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Complete Classification of Periodic Maps on Compact Surfaces

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Abstract. In the previous paper [9] and [10], we gave some classification theorems of periodic maps with isolated singular sets on compact surfaces. In this paper, we will give complete classification theorems of periodic maps on compact surfaces as similar formulae to those of [9] and [10].

§0. Introduction.

A homeomorphism $f: M \to M$ of a space M onto itself is called a periodic map on M with period n if $f^n = \text{identity}$ and $f^k \neq \text{identity}$ for $1 \leqq k < n$. We say that a periodic map f on M is *equivalent* to a periodic map f' on M' if there exists a homeomorphism $h: M \to M'$ such that $f'h = hf$.

We consider a pair (f, M), where M is a compact connected surface and f is a periodic map on M with period n. Let

$$\mathscr{S}_k(f) = \{x \in M \,;\, f^k(x) = x,\, f^i(x) \neq x \text{ for } 1 \leqq i < k\}\,, \quad \text{and} \quad \mathscr{S}(f) = \bigcup_{k<n} \mathscr{S}_k(f)\,,$$

which will be called the *singular set* of f. By Whyburn [8], we have the following:

PROPOSITION 0.1. *The orbit space M/f is a compact surface.*

PROPOSITION 0.2. *A connected component of $\mathscr{S}(f)$ is one of the following types;*
(1) *an isolated point in* $\mathring{M}$, where $\circ$ means the interior,
(2) *a simple loop in* $\mathring{M}$,
(3) *a proper simple arc in* M.

By $\mathscr{S}^0(f)$ we denote the subset of $\mathscr{S}(f)$ which consists of isolated points in $\mathring{M}$, and let $\mathscr{S}^1(f) = \mathscr{S}(f) - \mathscr{S}^0(f)$.

In [9] and [10], we showed some classification theorems for periodic maps on compact connected surfaces such that $\mathscr{S}^1(f) = \varnothing$, up to equivalence. In this paper, we show some classification theorems for periodic maps on compact connected surfaces

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such that $\mathscr{S}^1(f)\neq\emptyset$, up to equivalence. Therefore, we will complete the classification for periodic maps on compact surfaces, up to equivalence. The main idea of this paper is the following.

For a compact connected surface M and a periodic map f on M, if $\mathscr{S}^1(f)\neq\emptyset$, then by cutting M along $\mathscr{S}^1(f)$, we can decide a certain compact connected surface, say M_*, and a periodic map f_* on M_* naturally determined by f with $\mathscr{S}^1(f_*)=\emptyset$ and some necessary conditions on ∂M_*. We prove that the reducing operation of (f, M) to (f_*, M_*) is bijective, and applying the classification theorems of [9] and [10] to (f_*, M_*)'s, we can obtain our classification theorems for (f, M)'s.

After establishing definitions and notations, we give in §1 the main theorems. In §2 we show some fundamental lemmas for periodic maps on compact surfaces, which are of importance in the sequel. In particular, we discuss the reducing process of (f, M) to (f_*, M_*), and in §3 we determine the equivalence classes of periodic maps (f_*, M_*)'s. Applying our theorems, we discuss in §4 periodic maps on the Klein bottle and the torus, which will be of help to understand our theory, and in §5 we prove our main theorems (Theorems A.1, A.2, A.3, A.4, B.1 and B.2).

Moreover, applying our main theorems, we can construct all (f, M), and determine whether any two of them are equivalent or not.

We use the following notations in this paper: $\binom{a}{b}=a!/(b!(a-b)!)$, $\#A$ is the number of the elements of the set A, $\lfloor x \rfloor$ is the smallest integer $\geqq x$, $\lceil x \rceil$ is the largest integer $\leqq x$, $\varphi(x)$ is the Euler function and $\mu(d)$ is the Möbius function.

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§1. Definitions, notation and main results.

Let $P_n(M)=P_n$ be the set of elements (f, M) with $\mathscr{S}^1(f)\neq\emptyset$, where M is a compact connected surface and f is a periodic map on M with period n.

For an element (f, M) of P_n, there exist a natural quotient space $X=M/f$ and a natural quotient map $p\colon M\to X$. We put $p(\mathscr{S}(f))=S$.

The discussion is divided into two cases according as $M-\mathscr{S}^1(f)$ connected or disconnected.

Case A: $M-\mathscr{S}^1(f)$ **connected.** We denote by $\hat{P}_n$ the subset of P_n consisting of elements (f, M) with $M-\mathscr{S}^1(f)$ connected. For an element (f, M) of $\hat{P}_n$ we construct an element $(\hat{f}, \hat{M}, S_*)$ in §2, where $\hat{M}$ is a natural compactification of $M-\mathscr{S}^1(f)$, $\hat{f}$ is a periodic map on $\hat{M}$ with period n such that $\mathscr{S}^1(\hat{f})=\emptyset$, and $S_*=\hat{M}-(M-\mathscr{S}^1(f))$. We denote by $\hat{\boldsymbol{P}}_n$ the set of elements $(\hat{f}, \hat{M}, S_*)$, and define an equivalence relation $\underset{*}{\sim}$ on $\hat{\boldsymbol{P}}_n$ in §2. Then we have a bijection between the set of equivalence classes $\hat{P}_n/\sim$ and the set of equivalence classes $\hat{\boldsymbol{P}}_n/\underset{*}{\sim}$. So, we classify $\hat{\boldsymbol{P}}_n$.

By means of the orientability of M, $\hat{M}$ and X, we divide $\hat{P}_n$ into the following four sets:

$\hat{P}_n^{+-}=\{(f, M)\in \hat{P}_n$; M is non-orientable, $\hat{M}$ is orientable and X is orientable$\}$,

$\hat{P}_n^{-+}=\{(f, M)\in \hat{P}_n$; M is orientable, $\hat{M}$ is orientable and X is non-orientable$\}$,

$\hat{P}_n^{--}=\{(f, M)\in P_n$; M is non-orientable, $\hat{M}$ is orientable and X is non-orientable$\}$,

$\hat{P}_n^{oo}=\{(f, M)\in \hat{P}_n$; M is non-orientable, $\hat{M}$ is non-orientable and X is non-orientable$\}$.

Let $\hat{P}_n^{\varepsilon\varepsilon'}(\tilde{g}, \tilde{l}, \tilde{r}, \tilde{m}, \tilde{q}^+, \tilde{q}^-, \tilde{t}^+, \tilde{t}^-; \tilde{\boldsymbol{l}}, \tilde{\boldsymbol{m}}, \tilde{\boldsymbol{q}}^+, \tilde{\boldsymbol{q}}^-, \tilde{\boldsymbol{t}}^+, \tilde{\boldsymbol{t}}^-)$ be the set of elements (f, M) of $\hat{P}_n^{\varepsilon\varepsilon'}$, where $(\varepsilon, \varepsilon')=(+, -)$, $(-, +)$, $(-, -)$ or (o, o), satisfying the following nine conditions:

(1) M is a compact surface of genus $\tilde{g}$ and ∂M consists of $\tilde{l}$ components $D_1, D_2, \cdots, D_{\tilde{l}}$, and $\tilde{r}$ components $D^*_{\tilde{l}+1}, D^*_{\tilde{l}+2}, \cdots, D^*_{\tilde{l}+\tilde{r}}$, where $D_j \cap \mathscr{S}^1(f)=\varnothing$ $(1\leqq j\leqq \tilde{l})$ and $D^*_j \cap \mathscr{S}^1(f)\neq\varnothing$ $(\tilde{l}+1\leqq j\leqq \tilde{l}+\tilde{r})$.

(2) $\mathscr{S}^0(f)$ consists of $\tilde{m}$ points $S_1, S_2, \cdots, S_{\tilde{m}}$ in $\mathring{M}$.

(3) $\mathscr{S}^1(f)$ consists of the following sets;

(i) 2-sided loops $D_1^{o+}, D_2^{o+}, \cdots, D_{\tilde{q}^+}^{o+}$ in $\mathring{M}$,

(ii) 1-sided loops $D_1^{o-}, D_2^{o-}, \cdots, D_{\tilde{q}^-}^{o-}$ in $\mathring{M}$,

(iii) simple proper arcs in 2-sided Φ-sets $\Phi_1^+, \Phi_2^+, \cdots, \Phi_{\tilde{t}^+}^+$ in M,

(iv) simple proper arcs in 1-sided Φ-sets $\Phi_1^-, \Phi_2^-, \cdots, \Phi_{\tilde{t}^-}^-$ in M.

(The definition of the Φ-set is given in §2.)

(4) $\tilde{\boldsymbol{l}}=(\tilde{l}_a)_{a|n}$ is a vector of non-negative integers $\tilde{l}_a$, where

$$\tilde{l}_a=\#\{D_j\ ; f^a(D_j)=D_j \text{ and } f^b(D_j)\neq D_j \text{ for } 1\leqq\forall b<a\}$$

for each divisor a of n.

(5) $\tilde{\boldsymbol{m}}=(\tilde{m}_a)_{a|n}$ is a vector of non-negative integers $\tilde{m}_a$, where

$$\tilde{m}_a=\#\{S_k\ ;\ f^a(S_k)=S_k \text{ and } f^b(S_k)\neq S_k \text{ for } 1\leqq\forall b<a\}$$

for each divisor a of n.

(6) $\tilde{\boldsymbol{q}}^+=(\tilde{q}_a^+)_{a|n}$ is a vector of non-negative integers $\tilde{q}_a^+$, where

$$\tilde{q}_a^+=\#\{D_j^{o+}\ ;\ f^a(D_j^{o+})=D_j^{o+} \text{ and } f^b(D_j^{o+})\neq D_j^{o+} \text{ for } 1\leqq\forall b<a\}$$

for each divisor a of n.

(7) $\tilde{\boldsymbol{q}}^-=(\tilde{q}_a^-)_{a|n}$ is a vector of non-negative integers $\tilde{q}_a^-$, where

$$\tilde{q}_a^-=\#\{D_j^{o-}\ ;\ f^a(D_j^{o-})=D_j^{o-} \text{ and } f^b(D_j^{o-})\neq D_j^{o-} \text{ for } 1\leqq\forall b<a\}$$

for each divisor a of n.

(8) $\tilde{\boldsymbol{t}}^+=(\tilde{t}_a^+(\tilde{v}))_{a|n}$ is a vector of non-negative integers $\tilde{t}_a^+(\tilde{v})$, where

$$\tilde{t}_a^+(\tilde{v})=\#\{\Phi_w^+\ ;\ f^a(\Phi_w^+)=\Phi_w^+ \text{ and } f^b(\Phi_w^+)\neq\Phi_w^+ \text{ for } 1\leqq\forall b<a\}$$

for each divisor a of n, and $\tilde{v}$ is the number of arcs in $\Phi_w^+ \cap \mathscr{S}^1(f)$.

(9) $\tilde{t}^- = (\tilde{t}_a^-(\tilde{v}))_{a|n}$ is a vector of non-negative integers $\tilde{t}_a^-(\tilde{v})$, where

$$\tilde{t}_a^-(\tilde{v}) = \#\{\Phi_w^- \,;\, f^a(\Phi_w^-) = \Phi_w^- \text{ and } f^b(\Phi_w^-) \neq \Phi_w^- \text{ for } 1 \leqq \forall b < a\}$$

for each divisor a of n, and $\tilde{v}$ is the number of arcs in $\Phi_w^- \cap \mathscr{S}^1(f)$.

NOTATION. We denote $\hat{P}_n^{\varepsilon\varepsilon'}(\tilde{g}, \tilde{l}, \tilde{r}, \tilde{m}, \tilde{q}^+, \tilde{q}^-, \tilde{t}^+, \tilde{t}^-; \tilde{\boldsymbol{l}}, \tilde{\boldsymbol{m}}, \tilde{\boldsymbol{q}}^+, \tilde{\boldsymbol{q}}^-, \tilde{\boldsymbol{t}}^+, \tilde{\boldsymbol{t}}^-)$ by $\hat{P}_n^{\varepsilon\varepsilon'}(\mathscr{D})$ and denote by $\hat{\mathscr{P}}_n^{\varepsilon\varepsilon'}(\mathscr{D})$ the set of equivalence classes of $\hat{P}_n^{\varepsilon\varepsilon'}(\mathscr{D})$.

