W_g, and let $\Lambda_{g,i}$, $i=1, 2, \cdots, n$, be the connected components of the closure of $\Lambda_g \setminus (\Lambda_0 \cup D_2)$ and let $p_{2i}=\Lambda_{g,i} \cap \partial D_2$. Then there is a decomposition

$$(L, h)=(L_1, h_1)\natural(L_2, h_2)\natural\cdots\natural(L_n, h_n)$$

which satisfies the following properties. In each W_{h_i} take a point $p_{0i}\in\partial W_{h_i}$ and an arc Λ_{0i} whose initial point is p_{0i} and whose terminal point is p_{2i} so that it passes through no other critical values $q_{h_i}(S(h_i))$. Let $\Lambda'_{g,i}$ be a tree in W_{h_i} which naturally corresponds to $\Lambda_{g,i}$ in W_g. Then $\Lambda'_{g,i}\cup\Lambda_{0,i}$ is a transversal tree of h_i ($i=1, 2, \cdots, n$), if we forget the orientations.

Let Λ_{h_i} denote the tree $\Lambda'_{g,i}\cup\Lambda_{0i}$ with the new orientation as a transversal tree of h_i. An example of the changes induced on Λ_g during the S-operation centred at p_2 is illustrated in Figure 7.1.

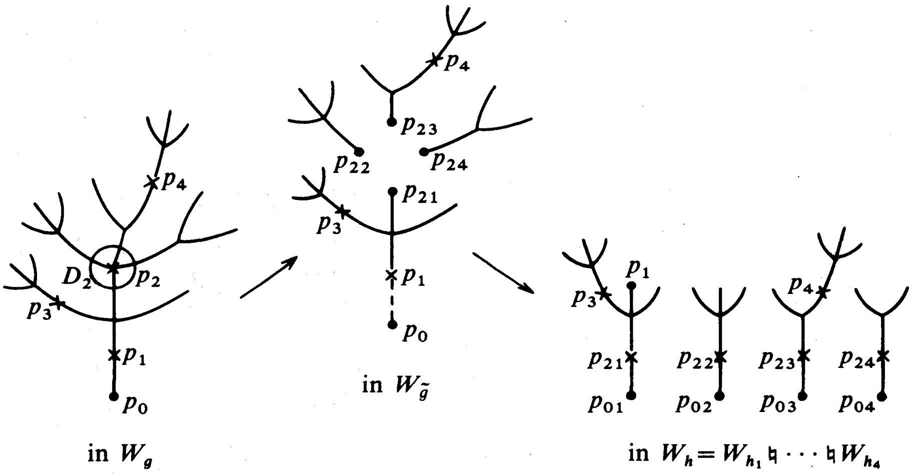


FIGURE 7.1

Note that each p_i in S except for p_2 has a unique corresponding point in some W_{h_i}, $i=1, 2, \cdots, n$. For convenience, we do not distinguish the original points from their corresponding points. Then, each Λ_{h_i} is divided into elementary trees at p_{2i} and at points in $S\setminus\{p_1, p_2\}$.

If Λ_{h_i} contains some p_j in S in its interior, then perform the S-operation centred at p_j to (L_i, h_i) and replace it with an S-equivalent pair (L'_i, h'_i).

In this way, by applying the procedure for all p_j's in S except for p_1, we can construct a pair $(N, f')=(N_1, f_1)\natural(N_2, f_2)\natural\cdots\natural(N_s, f_s)$ which is configuration trivial. Note that each elementary tree Λ_i of Λ_g except for the initial one is a transversal tree of some f_j, after added a suitable initial tree and given a suitable orientation.

Second step, cancelling and enumerating connected components of $S(f)$. Let (N, f) be a configuration trivial pair obtained from (N, f') by iterating C-operations so that the closure of no 1-region of W_f is diffeomorphic to the annulus. In the following $b_2(M)$ is denoted simply by b_2.

Case 1 where $\chi(M)$ is even. For each 1-region R of W_f, the boundary of $\bar{R}$ consists of one element of $q_f(S_-(f))$ and at least two elements of $q_f(S_+(f))$, by Lemma 5.1. This means $\# S_+(f) \geq 2\# S_-(f)$. From this and Lemma 3.1, one obtains $2\# S_-(f) \leq b_2$. This implies the required inequality, as in the proof of Theorem A.

Case 2 *where* f *is of type A*. Let C be the connected component of $q_f(S(f))$ which contains the image of the unique cusp. Let R_C be the 1-region of W_f which is adjacent to C. Note that $\partial\overline{\mathrm{R}_C}$ consists of C and k elements of $q_f(S_+(f))$ where $k \geq 1$, by Lemma 5.1. As in case 1, $\# S_+(f) - k \geq 2\# S_-(f)$, and one obtains $2\# S_-(f) \leq 1 + b_2 - 2k \leq b_2 - 1$. This implies the required inequality.

Case 3 *where* f *is of type B*. Let C and R_C be the same as in case 2. Note that $\partial\overline{\mathrm{R}_C}$ consists of C and l elements of $q_f(S_+(f))$ and an element of $q_f(S_-(f))$ where l is a non-negative integer. As in the previous cases, $\# S_+(f) - l \geq 2(\# S_-(f) - 1)$. One obtains $2\# S_-(f) \leq b_2 + 1 - 2l \leq b_2 + 1$, which implies the required inequality. q.e.d.

8. Proof of Theorem C.

To prove the theorem for the case $M \cong N \# C^2P \# \overline{C^2P}$, we need a result analogous to Theorem B. Let S'-equivalence be the relation defined by performing a connected sum with $C^2P \# \overline{C^2P}$ instead of $S^2 \times S^2$, in the definition of S-equivalence. Then an S'-operation is defined similarly as an S-operation. It is easily checked that Theorem B is valid if we replace "S-operation" with "S'-operation", and delete the phrase "*N is homeomorphic to M*". We call this result Theorem B$'$.