Then we have:

PROPOSITION A. (I) *If $\hat{P}_n^{-+}(\mathscr{D}) \neq \emptyset$, then we have the following conditions:*

$(0)_*$ *n is even and $n/2$ is odd.*

(1) $$\tilde{l} = \sum_{a|n} \tilde{l}_a, \quad \tilde{m} = \sum_{a|n} \tilde{m}_a, \quad \tilde{q}^+ = \sum_{a|n} \tilde{q}_a^+, \quad \tilde{q}^- = \sum_{a|n} \tilde{q}_a^-,$$

$$\tilde{t}^+ = \sum_{\tilde{v}} \sum_{a|n} \tilde{t}_a^+(\tilde{v}), \quad \tilde{t}^- = \sum_{\tilde{v}} \sum_{a|n} \tilde{t}_a^-(\tilde{v}) \quad \text{and} \quad \tilde{r} = \sum_{\tilde{v}} \sum_{a|n} (\tilde{v} \cdot \tilde{t}_a^+(\tilde{v}) + \tilde{v} \cdot \tilde{t}_a^-(\tilde{v})).$$

(2) $\tilde{l}_a \equiv 0 \pmod a$, $\tilde{m}_a \equiv 0 \pmod a$, $\tilde{q}_a^+ \equiv 0 \pmod a$, $\tilde{q}_a^- \equiv 0 \pmod a$, $\tilde{t}_a^+(\tilde{v}) \equiv 0 \pmod a$, *and* $\tilde{t}_a^+(\tilde{v}) \equiv 0 \pmod a$ *for each divisor a of n.*

$(4)_+$ *If $\tilde{q}_a^+ \neq 0$, then a is a divisor of $n/2$ and $2a$ is not a divisor of $n/2$.*

$(4)_0$ *$\tilde{q}_a^- = 0$ for each divisor a of n.*

$(5)_+$ *If $\tilde{t}_a^+(\tilde{v}) \neq 0$, then a is a divisor of $n/2$, $2a$ is not a divisor of $n/2$ and $2a\tilde{v}$ is a multiple of n.*

$(5)_0$ *$\tilde{t}_a^-(\tilde{v}) = 0$ for each divisor a of n.*

$(6)_{-+}$ $$g_{-+} = \frac{1}{n}\left\{2\tilde{g} + \sum_{a|n} (a-n)(l_a + m_a) - n(q+t) + 2n - 2\right\}$$

is a positive integer.

(II) *Except in the case of $(\varepsilon, \varepsilon') = (-, +)$, if $\hat{P}_n^{\varepsilon\varepsilon'}(\mathscr{D}) \neq \emptyset$, then we have the conditions* (1), (2), $(4)_+$ *and* $(5)_+$ *of* (I) *stated above, and*

(0) *n is even.*

$(4)_-$ *If $\tilde{q}_a^- \neq 0$, then a is a divisor of $n/2$.*

$(5)_-$ *If $\tilde{t}_a^-(\tilde{v}) \neq 0$, then a is a divisor of $n/2$, and $2a\tilde{v}$ is a multiple of n,*

$(6)_{+-}$ $$g_{+-} = \frac{1}{2n}\left\{\tilde{g} + \sum_{a|n} (a-n)(l_a + m_a) - n(q+t) + 2n - 2\right\}$$

is a non-negative integer, in the case of $(\varepsilon, \varepsilon') = (+, -)$, (6)$_{--}$ $$g_{--}=\frac{1}{n}\left\{\tilde{g}+\sum_{a\mid n}(a-n)(l_a+m_a)-n(q+t)+2n-2\right\}$$

is a positive integer, in the case of $(\varepsilon, \varepsilon')=(-, -)$,

(6)$_{oo}$ $$g_{oo}=\frac{1}{n}\left\{\tilde{g}+\sum_{a\mid n}(a-n)(l_a+m_a)-n(q+t)+2n-2\right\}$$

is a positive integer, in the case of $(\varepsilon, \varepsilon')=(o, o)$.

Here l_a, m_a, q *and* t *are given in the proof in* §5.

THEOREM A.1. (I) *If* g_{+-} *is a positive integer, then*

$$C^{+-}(n; \boldsymbol{l}, \boldsymbol{m}, \boldsymbol{q}, \boldsymbol{t})\equiv \#\hat{\mathscr{P}}_n^{+-}(\mathscr{D}) =\tfrac{1}{2}C^{+}(n; \boldsymbol{l}, \boldsymbol{m}, \boldsymbol{q}, \boldsymbol{t})+\tfrac{1}{2}Q(n; \boldsymbol{l}, \boldsymbol{m}, \boldsymbol{q}, \boldsymbol{t}).$$

(II) *If* g_{+-} *is equal to* 0, *then*

$$\#\hat{\mathscr{P}}_n^{+-}(\mathscr{D})=\sum_{d\mid a}\mu(d)C^{+-}(n/d; \boldsymbol{l}^{(d)}, \boldsymbol{m}^{(d)}, \boldsymbol{q}^{(d)}, \boldsymbol{t}^{(d)}),$$

where $l_{a/d}^{(d)}=l_a$, $m_{a/d}^{(d)}=m_a$, $q_{a/d}^{(d)}=q_a$, $t_{a/d}^{(d)}(v)=t_a(v)$, $\boldsymbol{l}^{(d)}=(l_{a/d}^{(d)})_{d\mid a\mid n}$, $\boldsymbol{m}^{(d)}=(m_{a/d}^{(d)})_{d\mid a\mid n}$, $\boldsymbol{q}^{(d)}=(q_{a/d}^{(d)})_{d\mid a\mid n}$ *and* $\boldsymbol{t}^{(d)}=(t_{a/d}^{(d)}(v))_{d\mid a\mid h}$.

(*For other notations, see the proof in* §5.)

THEOREM A.2. (*Notations are as above.*) *Under the conditions* (0), (1), (2), (4)$_+$, (4)$_-$, (5)$_+$ *and* (6)$_{oo}$ *in Proposition* A, *the necessary and sufficient conditions for* $\hat{P}_n^{oo}(\mathscr{D})$ *to be non-empty are the following*:

(a) *In case that* $g\geqq 3$,

(7)$_e$ $$\sum_{\substack{a\mid n\\ a;\,\mathrm{odd}}}\left(l_a+m_a+q_a^- +\sum_v t_a^-(v)\right) \quad \textit{is even}.$$

(b) *In case that* $g=1$, *the condition* (7)$_e$, *and* $d=1$.

(c) *In case that* $g=2$,

(I) $n/2$ *is odd and* d *is even,*

(II) *if* d *is odd, then the condition* (7)$_e$, *or*

(III) *if* d *is even and* $n/2$ *is even, then* $d/2$ *is odd and*

(7)$_o$ $$\sum_{\substack{a\mid n\\ a;\,\mathrm{odd}}}\left(l_a+m_a+q_a^- +\sum_v t_a^-(v)\right) \quad \textit{is odd},$$

where $d=$ g.c.d.$\{a\,;\, l_a\neq 0, m_a\neq 0, q_a\neq 0$ *or* $t_a(v)\neq 0$ $(1\leqq \exists v\leqq r)\}$.

Then the number of elements of $\hat{\mathscr{P}}_n^{oo}(\mathscr{D})$ *is given as follows*:

(a) *In case that* $g\neq 2$,

$$C(n; \boldsymbol{l}, \boldsymbol{m}, \boldsymbol{q}, \boldsymbol{t}) \qquad \textit{if } l_{n/2}+m_{n/2}+q_{n/2}+\sum_v t_{n/2}(v)\neq 0,$$

$$2\cdot C(n; \boldsymbol{l}, \boldsymbol{m}, \boldsymbol{q}, \boldsymbol{t}) \qquad \textit{if } l_{n/2}=m_{n/2}=q_{n/2}=t_{n/2}(v)=0 \quad \textit{for any } v\ (1\leqq v\leqq r).$$

(b) *In case that* $g=2$,

$$\left[\frac{\varphi(d)}{2}\right]\cdot C(n; \boldsymbol{l}, \boldsymbol{m}, \boldsymbol{q}, \boldsymbol{t}) \qquad \text{if } d \text{ is odd and } l_{n/2}+m_{n/2}+q_{n/2}+\sum_v t_{n/2}(v)\neq 0\,,$$

$$2\cdot\left[\frac{\varphi(d)}{2}\right]\cdot C(n; \boldsymbol{l}, \boldsymbol{m}, \boldsymbol{q}, \boldsymbol{t}) \qquad \text{if } d \text{ is odd and } l_{n/2}=m_{n/2}=q_{n/2}=t_{n/2}(v)=0 \text{ for any } v\ (1\leqq v\leqq r)\,,$$

$$\left[\frac{\varphi(d/2)}{2}\right]\cdot C(n; \boldsymbol{l}, \boldsymbol{m}, \boldsymbol{q}, \boldsymbol{t}) \qquad \text{if } d \text{ is even and } n/2 \text{ is odd,}$$

$$\left[\frac{\varphi(d/2)}{2}\right]\cdot C(n; \boldsymbol{l}, \boldsymbol{m}, \boldsymbol{q}, \boldsymbol{t}) \qquad \text{if } d \text{ is even, } n/2 \text{ is even and } l_{n/2}+m_{n/2}+q_{n/2}+\sum_v t_{n/2}(v)\neq 0\,,$$

$$2\cdot\left[\frac{\varphi(d/2)}{2}\right]\cdot C(n; \boldsymbol{l}, \boldsymbol{m}, \boldsymbol{q}, \boldsymbol{t}) \qquad \text{if } d \text{ is even, } n/2 \text{ is even and } l_{n/2}=m_{n/2}=q_{n/2}=t_{n/2}(v)=0 \text{ for any } v\ (1\leqq v\leqq r)\,.$$

Here $C(n; \boldsymbol{l}, \boldsymbol{m}, \boldsymbol{q}, \boldsymbol{t})$ *is equal to*

$$\prod_{\substack{a\mid n\\ a\neq n\\ a\neq n/2}}\left[\binom{\frac{\varphi(n/a)}{2}+l_a-1}{l_a}\binom{\frac{\varphi(n/a)}{2}+m_a-1}{m_a}\binom{\frac{\varphi(n/a)}{2}+q_a-1}{q_a}\cdot\prod_{v=1}^{r}\binom{\frac{\varphi(n/a)}{2}+t_a(v)-1}{t_a(v)}\right].$$

THEOREM A.3. *Under the conditions* (0), (1), (2), $(4)_+$, $(4)_-$, $(5)_+$, $(5)_-$ *and* $(6)_{--}$ *in Proposition* A, *the necessary and sufficient conditions for* $\hat{P}_n^{--}(\mathscr{D})$ *to be non-empty are the following*:

(a) *In case that* g *is odd and* $g\geqq 3$,

(i) $n/2$ *is even*;

$(3)_o$ *for each odd divisor* a *of* n, $l_a=0$, $m_a=0$, $q_a=0$ *and* $t_a(v)=0$ *for any* v $(1\leqq v\leqq r)$; *and*

$(\text{ii})_o$
$$\sum_{\substack{a\mid n\\ a;\,\text{even}\\ a/2;\,\text{odd}}}\Big(l_a+m_a+q_a+\sum_v t_a(v)\Big) \quad \text{is odd}\,.$$

(b) *In case that* $g=1$, *the conditions* (i), $(3)_o$, $(\text{ii})_o$ *of* (a), *and* $\frac{1}{2}\cdot d=1$.

(c) *In case that* g *is even, the conditons* (i), $(3)_o$ *of* (a), *and*

$(\text{ii})_e$
$$\sum_{\substack{a\mid n\\ a;\,\text{even}\\ a/2;\,\text{odd}}}\Big(l_a+m_a+q_a+\sum_v t_a(v)\Big) \quad \text{is even}\,,$$

where $d=\text{g.c.d.}\{a\,;\, l_a\neq 0,\ m_a\neq 0,\ q_a\neq 0 \text{ or } t_a(v)\neq 0\ (1\leqq \exists v\leqq r)\}$.

Then the number of elements of $\hat{\mathscr{P}}_n^{--}(\mathscr{D})$ *is given as follows;*

$$\left\lfloor \frac{\varphi(d)+\varphi(d/2)}{2} \right\rfloor \cdot C(n;\, \boldsymbol{l}, \boldsymbol{m}, \boldsymbol{q}, \boldsymbol{t}) \quad \text{if } n/d \text{ is odd},$$

$$\left\lfloor \frac{\varphi(d)}{2} \right\rfloor \cdot C(n;\, \boldsymbol{l}, \boldsymbol{m}, \boldsymbol{q}, \boldsymbol{t}) \quad \text{if } d\not\equiv 0 \pmod 4 \text{ and } l_{n/2}+m_{n/2}+q_{n/2}+\textstyle\sum_v t_{n/2}(v)\neq 0\,,$$

$$2\cdot\left\lfloor \frac{\varphi(d)+\varphi(d/2)}{2} \right\rfloor \cdot C(n;\, \boldsymbol{l}, \boldsymbol{m}, \boldsymbol{q}, \boldsymbol{t}) \quad \text{if } d\not\equiv 0 \pmod 4 \text{ and } l_{n/2}=m_{n/2}=q_{n/2}=t_{n/2}(v)=0 \text{ for any } v\ (1\leqq v\leqq r)\,,$$

$$\left\lfloor \frac{\varphi(d)}{2} \right\rfloor \cdot C(n;\, \boldsymbol{l}, \boldsymbol{m}, \boldsymbol{q}, \boldsymbol{t}) \quad \text{if } d\equiv 0 \pmod 4,\ n/d \text{ is even},\ \textstyle\sum_{a|n,\ a/d;\,\mathrm{odd}} (l_a+m_a+q_a+\sum_v t_a(v)) \text{ is even, and } l_{n/2}+m_{n/2}+q_{n/2}+\sum_v t_{n/2}(v)\neq 0\,,$$

$$2\cdot\left\lfloor \frac{\varphi(d)}{2} \right\rfloor \cdot C(n;\, \boldsymbol{l}, \boldsymbol{m}, \boldsymbol{q}, \boldsymbol{t}) \quad \text{if } d\equiv 0 \pmod 4,\ n/d \text{ is even},\ \textstyle\sum_{a|n,\ a/d;\,\mathrm{odd}} (l_a+m_a+q_a+\sum_v t_a(v)) \text{ is even, and } l_{n/2}=m_{n/2}=q_{n/2}=t_{n/2}(v)=0 \text{ for any } v\ (1\leqq v\leqq r)\,,$$

$$\left\lfloor \frac{\varphi(d/2)}{2} \right\rfloor \cdot C(n;\, \boldsymbol{l}, \boldsymbol{m}, \boldsymbol{q}, \boldsymbol{t}) \quad \text{if } d\equiv 0 \pmod 4,\ n/d \text{ is even},\ \textstyle\sum_{a|n,\ a/d;\,\mathrm{odd}} (l_a+m_a+q_a+\sum_v t_a(v)) \text{ is odd, and } l_{n/2}+m_{n/2}+q_{n/2}+\sum_v t_{n/2}(v)\neq 0\,,$$

$$2\cdot\left\lfloor \frac{\varphi(d/2)}{2} \right\rfloor \cdot C(n;\, \boldsymbol{l}, \boldsymbol{m}, \boldsymbol{q}, \boldsymbol{t}) \quad \text{if } d\equiv 0 \pmod 4,\ n/d \text{ is even},\ \textstyle\sum_{a|n,\ a/d;\,\mathrm{odd}} (l_a+m_a+q_a+\sum_v t_a(v)) \text{ is odd, and } l_{n/2}=m_{n/2}=q_{n/2}=t_{n/2}(v)=0 \text{ for any } v\ (1\leqq v\leqq r)\,.$$

THEOREM A.4. *Under the conditions* $(0)_*$, (1), (2), $(4)_+$, $(4)_o$, $(5)_+$, $(5)_o$ *and* $(6)_{-+}$ *in Proposition* A, *the necessary and sufficient conditions for* $\hat{P}_n^{-+}(\mathscr{D})$ *to be non-empty are the following*:

(a) *In case that* $g\geqq 2$,

(3) $l_a=0$ *and* $m_a=0$ *for each odd divisor* a *of* n.

(b) *In case that* $g=1$, *the condition* (3), *and* $\frac{1}{2}\cdot d=1$.

Then $\sharp\hat{\mathscr{P}}_n^{-+}(\mathscr{D})$ *is equal to*

$$C(n;\, \boldsymbol{l}, \boldsymbol{m}, \boldsymbol{q}, \boldsymbol{t}) \qquad \text{if } g\neq 2\,,$$

$$\lfloor \varphi(d)/2 \rfloor \cdot C(n;\, \boldsymbol{l}, \boldsymbol{m}, \boldsymbol{q}, \boldsymbol{t}) \qquad \text{if } g\neq 2\,,$$

where $d=\text{g.c.d.}\{a\,;\, l_a\neq 0,\ m_a\neq 0,\ q_a\neq 0,\ or\ t_a(v)\neq 0\ (1\leqq \exists v\leqq r)\}$.

Case B: $M-\mathscr{S}^1(f)$ **is disconnected.** We denote by $\ddot{P}_n$ the subset of P_n consisting of elements (f, M) with $M-\mathscr{S}^1(f)$ disconnected. For an element (f, M) of $\ddot{P}_n$, we construct an element $(\ddot{f}, \ddot{M}, S_*)$ in §2, where $\ddot{M}$ is the closure of a component of $M-\mathscr{S}^1(f)$, $\ddot{f}$ is a periodic map on $\ddot{M}$ with period $n/2$ such that $\mathscr{S}^1(\ddot{f})=\varnothing$, and $S_*=\ddot{M}\cap\mathscr{S}^1(f)$. In fact, $M-\mathscr{S}^1(f)$ has exactly two components, and $f(\ddot{M})=M_1=\overline{M-\ddot{M}}$, where $\overline{}$ means the closure. We denote by $\ddot{\boldsymbol{P}}_n$ the set of elements $(\ddot{f}, \ddot{M}, S_*)$ constructed in this way, and define an equivalence relation $\underset{*}{\sim}$ on $\ddot{\boldsymbol{P}}_n$ in §2. Then we have a bijection between the set of equivalence classes $\ddot{P}_n/\sim$ and the set of equivalence classes $\ddot{\boldsymbol{P}}_n/\underset{*}{\sim}$. So, we classify $\ddot{\boldsymbol{P}}_n$.

Let $\ddot{P}_n^+$ be the subset of $\ddot{P}_n$ consisting of elements (f, M) with M orientable, and let $\ddot{P}_n^o$ be the subset of $\ddot{P}_n$ consisting of elements (f, M) with M non-orientable. Clearly we have $\ddot{P}_n=\ddot{P}_n^+\cup\ddot{P}_n^o$. In this case, no 1-sided loops in $\mathscr{S}(f)$ exist and no 1-sided Φ-sets exist (see Corollary 2.1). (The definition of the Φ-set is given in §2.)

Let $\ddot{P}_n^\varepsilon(\tilde{g}, \tilde{l}, \tilde{r}, \tilde{m}, \tilde{q}, \tilde{t}; \tilde{\boldsymbol{l}}, \tilde{\boldsymbol{m}}, \tilde{\boldsymbol{q}}, \tilde{\boldsymbol{t}})$ be the set of elements (f, M) of $\ddot{P}_n^\varepsilon$, where $\varepsilon=+$ or o, satisfying the same conditions as (1), (2), (4) and (5) in the case of $\hat{P}_n^{\varepsilon\varepsilon'}(\mathscr{D})$, and

(3)′ $\mathscr{S}^1(f)$ consists of 2-sided loops $D_1^o, D_2^o, \cdots, D_{\tilde{q}}^o$ in $\mathring{M}$ and simple proper arcs in 2-sided Φ-sets $\Phi_1, \Phi_2, \cdots, \Phi_{\tilde{t}}$ in M.

(6)′ $\tilde{\boldsymbol{q}}=(\tilde{q}_a)_{a|n}$ is a vector of non-negative integers $\tilde{q}_a$, where

$$\tilde{q}_a=\#\{D_u^o\,;\, f^a(D_u^o)=D_u^o \text{ and } f^b(D_u^o)\neq D_u^o \text{ for } 1\leqq\forall b<a\}$$

for each divisor a of n.

(8)′ $\tilde{\boldsymbol{t}}=(\tilde{t}_a(\tilde{v}))_{a|n}$ is a vector of non-negative integers $\tilde{t}_a(\tilde{v})$, where

$$\tilde{t}_a(\tilde{v})=\#\{\Phi_w\,;\, f^a(\Phi_w)=\Phi_w \text{ and } f^b(\Phi_w)\neq\Phi_w \text{ for } 1\leqq\forall b<a\}$$

for each divisor a of n, and $\tilde{v}$ is the number of arcs in $\Phi_w\cap\mathscr{S}^1(f)$.

Notation. We denote $\ddot{P}_n^\varepsilon(\tilde{g}, \tilde{l}, \tilde{r}, \tilde{m}, \tilde{q}, \tilde{t}; \tilde{\boldsymbol{l}}, \tilde{\boldsymbol{m}}, \tilde{\boldsymbol{q}}, \tilde{\boldsymbol{t}})$ by $\ddot{P}_n^\varepsilon(\mathscr{D})$ and denote by $\ddot{\mathscr{P}}_n^\varepsilon(\mathscr{D})$ the set of equivalence classes of $\ddot{P}_n^\varepsilon(\mathscr{D})$.

Then we have:

Proposition B. *If* $\ddot{P}_n^\varepsilon(\mathscr{D})\neq\varnothing$, *then we have the following conditions*:

$(0)_*$ *n is even and n/2 is odd.*

(1) $\tilde{l}=\sum_{a|n}\tilde{l}_a,\quad \tilde{m}=\sum_{a|n}\tilde{m}_a,\quad \tilde{q}=\sum_{a|n}\tilde{q}_a,$

$$\tilde{t}=\sum_{\tilde{v}}\sum_{a|n}\tilde{t}_a(\tilde{v}) \quad and \quad \tilde{r}=\sum_{\tilde{v}}\sum_{a|n}\tilde{v}\cdot\tilde{t}_a(\tilde{v})\,.$$

(2) $\tilde{l}_a \equiv 0 \pmod a$, $\tilde{m}_a \equiv 0 \pmod a$, $\tilde{q}_a \equiv 0 \pmod a$ *and* $\tilde{t}_a(\tilde{v}) \equiv 0 \pmod a$, *for each divisor* a *of* n.

(3) *If* $\tilde{l}_a \neq 0$, *then* a *is even, and if* $\tilde{m}_a \neq 0$, *then* a *is even.*

(4) *If* $\tilde{q}_a \neq 0$, *then* a *is a divisor of* $n/2$.

(5) *If* $\tilde{t}_a(\tilde{v}) \neq 0$, *then* a *is a divisor of* $n/2$, *and* $2a\tilde{v}$ *is a multiple of* n.

$(6)_{2+}$
$$g_{2+} = \frac{1}{n}\left\{\tilde{g} + \sum_{a \mid n} \frac{a-n}{2}(l_{a/2} + m_{a/2}) - \frac{n}{2}(q+t) + n - 1\right\}$$

is a non-negative integer, if $\varepsilon = +$.

$(6)_{2o}$
$$g_{2o} = \frac{1}{n}\left\{\tilde{g} + \sum_{a \mid n}(a-n)(l_a + m_a) - n(q+t) + 2n - 2\right\}$$

is a positive integer, if $\varepsilon = o$.

Here, l_a, m_a, q *and* t *are given in the proof in* §5.