Case 1 *where* $b_2(N) \geq 1$. Let $g: N \to \boldsymbol{R}^2$ be a simple mapping with $g_g \leq 1$. Let (L, h) be a configuration trivial pair obtained from (N, g) by applying Theorem B or Theorem B$'$, according to the cases $M \cong N \# S^2 \times S^2$ or $N \# C^2P \# \overline{C^2P}$. For convenience, we assume that $\bar{h}: W_h \to \boldsymbol{R}^2$ is an embedding (see Remark (2) in section 2). We will construct the required pair (M, f) from (L, h) as follows.

Let C be one of the connected components of $q_h(S(h)) \setminus \partial W_h$ which bound the 0-region that is adjacent to ∂W_h. Such C exists; in fact, since $b_2(N) \geq 1$, we have $\# S(h) \geq 2$ by Theorem A. Let a be a point in ∂W_h. Take a tubular neighbourhood U of a in W_h so that $U \cap q_h(S(h)) = (U \cap C) \cup (U \cap \partial W_h)$ and that ∂U meets C transversely at two points. Let λ be the closure of $\partial U \setminus \partial W_h$. We may assume that λ is transverse to ∂W_h. By applying the arguments in [3], we see that $h^{-1}(U)$ is obtained from D^4 by a 1-handle attaching, thus diffeomorphic to $D^3 \times S^1$, and that $h^{-1}(\lambda) = \partial h^{-1}(U)$. We denote the closure of $L \setminus h^{-1}(U)$ by $\check{L}$.

Let $f_2: S^2 \times D^2 \to \boldsymbol{R}^2$ be a C^∞ mapping with two cusps constructed in Appendix 1 (see Figure 9.3). Then the two mappings $h|\partial\check{L}$ and $f_2|\partial(S^2 \times D^2)$ are right-left equivalent, which we show in Appendix 1. After changing h and f_2 both left equivalently, one can glue the two pairs $(\check{L}, h)$ and $(S^2 \times D^2, f_2)$ along their boundaries and obtains a pair (L', f') such that $f': L' \to \boldsymbol{R}^2$ is stable. Let f be the simple mapping obtained from f' by eliminating the two cusps (see the last step in the construction of a simple mapping on C^2P in Appendix 1). It is easily checked that $g_f \leq 1$, $\# S(f) = \# S(h) + 1$, and that (L', f) is configuration trivial. Therefore the proof is completed if one can choose the diffeomorphism of the gluing so that L' is diffeomorphic to M, which we prove in Appendix 1.

Case 2 *where* $b_2(N)=0$. We show that for such N, $N\# S^2\times S^2 \cong S^2\times S^2$ and $N\# \boldsymbol{C}^2P\#\overline{\boldsymbol{C}^2P}\cong \boldsymbol{C}^2P\#\overline{\boldsymbol{C}^2P}$. Then it is easily checked that the mappings given in Examples in Appendix 1 are the required ones (see Figure 9.1).

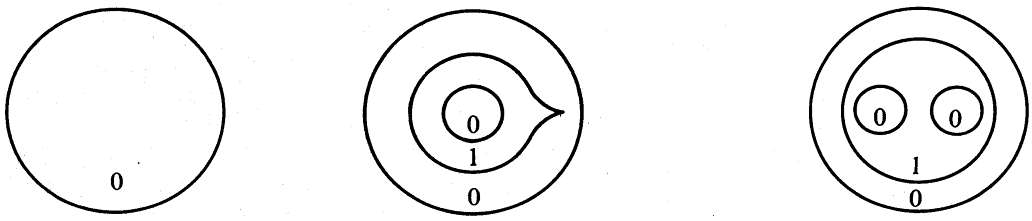
Applying Theorem B to (N, g), one obtains a pair (N', f') with $\# S(f')=1$. Then N' is diffeomorphic to S^4 (see Proposition 2.6 and Remark 2.7 in [6]). Therefore $S^2\times S^2\cong N'\# S^2\times S^2\cong N\# S^2\times S^2$ by Remark (2) in section 6. Next, let (N'', f'') be the pair obtained from (N, g) by applying Theorem B$'$. In the same way, we see that N'' is diffeomorphic to S^4 and $\boldsymbol{C}^2P\#\overline{\boldsymbol{C}^2P}\cong N''\#\boldsymbol{C}^2P\#\overline{\boldsymbol{C}^2P}\cong N\#\boldsymbol{C}^2P\#\overline{\boldsymbol{C}^2P}$. q.e.d.

Appendix 1.

In this section we give examples of simply connected 4-manifolds which admit simple mapping f's with $g_f\leq 1$ and we complete the proof of Theorem C using the same arguments as the construction of the examples.

EXAMPLES. (1) For $M=S^4$, $\boldsymbol{C}^2P$, $S^2\times S^2$ and $\boldsymbol{C}^2P\#\overline{\boldsymbol{C}^2P}$, we can construct a simple mapping $f\colon M\to\boldsymbol{R}^2$ with $g_f\leq 1$. The location of $q_f(S(f))$ in W_f, which is a 2-disc, is illustrated in Figure 9.1.

(2) Let M be a manifold obtained by performing finite connected sums of the manifolds above. Then there exists a simple mapping $f\colon M\to\boldsymbol{R}^2$ with $g_f\leq 1$.



The integers in the figures indicate the genus of the fibre over a point in the region.