THEOREM B.1. *Under the conditions* $(0)_*$, (1), (2), (3), (4), (5), *and* $(6)_{2+}$ *in Proposition* B,

(I) *If* g_{2+} *is a positive integer, then*

$$C^{2+}(n; \boldsymbol{l}, \boldsymbol{m}, \boldsymbol{q}, \boldsymbol{t}) \equiv \# \ddot{\mathscr{P}}_n^+(\mathscr{D}) = \tfrac{1}{2}C^+(n/2; \boldsymbol{l}, \boldsymbol{m}, \boldsymbol{q}, \boldsymbol{t}) + \tfrac{1}{2}Q(n/2; \boldsymbol{l}, \boldsymbol{m}, \boldsymbol{q}, \boldsymbol{t}).$$

(II) *If* g_{2+} *is equal to* 0, *then*

$$\#\mathscr{P}_n^+(\mathscr{D}) = \sum_{d \mid n} \mu(d) \cdot C^{2+}(n/d; \boldsymbol{l}^{(d)}, \boldsymbol{m}^{(d)}, \boldsymbol{q}^{(d)}, \boldsymbol{t}^{(d)}),$$

where the notations are the same as in Theorem A.1.

THEOREM B.2. *Under the conditions* $(0)_*$, (1), (2), (3), (4), (5) *and* $(6)_{2o}$ *in Proposition* B, *the necessary and sufficient conditions for* $\ddot{P}_n^o(\mathscr{D})$ *to be non-empty are the following;*

(a) $g_{2o} \neq 1$ *or* (b) $g_{2o} = 1$ *and* $d = 1$.

Then

$$\#\mathscr{P}_n^{2o}(\mathscr{D}) = \begin{cases} C^2(n; \boldsymbol{l}, \boldsymbol{m}, \boldsymbol{q}, \boldsymbol{t}) & \text{if } g_{2o} \neq 2, \\ \lfloor \varphi(d)/2 \rfloor \cdot C^2(n; \boldsymbol{l}, \boldsymbol{m}, \boldsymbol{q}, \boldsymbol{t}) & \text{if } g_{2o} \neq 2. \end{cases}$$

Here $d = \text{g.c.d.}\{a\,;\, l_a \neq 0,\ m_a \neq 0,\ q_a \neq 0,\ t_a(\tilde{v}) \neq 0\ (1 \leqq \exists v \leqq r)\}$ *and* $C^2(n; \boldsymbol{l}, \boldsymbol{m}, \boldsymbol{q}, \boldsymbol{t})$ *is equal to*

$$\prod_{\substack{a \mid n/2 \\ a \neq n/2}} \left[\binom{\frac{\varphi(n/(2a))}{2} + l_a - 1}{l_a}\binom{\frac{\varphi(n/(2a))}{2} + m_a - 1}{m_a}\binom{\frac{\varphi(n/(2a))}{2} + q_a - 1}{q_a} \cdot \prod_{v=1}^{r}\binom{\frac{\varphi(n/(2a))}{2} + t_a(v) - 1}{t_a(v)}\right].$$

§2. Reducing operation.

Let M be a compact connected surface and f be a periodic map on M with period n. Note that $p: M \to M/f = X$ is an n-fold cyclic branched covering with a branched set $p(\mathscr{S}(f)) = S$. We discuss the placement of $\mathscr{S}(f)$ in M.

By elementary properties of periodic maps, we have:

PROPOSITION 2.1. *A periodic map f on the* 1-*sphere S^1 is equivalent to*:
(1) *the reflection, that is, the map $(x, y) \to (x, -y)$, or*
(2) *free.*

PROPOSITION 2.2. *Let $x \in \mathscr{S}^1(f)$, and we suppose that there exists a neighborhood $V(x)$ of x such that $(\overline{V(x)}, \overline{V(x)} \cap \mathscr{S}(f))$ is homeomorphic to $([-1; 1] \times [-1; 1], [-1; 1] \times \{0\})$, where $[-1; 1] = \{x \in \boldsymbol{R}^1 ; -1 \leqq x \leqq 1\}$. Then, it holds that* (1) *n is even, and* (2) *$x \in \mathscr{S}_{n/2}(f)$.*

PROOF. Taking a sufficiently small neighborhood $V(x)$ of x if necessary, we may assume that $f(V(x)) = V(x)$. We put $x_0 = p(x)$, where $p: M \to M/f$ is the natural projection. We take a point $y_0 \notin \mathscr{S}(f)$ sufficiently near to x_0. If $p^{-1}(x_0) = \{x_1, x_2, \cdots, x_t\}$, then $p^{-1}(y_0) = \{y_1, y'_1, y_2, y'_2, \cdots, y_t, y'_t\}$. Hence, we have $2t = n$ since $y_0 \notin \mathscr{S}(f)$, completing the proof.

By Whyburn [8] and Proposition 2.1, we have the following.

PROPOSITION 2.3. *Let C be a connected component of $\mathscr{S}(f)$, and let $p: M \to M/f$ be the natural projection. Then, a neighborhood $V(C)$ and $p|_{V(C)}$ are characterized as follows*:

Type 1. *If C is an isolated point in $\mathring{M}$, a neighborhood $V(C)$ of C and $p|_{V(C)}$ are as shown in Fig.* 1.

Type 2. *If C is a simple loop in $\mathring{M}$, a neighborhood $V(C)$ of C and $p|_{V(C)}$ are as shown in Fig.* 2.

Type 3. *If C is a simple proper arc in M and D is a component of ∂M with $D \cap C \neq \emptyset$, then $D \cap \mathscr{S}(f) = D \cap \mathscr{S}^1(f)$ consists of exactly two points. Hence, a neighborhood $V(\Phi)$ of a connected component Φ of $\partial M \cup \mathscr{S}^1(f)$ and $p|_{\Phi}$ are as shown in Fig.* 3.

A connected component of $\partial M \cup \mathscr{S}^1(f)$ will be called a *Φ-set*.

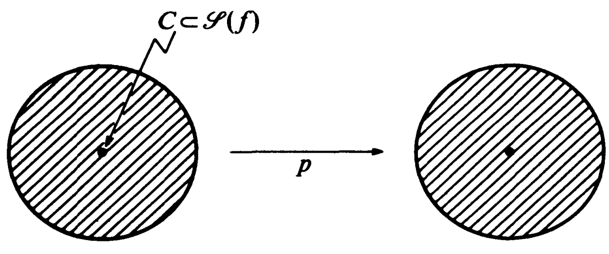

FIGURE 1

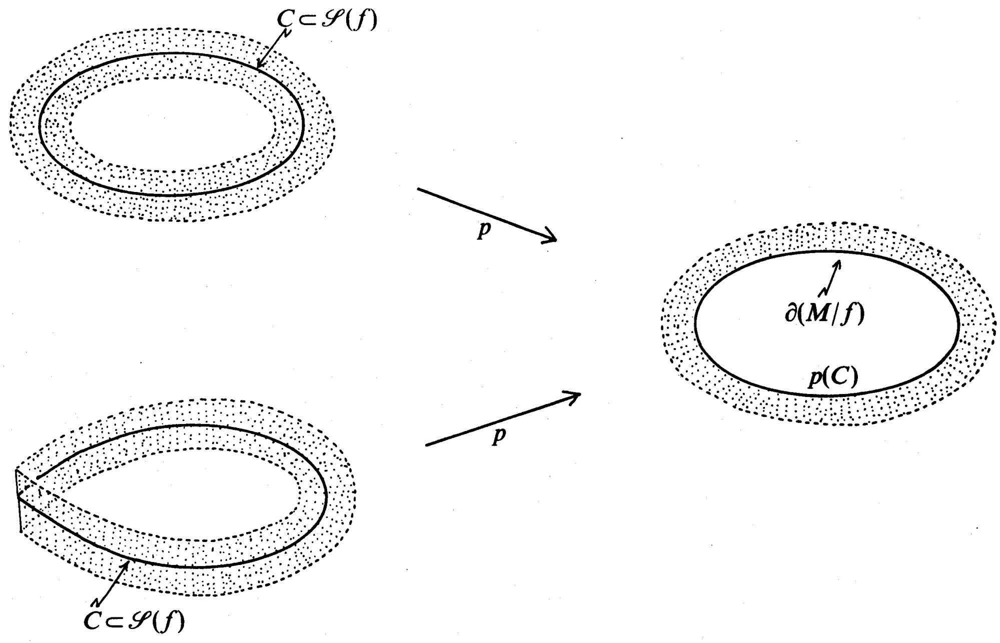

FIGURE 2

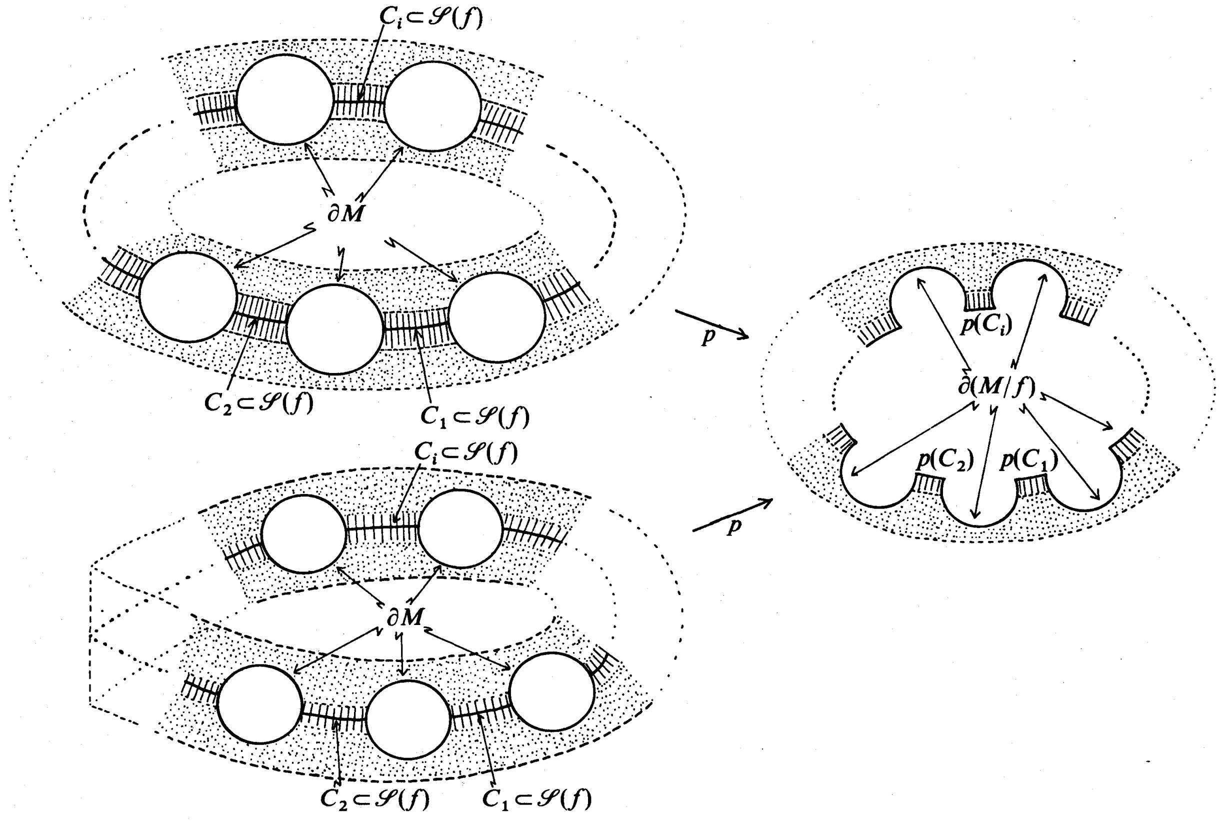

FIGURE 3

NOTATION. Let C be a connected component of $\mathscr{S}^1(f)$. If C is of type 2 and 2-sided in M, we denote it by $D_u^{\circ +}$, and if C is of type 2 and 1-sided in M, we denote it by $D_u^{\circ -}$.

Let Φ be a Φ-set of $\partial M \cup \mathscr{S}^1(f)$. If a neighborhood $V(\Phi)$ of Φ is homeomorphic to a punctured annulus, we call it a *2-sided Φ-set* and denote it by Φ_w^+, and if a neighborhood $V(\Phi)$ is homeomorphic to a Möbius strip, we call it a *1-sided Φ-set* and denote it by Φ_w^-.