FIGURE 9.1

PROOF OF (1). *Construction in the case of* S^4. Let $i\colon S^4\to\boldsymbol{R}^5$ be an embedding which maps S^4 onto the unit sphere. Let $\pi\colon\boldsymbol{R}^5\to\boldsymbol{R}^2$ be the projection given by $\pi(v, w, x, y, z)=(v, w)$. Then $\pi\circ i\colon S^4\to\boldsymbol{R}^2$ is the required simple mapping.

For the other manifolds, we construct the required mappings by the following steps. First we construct a mapping from B_k, the total space of a D^2 bundle over S^2 with Euler number k, into $\boldsymbol{R}^2$. Next we construct a mapping from the manifold M into

$\boldsymbol{R}^2$ using the decompositions $M=B_1\cup_\varphi D^4$, $B_0\cup_\phi B_0$ or $B_0\cup_{\phi'} B_0$ according to the cases where $M=\boldsymbol{C}^2P$, $S^2\times S^2$ or $\boldsymbol{C}^2P\sharp\overline{\boldsymbol{C}^2P}$.

Construction of a mapping f_2 from B_k into $\boldsymbol{R}^2$. Let f_1 be a C^∞-mapping from $D^2\times D^2$ onto a sector in $\boldsymbol{R}^2$ which satisfies the following conditions (see Appendix 2 for the construction).

(i) $f_1(D^2\times D^2)=\{(x,y)\mid x^2+y^2\le 1,\ x,y\ge 0\}$.

(ii) $f_1^{-1}(0,0)=\partial D^2\times\partial D^2$, $f_1^{-1}(0\times[0,1])=\partial D^2\times D^2$ and $f_1^{-1}([0,1]\times 0)=D^2\times\partial D^2$.

(iii) Set $\partial_0 f_1=f_1|(\partial D^2\times D^2)\colon \partial D^2\times D^2\to 0\times[0,1]$, and $\partial_1 f_1=f_1|(D^2\times\partial D^2)\colon D^2\times\partial D^2\to[0,1]\times 0$. Then $f_1|\mathrm{Int}(D^2\times D^2)$, $\partial_0 f_1$ and $\partial_1 f_1$ are stable mappings.

(iv) f_1 has exactly one cusp point in the interior of $D^2\times D^2$.

(v) Each regular fibre of f_1 is connected and is either a sphere or a torus.

(vi) Let $C(f_1)$ be the union of the critical values of $f_1|\mathrm{Int}(D^2\times D^2)$, $\partial_0 f_1$ and $\partial_1 f_1$. Then it consists of two connected components whose locations are illustrated in Figure 9.2.

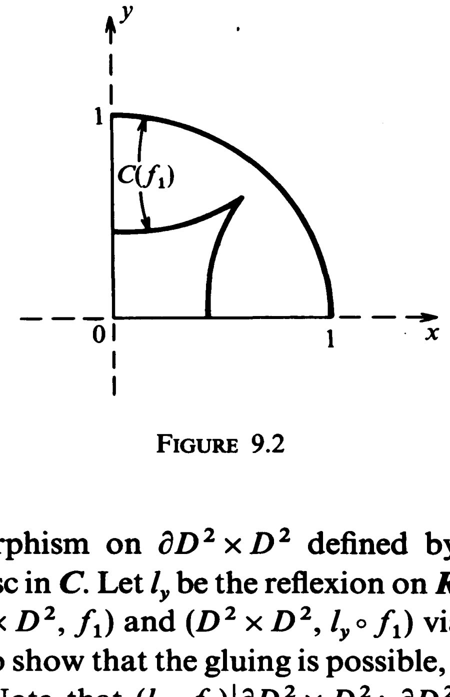


FIGURE 9.2

Let φ_k be a diffeomorphism on $\partial D^2\times D^2$ defined by $\varphi_k(z,w)=(\bar z, z^k\cdot w)$, where D^2 is regarded as the unit disc in $\boldsymbol{C}$. Let l_y be the reflexion on $\boldsymbol{R}^2$ given by $l_y(x,y)=(-x,y)$. We glue the two pairs $(D^2\times D^2, f_1)$ and $(D^2\times D^2, l_y\circ f_1)$ via φ_k and obtain a manifold B_k and a C^0-mapping f'_2. To show that the gluing is possible, we show that $f_1=l_y\circ f_1\circ\varphi_k$ on $\partial D^2\times D^2$, as follows. Note that $(l_y\circ f_1)|\partial D^2\times D^2\colon \partial D^2\times D^2\to 0\times[0,1]$ coincides with $\partial_0 f_1$. It is a Morse function which has two critical points of index 0 and of index 1. We can perturb φ_k slightly without changing its isotopy type so that it preserves all $\partial_0 f_1$-fibres, since φ_k preserves the meridian discs of $\partial D^2\times D^2$ (see Figure 3.1). Therefore one can glue the two pairs $(D^2\times D^2, f_1)$ and $(D^2\times D^2, l_y\circ f_1)$ via φ_k.

By the construction, the resulting manifold is diffeomorphic to B_k. Note that ∂B_k is a lens space, since it is obtained by gluing two solid tori. We see that it is diffeomorphic to $L(k,1)$, by calculation.

After we perturb f'_2 slightly, we obtain a C^∞-mapping $f_2\colon B_k\to\boldsymbol{R}^2$ such that both

$f_2|\mathrm{Int}\,B_k$ and $\partial f_2 = f_2|\partial B_k: \partial B_k \to [-1, 1] \times 0$ are stable mappings. Note that f_2 has two cusps in the interior of B_k. Let $C(f_2)$ be the union of the critical values of $f_2|\mathrm{Int}\,B_k$ and ∂f_2. The location of $C(f_2)$ is illustrated in Figure 9.3.