DEFINITION 2.1. Let P_m^* be the set of elements (f_*, M_*, S_*) satisfying,

(1) M_* is a compact connected surface,

(2) S_* is a subset of ∂M_* which consists of simple loops and simple arcs on ∂M_*,

(3) f_* is a periodic map on M_* with period m such that $\mathscr{S}^1(f_*)=\varnothing$ and $f_*(S_*)=S_*$.

Then we say that f_* is an *s-periodic map* on (M_*, S_*).

In the same way as the case of periodic maps, we define an s-equivalence relation on P_m^* as follows:

DEFINITION 2.2. Let (f_*, M_*, S_*) and (f'_*, M'_*, S'_*) be two elements of P_m^*. Then we say that (f_*, M_*, S_*) and (f'_*, M'_*, S'_*) are *s-equivalent*, denoted as $(f_*, M_*, S_*) \underset{*}{\approx} (f'_*, M'_*, S'_*)$, iff there exists a homeomorphism $h\colon M \to M'$ such that (1) $h(S_*)=S'_*$ and (2) $f'_*h=hf_*$.

LEMMA 2.1. *If (f, M) is an element of $\ddot{P}_n$ (that is, $M-\mathscr{S}^1(f)$ is not connected), then $M-\mathscr{S}^1(f)$ has exactly two connected components. We denote by M_* the closure of one of the connected components of $M-\mathscr{S}^1(f)$ and put $S_*=M_*-(M-\mathscr{S}^1(f))$. Then, it holds that $f(M_*)=\overline{M-M_*}$ and $n/2$ is odd. We put $f_*=f^2|_{M_*}$. Then f_* is an s-periodic map with period $n/2$ on (M_*, S_*).*

PROOF. Let M'_* be the closure of a connected component of $M-\mathscr{S}^1(f)$ which is distinct from M_*. We prove that $M_* \cap \mathscr{S}^1(f)=M'_* \cap \mathscr{S}^1(f)$. Let x_0 be a point in $M_* \cap M'_* \cap \mathscr{S}^1(f)$. We take a point y_0 sufficiently near to x_0 in M_* such that $y_0 \notin \mathscr{S}(f)$. Let x be an arbitrary point of $M_* \cap \mathscr{S}^1(f)$. We take a point y sufficiently near to x in M_* such that $y \notin \mathscr{S}(f)$. Then there exists an arc l in M_* which joins y_0 and y such that $l \cap \mathscr{S}^1(f)=\varnothing$. Let $h=f^{n/2}$ and $l'=h(l)$. Since $h(x_0)=x_0$ and $y'_0=h(y_0)\in M'_*$, l' is an arc in M'_* which joins $h(y_0)=y'_0$ and $h(y)=y'$. Since $l' \cap \mathscr{S}^1(f)=\varnothing$ and $y'_0 \in M'_*$, we have $y' \in M'_*$. Hence $x \in M'_* \cap \mathscr{S}^1(f)$, and $M_* \cap \mathscr{S}^1(f) \subset M'_* \cap \mathscr{S}^1(f)$. In the same way, $M'_* \cap \mathscr{S}^1(f) \subset M_* \cap \mathscr{S}^1(f)$. Hence $M_* \cap \mathscr{S}^1(f)=M'_* \cap \mathscr{S}^1(f)$.

Therefore, $\overline{M-\mathscr{S}^1(f)}=\mathring{M}_* \cup \mathring{M}'_*$, since M is connected. Clearly $f(M_*)=M'_*=\overline{M-M_*}$, and $\mathscr{S}^1(f)\subset M_*$. Hence $\mathscr{S}^1(f)\cap M_*=\mathscr{S}^1(f)=\mathscr{S}^1(f)\cap(M-M_*)$. Since $y \in M_*$, $f(y)\in M'_*$ and $f^{n/2}(y)=h(y)=y' \in M'_*$, we know that $n/2$ is odd. $f_*=f^2|_{M_*}$ is a periodic map on M_* with period $n/2$. By Proposition 2.3 and the definition of f_* and $\mathscr{S}_k^0(f_*)$, it holds that $\mathscr{S}^1(f_*)=\varnothing$ and $\mathscr{S}_k^0(f_*)=\mathscr{S}_{2k}^0(f)\cap M_*$. Since $f(\mathscr{S}^1(f))=\mathscr{S}^1(f)$ and $f_*=f^2|_{M_*}$, it holds that $f_*(\mathscr{S}^1(f)\cap M_*)=\mathscr{S}^1(f)\cap M_*$.

COROLLARY 2.1. *If (f, M) is an element of $\ddot{P}_n$, then no 1-sided loops in $\mathscr{S}^1(f)$ exist, and no 1-sided Φ-sets exist.*

If $M-\mathscr{S}^1(f)$ is connected and M is orientable, then there exists a 1-sided loop on the orbit space $X=M/f$. Hence we have;

PROPOSITION 2.4. *If (f, M) is an element of $\hat{P}_n$ (that is, $M-\mathscr{S}^1(f)$ is connected) and M is orientable, then the orbit space M/f is non-orientable.*

LEMMA 2.2. *For an element (f, M) of $\hat{P}_n$, we put M_* the natural compactification of $M-\mathscr{S}^1(f)$ and $S_*=M_*-(M-\mathscr{S}^1(f))$. We define a map $f_*: M_*\to M_*$ as follows. If $x\in M-S_*$ then $f_*(x)=f(x)$. If $x\in S_*$, we take a sequence $\{x_i\}$ of points x_i in M_*-S_* such that $\lim x_i=x$, and we define $f_*(x)=\lim f(x_i)$. Then f_* is an s-periodic map with period n on (M_*, S_*) with $\mathscr{S}^0_k(f_*)=\mathscr{S}^0_k(f)$.*

DEFINITION 2.3. For $(f, M)\in P_n$, we have a unique element (f_*, M_*, S_*) in $\boldsymbol{P}^*_m$ by Lemma 2.1 $(m=n/2)$ or Lemma 2.2 $(m=n)$. We define a map $RD: P_n\to \boldsymbol{P}^*_m$ by $RD(f, M)=(f_*, M_*, S_*)$.

By abuse of notation, we also have the mapping RD from an arbitrary element of P_n to its RD-image in $\boldsymbol{P}^*_m$ in an obvious manner. Note that such an RD is not necessarily univalent.

By the following four lemmas, we will show that RD is a bijection of the equivalence classes of the set P_n onto the s-equivalence classes of the set $\boldsymbol{P}^*_m$.

LEMMA 2.3. *Let (f, M) and (f', M') be elements of $\ddot{P}_n$ and we put $RD(f, M)=(f_*, M_*, S_*)$ and $RD(f', M')=(f'_*, M'_*, S'_*)$. Then, $(f, M)\sim(f', M')$ if and only if $(f_*, M_*, S_*)\underset{*}{\sim}(f'_*, M'_*, S'_*)$.*

PROOF. If $(f, M)\sim(f', M')$, it is clear that $(f_*, M_*, S_*)\underset{*}{\sim}(f'_*, M'_*, S'_*)$. Conversely, if $(f_*, M_*, S_*)\underset{*}{\sim}(f'_*, M'_*, S'_*)$, then there exists a homeomorphism h of (M_*, S_*) onto (M'_*, S'_*). We define a map H of M to M' as follows; if $x\in M_*$, then we define by $H(x)=h(x)$ and if $x\notin M_*$, then we define by $H(x)=f'^{n/2}hf^{n/2}(x)$. Then H is a homeomorphism of M onto M' such that $f'H=Hf$. Hence $(f, M)\sim(f', M')$.

LEMMA 2.4. *Let (f, M) and (f', M') be elements of $\hat{P}_n$ and we put $RD(f, M)=(f_*, M_*, S_*)$ and $RD(f', M')=(f'_*, M'_*, S'_*)$. Then, $(f, M)\sim(f', M')$ if and only if $(f_*, M_*, S_*)\underset{*}{\sim}(f'_*, M'_*, S'_*)$.*

PROOF. If $(f, M)\sim(f', M')$, then there exists a homeomorphism h of M onto M' such that $hf=f'h$. We define a map $H: M_*\to M'_*$ as follows. If $x\notin S_*$, we define by $H(x)=h(x)$ and if $x\in S_*$, we take a sequence $\{x_i\}$ of points in M_*-S_* such that $\lim x_i=x$, and we define $H(x)=\lim h(x_i)$. Then H is a homeomorphism of (M_*, S_*) onto (M'_*, S'_*) such that $Hf_*=f'_*H$. Hence (f_*, M_*, S_*) is s-equivalent to (f'_*, M'_*, S'_*).

Conversely, if (f_*, M_*, S_*) is s-equivalent to (f'_*, M'_*, S'_*) then there exists a homeomorphism h of (M_*, S_*) onto (M'_*, S'_*) such that $hf_* = f_*h$. Let $q: M_* \to M_*/f_* = M$ and $q': M'_* \to M'_*/f'_* = M'$ be the natural projections. We define a map $H: M \to M'$ as follows. For a point x of M, there is a point y of M_* such that $q(y) = x$. So we define by $H(x) = q'h(y)$. Then H is a well-defined homeomorphism such that $Hf = f'H$. Hence $(f, M) \sim (f', M')$.

Let (f_*, M_*, S_*) be an element of $\boldsymbol{P}^*_m$. We take a copy M'_* of M_*, let $i: M_* \to M'_*$ be the identification homeomorphism, and put $M' = M_* \cup_j M'_*$ where $j = i|_{S_*}$. We define a map f' on M' by putting

$$f'(x) = \begin{cases} if_*^{(m+1)/2}(x) & \text{if} \quad x \in M_* \\ f_*^{(m+1)/2} i^{-1}(x) & \text{if} \quad x \in M'_* \end{cases}.$$

From the definition of f', we have:

LEMMA 2.5. *f' is a periodic map on M' with period $2m$ such that $\mathscr{S}^1(f') = S_*$ and $M' - \mathscr{S}^1(f')$ is not connected. Moreover, if (f_*, M_*, S_*) is constructed by Lemma 2.1 from a periodic map f on M with period $2m$, then we have that (f', M') is equivalent to (f, M).*

Let (f_*, M_*, S_*) be an element of $\boldsymbol{P}^*_n$ with even n. We define an equivalence relation $\mathscr{R}$ on M_* as follows: $x\mathscr{R}y$ iff (i) $x = y$ or $y = f^{n/2}(x)$ (if $x, y \in S_*$), and (ii) $x = y$ (otherwise). Let M' be the quotient space $M/\mathscr{R}$ and let $q: M_* \to M'$ be the natural quotient map. For a point x of M', there is a point y of M_* such that $q(y) = x$. Hence we define a map $f': M' \to M'$ by $f'(x) = qf_*(y)$. Then, f' is well-defined and we have:

LEMMA 2.6. *f' is a periodic map on M' with period n such that $\mathscr{S}^1(f') = q(S_*)$, $\mathscr{S}^0_k(f') = q(\mathscr{S}^0_k(f_*))$ and $M' - S^1(f')$ is connected. Moreover, if (f_*, M_*, S_*) is constructed by Lemma 2.2 from a periodic map f on M with period n, then (f', M') is equivalent to (f, M).*

NOTATION. (1) Since P_n is the disjoint union of $\hat{P}_n$ and $\ddot{P}_n$, the function RD may be regarded as a function from $\hat{P}_n$ to $\boldsymbol{P}^*_n$ and a function from $\ddot{P}_n$ to $\boldsymbol{P}^*_{n/2}$. We denote $\boldsymbol{P}^*_n$ by $\hat{\boldsymbol{P}}_n$ and $\boldsymbol{P}^*_{n/2}$ by $\ddot{\boldsymbol{P}}_{n/2}$.

(2) We denote the image $RD(D_i)$ by $\hat{D}_i$, and the image $RD(S_j)$ by $\hat{S}_j$, where D_i is a component of ∂M with $D_i \cap \mathscr{S}(f) = \emptyset$, and S_j is a component of $\mathscr{S}^0(f)$.