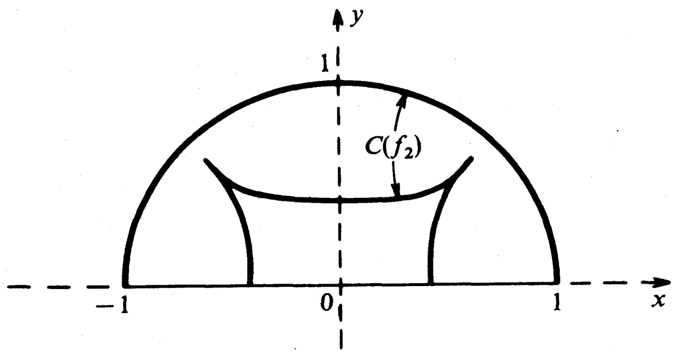


FIGURE 9.3

A construction in the case of C^2P. Let h: $D^4 \to \boldsymbol{R}^2$ be a C^∞-mapping satisfying the following conditions (see Appendix 2 for a construction).

(i)′ $h(D^4) = \{(x, y) \mid x^2 + y^2 \leq 1, x \geq 0\}$.
(ii)′ $h^{-1}(0 \times [-1, 1]) = \partial D^4$.
(iii)′ Set $\partial h = h|\partial D^4$: $\partial D^4 \to 0 \times [-1, 1]$. Then both $h|\mathrm{Int}\,D^4$ and ∂h are stable mappings.
(iv)′ h has exactly one cusp point in the interior of D^4.
(v)′ Each regular fibre of h is connected and is either a sphere or a torus.
(vi)′ Let $C(h)$ be the union of the critical values of $h|\mathrm{Int}\,D^4$ and ∂h. Then it consists of two connected components whose locations are illustrated in Figure 9.4.

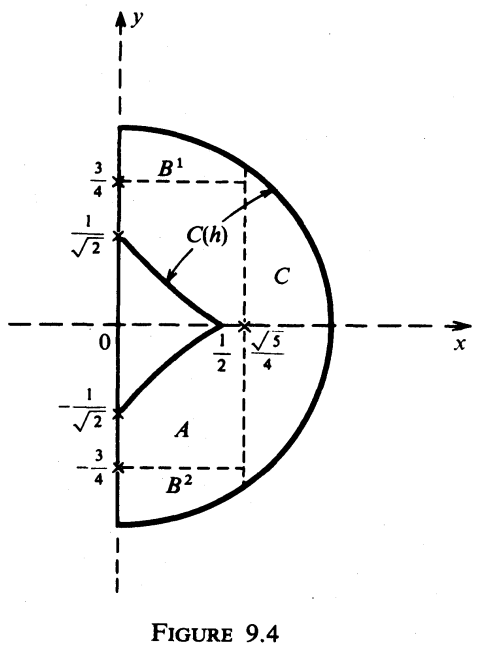


FIGURE 9.4

Let l_r be a slight perturbation of the $-\pi/2$-rotation on $\boldsymbol{R}^2$ centred at the origin such that l_r maps the critical values of ∂h to those of ∂f_2 and that $f_2(B_1)$ and $l_r \circ h(D^4)$ form a disc. We will glue the pairs (B_1, f_2) and $(D^4, l_r \circ h)$ as before. We must show that there is a diffeomorphism φ: $\partial B_1 \to \partial D^4$ such that $f_2 = l_r \circ h \circ \varphi$ on ∂B_1.

Let C be the connected component of $C(f_2)$ which contains the image of the cusps. Let C_0 be the connected component of $C \setminus f_2(\{\text{cusps}\})$ which meets the negative part of the x-axis, and C_1 the one which meets the positive part of the x-axis. Let C' be the connected component of $C(h)$ which contains the image of the cusp. Let C'_0 be the connected component of $C' \setminus h(\{\text{cusp}\})$ which meets the negative part of the y-axis, and C'_1 the other. Take longitudes and meridians (l_0, m_0) and (l_1, m_1) of $f_2^{-1}(0, 0)$ with respect to $(C_0, [-1, 0] \times 0)$ and $(C_1, [0, 1] \times 0)$ respectively. Take longitudes and meridians (l'_0, m'_0) and (l'_1, m'_1) of $h^{-1}(0, 0)$ with respect to $(C'_0, 0 \times [-1, 0])$ and $(C'_1, 0 \times [0, 1])$ respectively.

We fix orientations of B_1 and D^4, and give orientations to the longitudes and meridians as follows. Set $\partial_0 = f_2^{-1}([-1, 0] \times 0)$ and set $\partial_1 = f_2^{-1}([0, 1] \times 0)$. We give orientations to ∂_0 and ∂_1 so that each of them coincides with the orientation of ∂B_1 in the interiors. Then give orientations to l_i and m_i so that, lined in this order, they coincide with the orientation of $f_2^{-1}(0, 0)$ as the boundary of ∂_i for $i = 0, 1$. (This says nothing on the choice of the orientation of m_i. We fix one.) In the same way, we orient l'_i and m'_i by using the orientation of D^4.

Let

$$A = \begin{pmatrix} r & p \\ s & q \end{pmatrix}$$

and

$$B = \begin{pmatrix} z & u \\ w & v \end{pmatrix}$$

be the matrices given by $[l_1] = r[l_0] + s[m_0]$ and $[m_1] = p[l_0] + q[m_0]$ in $H_1(f_2^{-1}(0, 0), \boldsymbol{Z})$, and $[-l'_1] = z[-l'_0] + w[m'_0]$ and $[m'_1] = u[-l'_0] + v[m'_0]$ in $H_1(h^{-1}(0, 0), \boldsymbol{Z})$.

One can take an orientation reversing diffeomorphism φ_0 from $f_2^{-1}([-1, 0] \times 0)$ to $h^{-1}(0 \times [-1, 0])$ such that $f_2 | f_2^{-1}([-1, 0] \times 0) = l_r \circ h \circ \varphi_0$. Then the induced isomorphism $(\varphi_0 | f_2^{-1}(0, 0))_*$ from $H_1(f_2^{-1}(0, 0), \boldsymbol{Z})$ to $H_1(h^{-1}(0, 0), \boldsymbol{Z})$ has a matrix representation of the form $\pm T_c$, $c \in \boldsymbol{Z}$, for the basis $\langle [l_0], [m_0] \rangle$ of the domain and $\langle [-l'_0], [m'_0] \rangle$ of the image, by Proposition 3.2. In the same way, let φ_1 be an orientation reversing diffeomorphism from $f_2^{-1}([0, 1] \times 0)$ to $h^{-1}(0 \times [0, 1])$ such that $f_2 | f_2^{-1}([0, 1] \times 0) = l_r \circ h \circ \varphi_1$. Let $\pm T_d$, $d \in \boldsymbol{Z}$, be the matrix representation of the induced isomorphism $(\varphi_1 | f_2^{-1}(0, 0))_*$ for the bases $\langle [l_1], [m_1] \rangle$ and $\langle [-l'_1], [m'_1] \rangle$.

The two diffeomorphisms φ_0 and φ_1 can be glued so as to define an orientation reversing diffeomorphism φ from ∂B_1 to ∂D^4 if and only if the following $(*)$ holds.

$$(*) \qquad \pm T_c \cdot A = B \cdot (\pm T_d).$$

By a direct calculation, $(*)$ holds for some integers c and d if and only if $p = \pm u$, $q \equiv \pm v (\mathrm{mod}\,|p|)$ and $r \equiv \pm z (\mathrm{mod}\,|p|)$. Now we have $|p| = |u| = 1$, and hence these conditions are satisfied. In fact ∂B_1 is identified with $S^1 \times D^2 \cup_A S^1 \times D^2 = L(p, q)$ and since ∂B_1 is also identified with S^3, p must be ± 1. In the same way, u must be ± 1. Therefore one can glue φ_0 and φ_1. The resulting diffeomorphism $\varphi : \partial B_1 \to \partial D^4$ reverses orientation and satisfies $f_2 = l_r \circ h \circ \varphi$ on ∂B_1. Now glue the pairs (B_1, f_2) and $(D^4, l_r \circ h)$ via φ so as to obtain $\boldsymbol{C}^2P$ and a C^0-mapping f'_3 from $\boldsymbol{C}^2P$ into $\boldsymbol{R}^2$. After a slight perturbation, f'_3 becomes a C^∞-stable mapping f_3.

Next, one can eliminate two of the three cusps of f_3 by applying the method of [4] carefully to f_3, and obtains a simple mapping f. In fact take the joining curve (which is defined in (4.4) of [4]) that connects two of the three cusps so that its image does not meet the critical values of f_3 except for the two cusps. Then one can eliminate the two cusps without creating any crossings of $f(S(f))$ and without changing the maximum genus of regular map-fibres. The mapping f thus obtained is the required one (refer to Figure 9.1 (b)).

Constructions for the cases of $S^2 \times S^2$ *and* $\boldsymbol{C}^2P \# \overline{\boldsymbol{C}^2P}$. Let l_x be the reflexion on $\boldsymbol{R}^2$ given by $l_x(x, y) = (x, -y)$. We will glue the pairs (B_0, f_2) and $(B_0, l_x \circ f_2)$. Assuming that the gluing is possible, let f_3 be the resulting mapping from the resulting manifold into $\boldsymbol{R}^2$. Then one can eliminate the four cusps of f_3 in pairs and obtains a required simple mapping as in the previous cases. Therefore it suffices to show the following: There are two diffeomorphisms φ_0 (resp. φ'_0) on $f_2^{-1}([-1, 0] \times 0)$ and φ_1 (resp. φ'_1) on $f_2^{-1}([0, 1] \times 0)$ which can be glued such that the resulting diffeomorphism ϕ (resp. ϕ') on ∂B_0 is orientation reversing and satisfies, (a) $f_2 | \partial B_0 = l_x \circ f_2 \circ \phi$ (resp. $f_2 | \partial B_0 = l_x \circ f_2 \circ \phi'$), and (b) $B_0 \cup_\phi B_0 \cong S^2 \times S^2$, $B_0 \cup_{\phi'} B_0 \cong \boldsymbol{C}^2P \# \overline{\boldsymbol{C}^2P}$.

Take C_i, l_i, m_i $(i = 0, 1)$ and integers p, q, r, s which are the elements of a matrix A, similarly as in the cases of B_1. Since ∂B_0 is identified with $L(p, q)$ and with $S^1 \times S^2$, p is zero and hence $q = \pm 1$. Therefore one may assume that $A = I'$ by an appropriate choice of m_1 and l_1, where I' is the matrix defined by

$$I' = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Let φ_0 (resp. φ'_0) be a diffeomorphism on $f_2^{-1}([-1, 0] \times 0)$ such that the induced isomorphism $(\varphi_0 | f_2^{-1}(0, 0))_*$ (resp. $(\varphi'_0 | f_2^{-1}(0, 0))_*$) on $H_1(f_2^{-1}(0, 0), \boldsymbol{Z})$ has the matrix representation T_0 (resp. T_1) for the basis $\langle [l_0], [m_0] \rangle$ of the domain and $\langle [-l_0], [m_0] \rangle$ of the image. Let φ_1 (resp. φ'_1) be a diffeomorphism on $f_2^{-1}([0, 1] \times 0)$ such that the induced isomorphism $(\varphi_1 | f_2^{-1}(0, 0))_*$ (resp. $(\varphi'_1 | f_2^{-1}(0, 0))_*$) on $H_1(f_2^{-1}(0, 0), \boldsymbol{Z})$ has the matrix representation T_0 (resp. T_{-1}) for the bases $\langle [l_1], [m_1] \rangle$ and $\langle [-l_1], [m_1] \rangle$. We can take these mappings so that they preserve all f_2-fibres, by Proposition 3.2.