(3) If D^{o+}_u is a 2-sided loop in $\mathscr{S}^1(f)$, then the image $RD(D^{o+}_u)$ has exactly two components. So we denote $RD(D^{o+}_u)$ by $\hat{D}^{o+}_{u,1} \cup \hat{D}^{o+}_{u,2}$. If D^{o-}_u is a 1-sided loop in $\mathscr{S}^1(f)$, then the $RD(D^{o-}_u)$ has exactly one component, and so we denote it by $\hat{D}^{o-}_u$.

(4) If a connected component Φ^+_w of $\partial M \cup \mathscr{S}^1(f)$ is a 2-sided Φ-set, then the image $RD(\Phi^+_w)$ has exactly two components. We denote $RD(\Phi^+_w)$ by $\hat{\Phi}^+_{w,1} \cup \hat{\Phi}^+_{w,2}$. If a connected component Φ^-_w of $\partial M \cup \mathscr{S}^1(f)$ is a 1-sided Φ-set, then the $RD(\Phi^-_w)$ has exactly one component, and so we denote it by $\hat{\Phi}^-_w$.

(5) For each $(\varepsilon, \varepsilon') = (+, -), (-, +), (-, -)$ or (o, o), $\hat{P}^{\varepsilon\varepsilon'}_n(\mathscr{D})$ is a subset of $\hat{P}_n$, and so we denote by $\hat{\boldsymbol{P}}^{\varepsilon\varepsilon'}_n(\mathscr{D})$ the image $RD(\hat{P}^{\varepsilon\varepsilon'}_n(\mathscr{D}))$.

For an element $(f, M)\in \hat{P}_n^{\varepsilon\varepsilon'}(\mathscr{D})$, we put $RD(f, M)=(\hat{f}, \hat{M}, S_*)\in \hat{\boldsymbol{P}}_n^{\varepsilon\varepsilon'}(\mathscr{D})=\hat{\boldsymbol{P}}_n^{\varepsilon\varepsilon'}(\hat{g}, \hat{l}, \hat{r}, \hat{m}, \hat{q}^+, \hat{q}^-, \hat{t}^+, \hat{t}^-; \hat{\boldsymbol{l}}, \hat{\boldsymbol{m}}, \hat{\boldsymbol{q}}^+, \hat{\boldsymbol{q}}^-, \hat{\boldsymbol{t}}^+, \hat{\boldsymbol{t}}^-)$. It should be noticed that $(\hat{f}, \hat{M}, S_*)$ satisfies the following conditions:

(1) $\hat{M}$ is a compact surface of genus $\hat{g}$ with $\hat{l}+\hat{q}^+ +\hat{q}^- +\hat{t}^+ +\hat{t}^-$ boundary components.

(2) $\partial\hat{M}$ consists of the following sets;

(i) $\hat{D}_1, \hat{D}_2, \cdots, \hat{D}_{\hat{l}}$; (ii) $\hat{D}_1^{\circ +}, \hat{D}_2^{\circ +}, \cdots, \hat{D}_{\hat{q}+}^{\circ +}$; (iii) $\hat{D}_1^{\circ -}, \hat{D}_2^{\circ -}, \cdots, \hat{D}_{\hat{q}-}^{\circ -}$;

(iv) $\hat{\Phi}_1^+, \hat{\Phi}_2^+, \cdots, \hat{\Phi}_{\hat{t}+}^+$; and (v) $\hat{\Phi}_1^-, \hat{\Phi}_2^-, \cdots, \hat{\Phi}_{\hat{t}-}^-$.

(3) $\mathscr{S}_0(\hat{f})$ consists of $\hat{m}$ points $\hat{S}_1, \hat{S}_2, \cdots, \hat{S}_{\hat{m}}$ in $\mathring{\hat{M}}$.

(4) We have the vectors $\hat{\boldsymbol{l}}=(\hat{l}_a)_{a|n}$, $\hat{\boldsymbol{m}}=(\hat{m}_a)_{a|n}$, $\hat{\boldsymbol{q}}^+=(\hat{q}_a^+)_{a|n}$, $\hat{\boldsymbol{q}}^-=(\hat{q}_a^-)_{a|n}$, $\hat{\boldsymbol{t}}^+=(\hat{t}_a^+(\hat{v}))_{a|n,\hat{v}}$ and $\hat{\boldsymbol{t}}^-=(\hat{t}_a^-(\hat{v}))_{a|n,\hat{v}}$ defined similarly as $\tilde{\boldsymbol{l}}, \tilde{\boldsymbol{m}}, \tilde{\boldsymbol{q}}^+, \tilde{\boldsymbol{q}}^-, \tilde{\boldsymbol{t}}^+$ and $\tilde{\boldsymbol{t}}^-$, respectively, in Section 1, where we replace each letter $f, D_j, S_k, D_j^{\circ\pm}, \Phi_w^\pm$ or v by the same letter with $\hat{}$, $\hat{v}$ being the number of arcs in $\hat{\Phi}_w^\pm \cap S_*$.

We denote by $\hat{\mathscr{P}}_n^{\varepsilon\varepsilon'}(\mathscr{D})$ the set of equivalence classes of $\hat{\boldsymbol{P}}_n^{\varepsilon\varepsilon'}(\mathscr{D})$.

Proposition 2.5. *Under the above conditions and notation, $\hat{g}, \hat{l}, \hat{r}, \hat{m}, \hat{q}^+, \hat{q}^-, \hat{t}^+, \hat{t}^-, \hat{\boldsymbol{l}}, \hat{\boldsymbol{m}}, \hat{\boldsymbol{q}}^+, \hat{\boldsymbol{q}}^-, \hat{\boldsymbol{t}}^+$ and $\hat{\boldsymbol{t}}^-$ satisfy the following equations.*

(1)
$$\hat{l}_a=\tilde{l}_a\,,\quad \hat{m}_a=\tilde{m}_a\,,\quad \hat{q}_{2a}^+=2\cdot\tilde{q}_a^+\,,\quad \hat{q}_a^-=\tilde{q}_a^-\,,$$
$$\hat{t}_{2a}^+(\hat{v})=2\cdot\tilde{t}_a^+(\hat{v})\,,\quad \hat{t}_a^-(\hat{v})=\tilde{t}_a^-(\hat{v})=\tilde{t}_a^-(\hat{v}/2)\,.$$

(2)
$$\hat{l}=\sum_{a|n}\hat{l}_a\,,\quad \hat{m}=\sum_{a|n}\hat{m}_a\,,\quad \hat{q}^+=\sum_{a|n}\hat{q}_a^+\,,\quad \hat{q}^-=\sum_{a|n}\hat{q}_a^-\,,$$
$$\hat{t}^+=\sum_{\hat{v}}\sum_{a|n}\hat{t}_a^+(\hat{v})\,,\quad \hat{t}^-=\sum_{\hat{v}}\sum_{a|n}\hat{t}_a^-(\hat{v})\,,\quad \hat{r}=\sum_{\hat{v}}\sum_{a|n}\hat{v}\cdot(\hat{t}_a^+(\hat{v})+(\hat{t}_a^-(\hat{v}))\,.$$

(3)
$$\tilde{g}=\begin{cases} 2\hat{g}+2\tilde{q}^+ +2\tilde{t}^+ +\tilde{q}^- +\tilde{t}^- & \text{if } \varepsilon'=-\,,\\ \hat{g}+\tilde{q}^+ +\tilde{t}^+ +\tilde{q}^- +\tilde{t}^- & \text{if } \varepsilon'=+\,,\\ \hat{g}+2\tilde{q}^+ +2\tilde{t}^+ +\tilde{q}^- +\tilde{t}^- & \text{if } \varepsilon'=\mathrm{o}\,.\end{cases}$$

Since $\ddot{P}_n^\varepsilon(\mathscr{D})$ is a subset of $\ddot{P}_n$, we denote by $\ddot{\boldsymbol{P}}_n^\varepsilon(\mathscr{D})$ the image $RD(\ddot{P}_n^\varepsilon(\mathscr{D}))$. For an element $(f, M)\in \ddot{P}_n^\varepsilon(\mathscr{D})$, we put $RD(f, M)=(\ddot{f}, \ddot{M}, S_*)\in \ddot{\boldsymbol{P}}_n^\varepsilon(\mathscr{D})=\ddot{\boldsymbol{P}}_n^\varepsilon(\hat{g}, \hat{l}, \hat{r}, \hat{m}, \hat{q}, \hat{t}; \hat{\boldsymbol{l}}, \hat{\boldsymbol{m}}, \hat{\boldsymbol{q}}, \hat{\boldsymbol{t}})$. It should be noticed that $(\ddot{f}, \ddot{M}, S_*)$ satisfies the following conditions:

(1) $\ddot{M}$ is a compact surface of genus $\hat{g}$ with $\hat{l}+\hat{q}+\hat{t}$ boundary components.

(2) $\partial\ddot{M}$ consists of the following sets;

(i) $\hat{D}_1, \hat{D}_2, \cdots, \hat{D}_{\hat{l}}$, (ii) $\hat{D}_1^\circ, \hat{D}_2^\circ, \cdots, \hat{D}_{\hat{q}}^\circ$, and (iii) Φ-sets $\hat{\Phi}_1, \hat{\Phi}_2, \cdots, \hat{\Phi}_{\hat{t}}$.

(3) $\mathscr{S}^0(f)$ consists of $\hat{m}$ points $\hat{S}_1, \hat{S}_2, \cdots, \hat{S}_{\hat{m}}$ in $\partial\ddot{M}$.

(4) We have the vectors $\hat{\boldsymbol{l}}=(\hat{l}_a)_{a|n}$, $\hat{\boldsymbol{m}}=(\hat{m}_a)_{a|n}$, $\hat{\boldsymbol{q}}=(\hat{q}_a)_{a|n}$ and $\hat{\boldsymbol{t}}=(\hat{t}_a(\hat{v}))_{a|n,\hat{v}}$ defined similarly, except that we take as a each divisor of $n/2$ instead of each divisor of n.

We denote by $\ddot{\mathscr{P}}_n^\varepsilon(\mathscr{D})$ the set of equivalence classes of $\ddot{\boldsymbol{P}}_n^\varepsilon(\mathscr{D})$.

PROPOSITION 2.6. *Under the above conditions and notations, $\hat{g}$, $\hat{l}$, $\hat{r}$, $\hat{m}$, $\hat{q}$, $\hat{t}$, $\hat{\boldsymbol{l}}$, $\hat{\boldsymbol{m}}$, $\hat{\boldsymbol{q}}$ and $\hat{\boldsymbol{t}}$ satisfy the following equations*:

(1) $$2\hat{l}_a=\tilde{l}_{2a}\,,\quad 2\hat{m}_a=\tilde{m}_{2a}\,,\quad \hat{q}_a=\tilde{q}_a\,,\quad \hat{t}_a(\hat{v})=\tilde{t}_a(\tilde{v})\,.$$

(2) $$\hat{l}=\sum_{a|n}\hat{l}_a\,,\quad \hat{m}=\sum_{a|n}\hat{m}_a\,,\quad \hat{q}=\sum_{a|n}\hat{q}_a\,,\quad \hat{t}=\sum_{\hat{v}}\sum_{a|n}\hat{t}_a(\hat{v})\,,\quad \hat{r}=\sum_{\hat{v}}\sum_{a|n}\hat{v}\cdot\hat{t}_a(\hat{v})\,.$$

(3) $$\tilde{g}=\begin{cases}2\hat{g}+\hat{q}+\hat{t}-1 & \text{if } \varepsilon=+\,,\\ 2\hat{g}+2\hat{q}+2\hat{t}-2 & \text{if } \varepsilon=\mathrm{o}\,.\end{cases}$$

§3. Determination of the equivalence classes of $\hat{P}_n(X, S)$ and $\ddot{P}_n(X, S)$.