Since the condition $(*)$ is verified for $A=B=I'$, one can glue φ_0 and φ_1 (resp. φ'_0 and φ'_1). Let ϕ (resp. ϕ') be the resulting diffeomorphism. Then they are orientation reversing diffeomorphisms on ∂B_0 and satisfy condition (a).

Now we check the condition (b). Note that one can identify $S^1\times D^2\cup_A S^1\times D^2$ with $S^1\times S^2$ since both of them are identified with ∂B_0. Via the identification, $\{t\}\times D^2\cup\{t\}\times D^2$ corresponds to $\{t\}\times S^2$ $(t\in S^1)$. We say a point p_t (resp. n_t) in $\{t\}\times S^2$ is a north pole (resp. south pole) if it is identified with $(t, 0)$ of the former $\{t\}\times D^2$ (resp. the latter $\{t\}\times D^2$). Since we may assume that φ_i and φ'_i $(i=0, 1)$ induce diffeomorphisms on $S^1\times D^2$ which preserve $\{t\}\times D^2$ $(t\in S^1)$, ϕ and ϕ' satisfy $\phi(\{t\}\times S^2)=\{t\}\times S^2$ and $\phi'(\{t\}\times S^2)=\{t\}\times S^2$. Moreover ϕ does not twist $\{t\}\times S^2$ according as t moves, since $\varphi_i(l_i)$ $(i=0, 1)$ does not turn around $S^1\times\{0\}\subset S^1\times D^2$ and ϕ' twists $\{t\}\times S^2$ once around the north and south poles according as t moves, since $\varphi'_i(l_i)$ $(i=0, 1)$ turns once around $S^1\times\{0\}\subset S^1\times D^2$. Therefore $B_0\cup_\phi B_0$ is the total space of the trivial S^2 bundle over S^2 and $B_0\cup_{\phi'} B_0$ is that of the non-trivial S^2 bundle over S^2, which is diffeomorphic to $\boldsymbol{C}^2P\sharp\overline{\boldsymbol{C}^2P}$ ([9]). q.e.d.

PROOF OF (2). Let (M_i, f_i) be one of the pairs of a manifold and a simple mapping constructed in (1). By eliminating extra cusps of $\sharp_i f_i$: $\sharp_i M_i\to\boldsymbol{R}^2$ as in (1), one obtains a simple mapping f: $\sharp_i M_i\to\boldsymbol{R}^2$ with $g_f\leq 1$. q.e.d.

REMARK. By using the method of (2), one can construct a simple mapping on $\boldsymbol{C}^2P\sharp\overline{\boldsymbol{C}^2P}$. Let f be a simple mapping on $\boldsymbol{C}^2P$ constructed in (1). Note that $\overline{\boldsymbol{C}^2P}$ is obtained by gluing B_{-1} and D^4. One can construct a simple mapping g on $\overline{\boldsymbol{C}^2P}$, in the same way as f. Then, from $f\sharp g$, one obtains a simple mapping on $\boldsymbol{C}^2P\sharp\overline{\boldsymbol{C}^2P}$.

COMPLETION OF THE PROOF OF THEOREM C. We show that there is a diffeomorphism Φ: $\partial\check{L}\to\partial(S^2\times D^2)$ such that, (a′) $h|\partial\check{L}$ and $(f_2|\partial(S^2\times D^2))\circ\Phi$ are left equivalent, and (b′) $\check{L}\cup_\Phi S^2\times D^2$ is diffeomorphic to M.

Take a point a in λ so that $q_h^{-1}(a)$ is a torus. Then divide λ into two closed arcs I_0 and I_1 with $I_0\cap I_1=\{a\}$. Since I_0 and I_1 meet C transversely at each single point, one can choose a longitude and a meridian (l_i, m_i) of $q_h^{-1}(a)$ with respect to (C, I_i), $i=0, 1$. One may assume that $[m_1]=[m_0]$ and $[l_1]=-[l_0]$ as in the proof of (1), since $\partial\check{L}$ is diffeomorphic to $S^1\times S^2$. Let (l'_i, m'_i) be the longitude and meridian of $f_2^{-1}(0, 0)$ chosen for B_0 in the proof of (1). Then one can define diffeomorphisms ϕ and ϕ' from $\partial\check{L}$ to $\partial(S^2\times D^2)$ in the same way as in the proof of (1), using these longitudes and meridians. The condition (a′) is checked in the same way. For the condition (b′), one can show that either $\check{L}\cup_\phi S^2\times D^2$ or $\check{L}\cup_{\phi'} S^2\times D^2$ is diffeomorphic to $N\sharp S^2\times S^2$ and that the other is diffeomorphic to $N\sharp\boldsymbol{C}^2P\sharp\overline{\boldsymbol{C}^2P}$. Therefore either ϕ or ϕ' can be taken as Φ. q.e.d.

Appendix 2. Construction of the mappings used in Appendix 1.

In this section we construct the mappings $f_1: D^2 \times D^2 \to \boldsymbol{R}^2$ and $h: D^4 \to \boldsymbol{R}^2$ which were used in Appendix 1.

Once h is constructed, one can construct f_1 from h as follows. Set $E = \{(x, y) \in \boldsymbol{R}^2 \mid x^2 + y^2 \leq 1, x \geq 0\}$ and let S be the sector in E enclosed by the two lines $l_1: x = y$ and $l_2: x = -y$. One may assume that the line $x = ty$ and $C(h)$ meet transversely for $t \in [-1, 1]$. Then $h^{-1}(S)$ is a manifold with boundary $h^{-1}(l_1 \cap E) \cup h^{-1}(l_2 \cap E)$ and corner $h^{-1}(0, 0)$. We show that $h^{-1}(S)$ is diffeomorphic to $D^2 \times D^2$.

By an appropriate choice of coordinates, one may assume that $h^{-1}(E) = \{(z, w) \in \boldsymbol{C}^2 \mid |z|^2 + |w|^2 \leq 1\}$ and $h^{-1}(0 \times [-1, 0])$(resp. $h^{-1}(0 \times [1, 0])) = \{(z, w) \in \boldsymbol{C}^2 \mid |z|^2 + |w|^2 = 1, |z| \leq |w|$ (resp. $|z| \geq |w|)\}$. Note that $h^{-1}(E \cap \{(x, y) \mid x < y\})$ is an open tubular neighbourhood of $h^{-1}(0 \times [1, 0]) \setminus h^{-1}(0, 0)$. Therefore $h^{-1}(E \cap \{(x, y) \mid x < y\})$ is isotopic to $\{(z, w) \in \boldsymbol{C}^2 \mid |z|^2 + |w|^2 \leq 1, |z| > 1/\sqrt{2}\}$ by the uniqueness of the tubular neighbourhood. In the same way, $h^{-1}(E \cap \{(x, y) \mid x < -y\})$ is isotopic to $\{(z, w) \in \boldsymbol{C}^2 \mid |z|^2 + |w|^2 \leq 1, |w| > 1/\sqrt{2}\}$. Therefore $h^{-1}(S)$ is diffeomorphic to $\{(z, w) \in \boldsymbol{C}^2 \mid |z|^2 + |w|^2 \leq 1, |z| \leq 1/\sqrt{2}, |w| \leq 1/\sqrt{2}\}$, and hence is diffeomorphic to $D^2 \times D^2$.

It is easily checked that the restriction of h to $h^{-1}(S)$ satisfies the conditions (i) through (vi) after composed with a suitable diffeomorphism on $\boldsymbol{R}^2$. Therefore we have only to construct h.

We divide E into four pieces and construct h step by step over these pieces. Set $A = [0, \sqrt{5}/4] \times [-3/4, 3/4]$, which will include the connected component of $C(h)$ that contains the image of the cusp. Let B^1 and B^2 be the two connected components of $E \cap \{(x, y) \mid x \leq \sqrt{5}/4, |y| \geq 3/4\}$ and set $C = E \cap \{(x, y) \mid x \geq \sqrt{5}/4\}$ (see Figure 9.4). Then $h^{-1}(A)$ and $h^{-1}(A \cup B^1 \cup B^2)$ will be diffeomorphic to $[0, \sqrt{5}/4] \times [-1, 1] \times S^2$ and $[0, \sqrt{5}/4] \times S^3$ respectively, and $h^{-1}(C)$ will be diffeomorphic to D^4.

First step, construction on $[0, \sqrt{5}/4] \times [-1, 1] \times S^2$. Let $H: [0, 3/4] \times \boldsymbol{R}^3 \to \boldsymbol{R}^2$ be the stable mapping defined by $H(u, x, y, z) = (u, x^3 + 3(u - 1/2)x + y^2 - z^2)$. H has a cusp point at $(1/2, 0, 0, 0)$ and the critical values $C(H)$ divide the image of H into two regions. If $(u, a) \in \mathrm{Im}(H)$ is in the same region as $(0, 0)$, then $H^{-1}(u, a)$ is diffeomorphic to $T^2 \setminus D^2$. If $(u, a) \in \mathrm{Im}(H)$ is in the other region, then $H^{-1}(u, a)$ is diffeomorphic to $S^2 \setminus D^2$ (see Figure 10.1).

Now we restrict H to the subset F of $\boldsymbol{R}^4$ defined by $F = \bigcup_{(u,a)} F_{u,a}$, $(u, a) \in [0, \sqrt{5}/4] \times [-1, 1]$ where $F_{u,a} = u \times [-3/2, 3/2] \times Q \cap H^{-1}(u, a) \subset \boldsymbol{R}^4$ with $Q = \{(y, z) \in \boldsymbol{R}^2 \mid |y^2 - z^2| \leq 5, |y \pm z| \leq 3\}$ (see Figure 10.1).

The following properties are checked in an elementary way. For all $u \in [0, \sqrt{5}/4]$,

(1) $F_{u,a}$ is connected for $a \in [-1, 1]$,

(2) $\partial F_{u,a}$ is a circle with eight corners for $a \in [-1, 1]$,

(3) $F_{u,\pm 1}$ is a manifold with eight corners on the boundary, and its interior is diffeomorphic to $\mathrm{Int}\, D^2$,