Let X_g be a compact connected orientable surface of genus g, and we take a set S^1 of simple loops and simple arcs on ∂X_g. We divide the components of ∂X_g into the following three types;

(i) the components $\vec{d}_1, \vec{d}_2, \cdots, \vec{d}_l$ of ∂X_g such that $\vec{d}_j \cap S^1=\varnothing$,

(ii) the components $\vec{d}^{\circ}_1, \vec{d}^{\circ}_2, \cdots, \vec{d}^{\circ}_q$ of ∂X_g such that $\vec{d}^{\circ}_u \in S^1$,

(iii) the components $\vec{d}^{(v)}_1, \vec{d}^{(v)}_2, \cdots, \vec{d}^{(v)}_{t(v)}$ of ∂X_g such that $\vec{d}^{(v)}_w \cap S^1$ consists of v arcs on ∂X_g $(v=1, 2, \cdots, r)$.

We take a standard model for X_g in the same way as in [9] Fig. 1, and simple oriented loops $a_1, b_1, a_2, b_2, \cdots, a_g, b_g, d_1, d_2, \cdots, d_l, d^{\circ}_1, d^{\circ}_2, \cdots, d^{\circ}_q, d^{(1)}_1, d^{(1)}_2, \cdots, d^{(1)}_{t(1)}, d^{(2)}_1, d^{(2)}_2, \cdots, d^{(2)}_{t(2)}, \cdots, d^{(r)}_1, d^{(r)}_2, \cdots, d^{(r)}_{t(r)}$ on X_g as shown in Fig. 4. Let S^0 be a set of finite points $\tilde{s}_1, \tilde{s}_2, \cdots, \tilde{s}_m$ in $\mathring{X}_g$, and take simple oriented loops $s_1, s_2, \cdots, s_m$ on X_g as shown in Fig. 4. We put $S=S^1 \cup S^0$.

Let X^-_{2g+1} (resp. X^-_{2g+2}) be a compact connected non-orientable surface of genus $2g+1$ (resp. $2g+2$) and take a set S^1 of simple loops and simple arcs on ∂X^-_{2g+1} (resp. ∂X^-_{2g+2}). Let S^0 be a set of finite points in $\mathring{X}^-_{2g+1}$ (resp. $\mathring{X}^-_{2g+2}$). We take a standard model for X^-_{2g+1} (resp. X^-_{2g+2}) as shown in Fig.5 (resp. Fig.6). That is, X^-_{2g+1} (resp. X^-_{2g+2}) consists of a compact connected orientable surface X_g with the interior of a disk Δ removed and a Möbius strip attached by its boundary to $\partial\Delta$ (resp. X^-_{2g+2} consists of a compact orientable surface X_g with the interior of two disjoint disks $\Delta_1 \cup \Delta_2$ removed and two Möbius strips attached by their boundaries to $\partial\Delta_1 \cup \partial\Delta_2$). Furthermore, in addition to simple loops on X_g, we take a simple loop c (resp. simple loops c_1, c_2) on X^-_{2g+1} (resp. X^-_{2g+2}) as shown in Fig.5 (resp. Fig.6).

To avoid the multiplicity of brackets, we refer to loops rather than to homology classes. We take the set of the above homology classes as a generating set of the first integral homology group $H_1(X-S^0)$ of $X-S^0$, where X is X_g, X^-_{2g+1} or X^-_{2g+2}.

For X and S, we denote by $\hat{P}_n(X, S)$ the set of elements $(\hat{f}, \hat{M}, S_*)$ of $\hat{\boldsymbol{P}}_n$ such that $\hat{M}/\hat{f}=X$ and $p: \hat{M}\to X$ is an n-fold cyclic branched covering with the branched set $p(S_*)=S$, and by $\ddot{P}_n(X, S)$ the set of elements $(\ddot{f}, \ddot{M}, S_*)$ of $\ddot{\boldsymbol{P}}_n$ such that $\ddot{M}/\ddot{f}=X$ and $p: \ddot{M}\to X$ is an n-fold cyclic branched covering with the branched set $p(S_*)=S$.

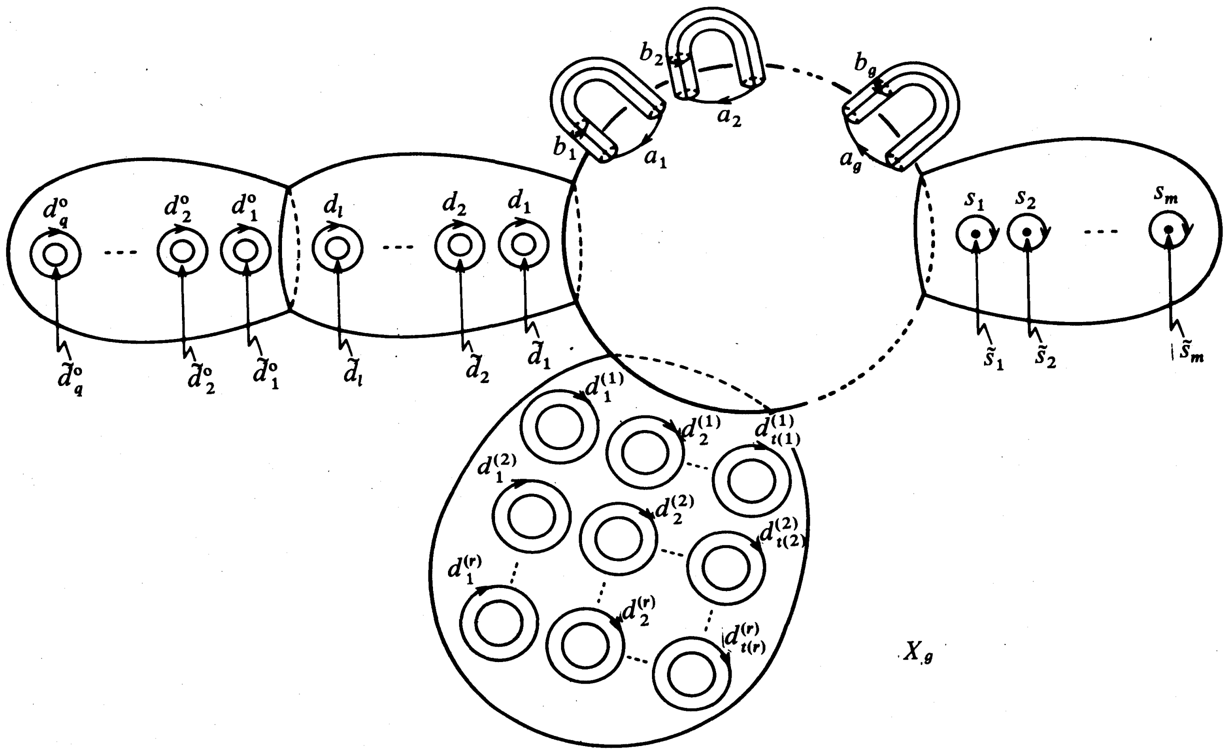

FIGURE 4

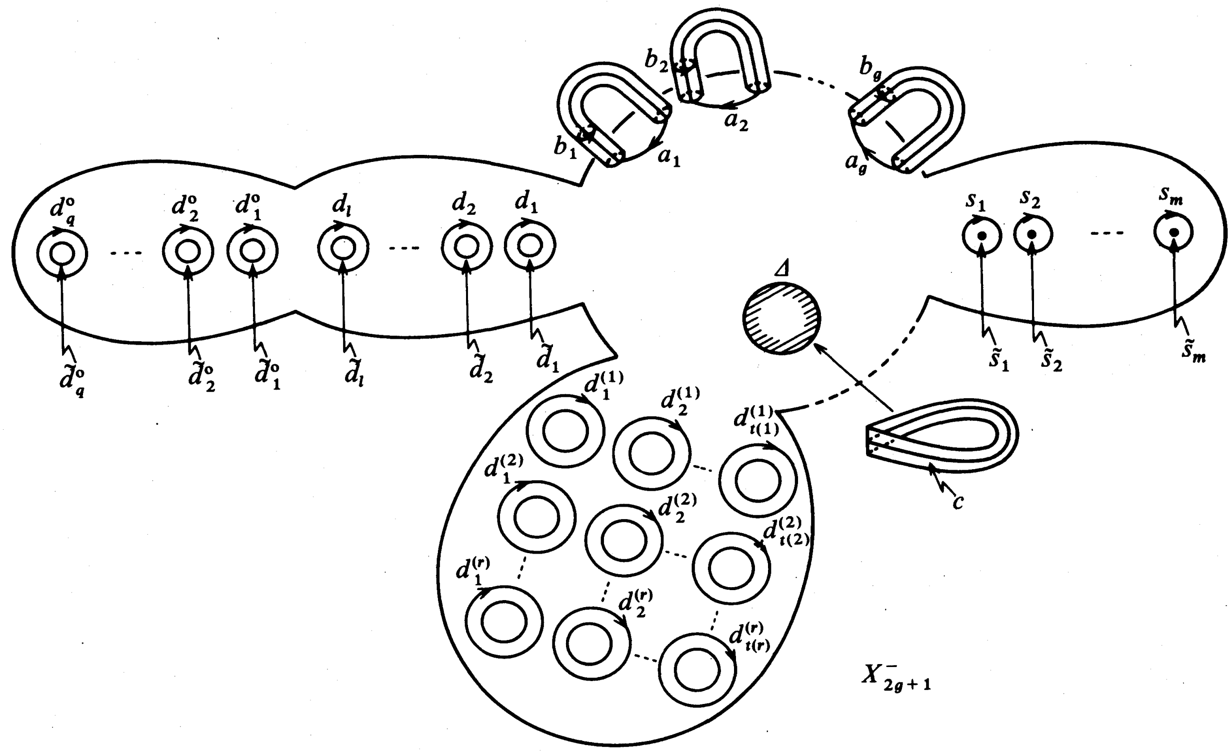

FIGURE 5

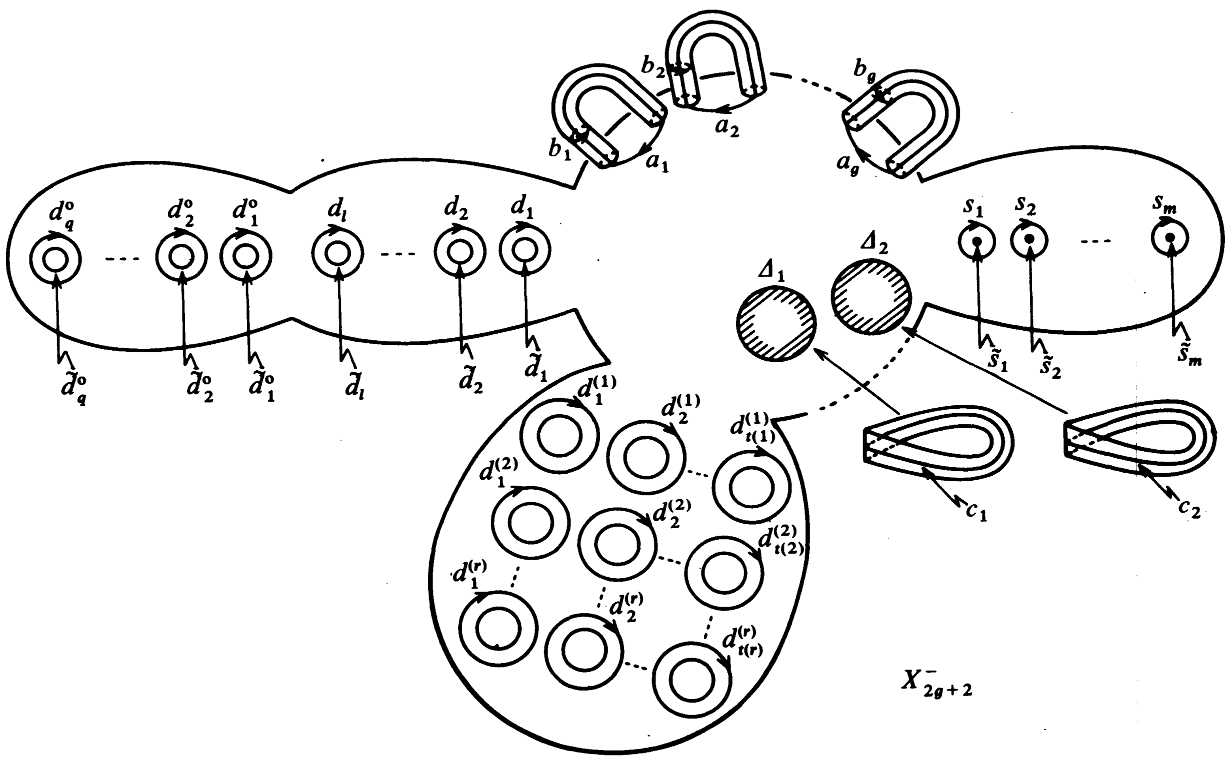

FIGURE 6

To determine the equivalence classes of $\hat{P}_n(X, S)$ and $\ddot{P}_n(X, S)$, the following is useful.