(4) $F_{u,a} \cap (u, \pm 3/2) \times Q \neq \varnothing$ for $a \in [-1, 1]$,

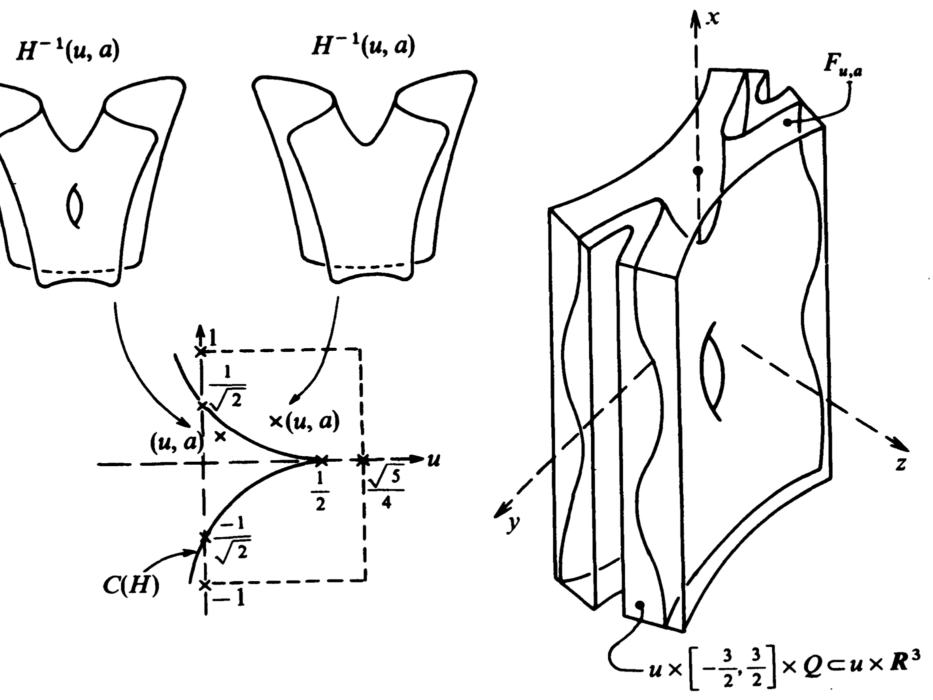
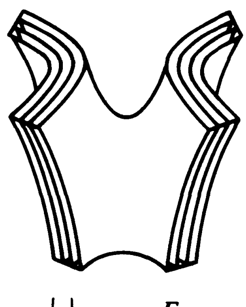


FIGURE 10.1



$\bigcup_{a\in[-1,1]}F_{u,a}$

FIGURE 10.2

(5) let $\Phi_{u,t}\colon \bigcup_{a\in[-1,1]}F_{u,a}\cap(u,t)\times Q\to\boldsymbol{R}$ be the mapping defined by $\Phi_{u,t}(u,t,y,z)=y^2-z^2$, then $\mathrm{Im}(\Phi_{u,t})$ is a closed subinterval of $(-5,5)$ for all $t\in[-3/2,3/2]$,

(6) if $(u,3/2,y,z)\in F_{u,a}$, then $y^2-z^2<0$ for $a\in[-1,1]$,

(7) if $(u,-3/2,y,z)\in F_{u,a}$, then $y^2-z^2>0$ for $a\in[-1,1]$,

(8) $\bigcup_{a\in[-1,1]}F_{u,a}$ contains the singular points of $g_u(x,y,z)=x^3+3(u-1/2)x+y^2-z^2$.

We see that $\bigcup_{a\in[-1,1]}F_{u,a}$ is a thick saddle, by (4) through (7), and that $H|\bigcup_{(u,a)\in[0,\sqrt{5}/4]\times[-1,1]}\partial F_{u,a}$ is a trivial fibration with fibre $\partial F_{u,a}$, by (2) (see Figure 10.2). Therefore one can naturally extend $H|F$ to $H_1\colon[0,\sqrt{5}/4]\times[-1,1]\times S^2\to[0,\sqrt{5}/4]\times[-1,1]$ by attaching the pair $([0,\sqrt{5}/4]\times[-1,1]\times D^2,\pi)$ to $(F,H|F)$, where $\pi\colon[0,\sqrt{5}/4]\times[-1,1]\times D^2\to[0,\sqrt{5}/4]\times[-1,1]$ is the projection. Note that the restriction of H_1 to $[0,\sqrt{5}/4]\times\{\pm1\}\times S^2$ is the projection with sphere-fibres by

(3), and that H_1 has a cusp point in $(1/2, 0)\times S^2$ by (8).

Second step, extension to $[0, \sqrt{5}/4]\times S^3$. Define $k\colon D_1^3\to[1, 2]$ by $k(x, y, z)=2-x^2-y^2-z^2$, where D_1^3 denotes the unit 3-ball centred at the origin. After slight perturbation of mappings, one can glue the three pairs $([0, \sqrt{5}/4]\times D_1^3, \mathrm{id}\times k)$, $([0, \sqrt{5}/4]\times[-1, 1]\times S^2, H_1)$ and $([0, \sqrt{5}/4]\times D_1^3, \mathrm{id}\times(-k))$ one after the other along $[0, \sqrt{5}/4]\times\{\pm 1\}\times S^2$ so as to obtain a C^∞-mapping $H_2\colon [0, \sqrt{5}/4]\times S^3\to[0, \sqrt{5}/4]\times[-2, 2]$.

Third step, extension to D^4. Define $l\colon D_1^4\to \boldsymbol{R}^2$ by $l(u, x, y, z)=(1-((4-\sqrt{5})/4)\cdot(u^2+x^2+y^2+z^2), 2u)$ where D_1^4 denotes the unit 4-ball centred at the origin. Then $l|\partial D_1^4\colon \partial D_1^4\to\sqrt{5}/4\times[-2, 2]$ and $H_2|\sqrt{5}/4\times S^3\colon \sqrt{5}/4\times S^3\to\sqrt{5}/4\times[-2, 2]$ are right-equivalent Morse functions. Therefore after slight perturbations of l and H_2, one can glue the pairs $([0, \sqrt{5}/4]\times S^3, H_2)$ and (D_1^4, l) so as to obtain a C^∞-mapping $H_3\colon D^4\to\boldsymbol{R}^2$.

It is obvious from the construction that $h=\psi\circ H_3$ satisfies the conditions (i)′ through (vi)′, for an appropriate diffeomorphism ψ on $\boldsymbol{R}^2$. Therefore we have constructed the required mapping h. q.e.d.

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