DEFINITION 3.1. Let $[H_1(X-S^0); Z_n]^*$ be a set of homomorphism ω of $H_1(X-S^0)$ onto the cyclic group Z_n of order n such that $\omega(s_k)\neq 0$ for every $s_k \in H_1(X-S^0)$, where X is X_g, X^-_{2g+1} or X^-_{2g+2}. We say that two elements ω_1 and ω_2 of $[H_1(X-S^0); Z_n]^*$ are *$\mathscr{A}$-equivalent*, denoted as $\omega_1 \underset{\mathscr{A}}{\sim} \omega_2$, if there exists a homeomorphism h of (X, S) onto (X, S) such that $\omega_1 h_* = \omega_2$, where h_* is the automorphism of $H_1(X-S^0)$ induced by $h|_{X-S^0}$.

To avoid the multiplicity of $*$, we also use h instead of h_*, if there is no confusion.

Using a branched covering theory, we obtain the following, in a similar way to P. A. Smith [6];

PROPOSITION 3.1. *There is a bijection between the set of equivalence classes of $\ddot{P}_n(X, S) \cup \hat{P}_n(X, S)$ and the set of $\mathscr{A}$-equivalence classes of $[H_1(X-S^0); Z_n]^*$.*

We express $[H_1(X-S^0); Z_n]^*$ by a set $Z^\varepsilon_n(g; l, q, t(1), t(2), \cdots, t(r), m) = Z^\varepsilon_n(g; \mathscr{L})$ of systems of integers, where $\varepsilon = +$ when $X = X_g$ or $\varepsilon = -$ when $X = X^-_g$, which is defined as follows:

DEFINITION 3.2. Let $Z_n^\varepsilon(g;\mathscr{X})=Z_n^\varepsilon(g;l,q,t(1),t(2),\cdots,t(r),m)$ be a set of systems

$$\zeta=(\alpha_1,\beta_1,\alpha_2,\beta_2,\cdots,\alpha_g,\beta_g,\boldsymbol{\gamma},\boldsymbol{\delta},\boldsymbol{\eta},\boldsymbol{\lambda}^{(1)},\boldsymbol{\lambda}^{(2)},\cdots,\boldsymbol{\lambda}^{(r)},\boldsymbol{\theta})$$

of integers where

$$\boldsymbol{\gamma}=\begin{cases}\varnothing & \text{if } \varepsilon=+\,,\\ \gamma & \text{if } \varepsilon=- \text{ and } g \text{ is odd}\,,\\ (\gamma_1,\gamma_2) & \text{if } \varepsilon=- \text{ and } g \text{ is even}\,,\end{cases}$$

$$\boldsymbol{\delta}=(\delta_1,\delta_2,\cdots,\delta_l)\,,\qquad \boldsymbol{\eta}=(\eta_1,\eta_2,\cdots,\eta_q)\,,$$

$$\boldsymbol{\lambda}^{(v)}=(\lambda_1^{(1)},\lambda_2^{(2)},\cdots,\lambda_{t(v)}^{(v)})\qquad \text{for every}\quad 1\leqq v\leqq r\,,$$

$$\boldsymbol{\theta}=(\theta_1,\theta_2,\cdots,\theta_m)\,,$$

satisfying the following two conditions:

(1) $2\tilde{\gamma}+\delta_1+\delta_2+\cdots+\delta_l+\eta_1+\eta_2+\cdots+\eta_q+\lambda_1^{(1)}+\lambda_2^{(1)}+\cdots+\lambda_{t(1)}^{(1)}+\lambda_1^{(2)}+\lambda_2^{(2)}+\cdots+\lambda_{t(2)}^{(2)}+\cdots+\lambda_1^{(r)}+\lambda_2^{(r)}+\cdots+\lambda_{t(r)}^{(r)}+\theta_1+\theta_2+\cdots+\theta_m\equiv 0 \pmod n$; where $\tilde{\gamma}=0$ (when $\varepsilon=+$), $\tilde{\gamma}=\gamma$ (when $\varepsilon=-$ and g is odd), or $\tilde{\gamma}=\gamma_1+\gamma_2$ (when $\varepsilon=-$ and g is even).

(2) g.c.d.$\{\alpha_1,\beta_1,\alpha_2,\beta_2,\cdots,\alpha_g,\beta_g,\tilde{\gamma},\delta_1,\delta_2,\cdots,\delta_l,\eta_1,\eta_2,\cdots,\eta_q,\lambda_1^{(1)},\lambda_2^{(1)},\cdots,\lambda_{t(1)}^{(1)},\lambda_1^{(2)},\lambda_2^{(2)},\cdots,\lambda_{t(1)}^{(2)},\cdots,\lambda_1^{(r)},\lambda_2^{(r)},\cdots,\lambda_{t(r)}^{(r)},\theta_1,\theta_2,\cdots,\theta_m,n\}=1$.

Let ω be an element of $[H_1(X-S^0);\boldsymbol{Z}_n]^*$, $X=X_g$ or $X=X_g^-$. If $\omega(a_i)=\alpha_i$, $\omega(b_i)=\beta_i$, $\omega(d_j)=\delta_j$, $\omega(c)=\gamma$ (or $\omega(c_1)=\gamma_1$, $\omega(c_2)=\gamma_2$), $\omega(d_u^\circ)=\eta_u$, $\omega(d_w^{(v)})=\lambda_w^{(v)}$, and $\omega(s_k)=\theta_k$ $(1\leqq i\leqq g,\ 1\leqq j\leqq l,\ 1\leqq u\leqq q,\ 1\leqq v\leqq r,\ 1\leqq w\leqq t(v),\ 1\leqq k\leqq m)$, ω is represented by an element $(\alpha_1,\beta_1,\alpha_2,\beta_2,\cdots,\alpha_g,\beta_g,\boldsymbol{\gamma},\boldsymbol{\delta},\boldsymbol{\eta},\boldsymbol{\lambda}^{(1)},\boldsymbol{\lambda}^{(2)},\cdots,\boldsymbol{\lambda}^{(r)},\boldsymbol{\theta})$ of $Z_n^\varepsilon(g,\mathscr{X})=Z_n^\varepsilon(g;l,q,t(1),t(2),\cdots,t(r),m)$.

In each case, we denote by Σ the bijection of $[H_1(X_g-S^0);\boldsymbol{Z}_n]^*$ onto $Z_n^+(g,\mathscr{X})$ (resp. of $[H_1(X_g^--S^0);\boldsymbol{Z}_n]^*$ onto $Z_n^-(g,\mathscr{X})$). We will define an equivalence relation $\underset{*}{\sim}$ on $Z_n^+(g,\mathscr{X})$ (resp. $Z_n^-(g,\mathscr{X})$) by the $\mathscr{A}$-equivalence relation $\underset{\mathscr{A}}{\sim}$ on $[H_1(X_g-S^0);\boldsymbol{Z}_n]^*$ (resp. $[H_1(X_g^--S^0);\boldsymbol{Z}_n]^*$), as follows:

DEFINITION 3.3. We say that two elements ζ_1 and ζ_2 of $Z_n^+(g,\mathscr{X})$ (resp. $Z_n^-(g,\mathscr{X})$) are *equivalent*, denoted by $\zeta_1\underset{*}{\sim}\zeta_2$, iff $\Sigma^{-1}(\zeta_1)\underset{\mathscr{A}}{\sim}\Sigma^{-1}(\zeta_2)$.

We can easily check that Σ is a bijection of the set of the $\mathscr{A}$-equivalence classes of $[H_1(X_g-S^0);\boldsymbol{Z}_n]^*$ (resp. $[H_1(X_g^--S^0);\boldsymbol{Z}_n]^*$), and the set of the equivalence classes of $Z_n^+(g,\mathscr{X})$ (resp. $Z_n^-(g,\mathscr{X})$).

To determine the equivalence classes of $\hat{P}_n(X,S)$ or $\ddot{P}_n(X,S)$, it suffices to determine the equivalence classes of $Z_{n/2}^+(g,\mathscr{X})$, $Z_{n/2}^-(g,\mathscr{X})$, $Z_n^+(g,\mathscr{X})$, or $Z_n^-(g,\mathscr{X})$.

To determine the equivalence classes of $Z_{n/2}^+(g,\mathscr{X})$ and $Z_n^+(g,\mathscr{X})$, we use the following equivalence relation $\underset{\eta}{\sim}$.

DEFINITION 3.4. (I) An element $(\boldsymbol{\delta},\boldsymbol{\eta},\boldsymbol{\lambda}^{(1)},\boldsymbol{\lambda}^{(2)},\cdots,\boldsymbol{\lambda}^{(r)},\boldsymbol{\theta})$ of $Z_n^+(0;\mathscr{X})$ is *η-equivalent* to an element $(\tilde{\boldsymbol{\delta}},\tilde{\boldsymbol{\eta}},\tilde{\boldsymbol{\lambda}}^{(1)},\tilde{\boldsymbol{\lambda}}^{(2)},\cdots,\tilde{\boldsymbol{\lambda}}^{(r)},\tilde{\boldsymbol{\theta}})$ of $Z_n^+(0;\mathscr{X})$, if and only if it holds one of the following two conditions:

(i) $(\delta, \eta, \lambda^{(1)}, \lambda^{(2)}, \cdots, \lambda^{(r)}, \theta)=(\tilde{\delta}, \tilde{\eta}, \tilde{\lambda}^{(1)}, \tilde{\lambda}^{(2)}, \cdots, \tilde{\lambda}^{(r)}, \tilde{\theta})$.

(ii) (a) If $\delta_1=\delta_2=\cdots=\delta_j=0<\delta_{j+1}$ and $\tilde{\delta}_1=\tilde{\delta}_2=\cdots=\tilde{\delta}_{j'}=0<\tilde{\delta}_{j'+1}$ then $j=j'$ and moreover $n-\delta_l=\tilde{\delta}_{j+1}, n-\delta_{l-1}=\tilde{\delta}_{j+2}, \cdots, n-\delta_{l-i+1}=\tilde{\delta}_{j+i}, \cdots, n-\delta_{j+1}=\tilde{\delta}_l$,
(b) if $\eta_1=\eta_2=\cdots=\eta_u=0<\eta_{u+1}$ and $\tilde{\eta}_1=\tilde{\eta}_2=\cdots=\tilde{\eta}_{u'}=0<\tilde{\eta}_{u'+1}$ then $u=u'$ and moreover $n-\eta_q=\tilde{\eta}_{u+1}, n-\eta_{q-1}=\eta_{u+2}, \cdots, n-\eta_{q-i+1}=\tilde{\eta}_{u+i}, \cdots, n-\eta_{u+1}=\tilde{\eta}_q$,
(c) if $\lambda_1^{(v)}=\lambda_2^{(v)}=\cdots=\lambda_{w(v)}^{(v)}=0<\lambda_{w(v)+1}^{(v)}$ and $\tilde{\lambda}_1^{(v)}=\tilde{\lambda}_2^{(v)}=\cdots=\tilde{\lambda}_{w(v)'}^{(v)}=0<\tilde{\lambda}_{w(v)'+1}^{(v)}$ for any integer v $(1\leqq v\leqq r)$, then $w(v)=w(v)'$, and moreover $n-\lambda_{t(v)}^{(v)}=\tilde{\lambda}_{w(v)+1}^{(v)}, n-\lambda_{t(v)-1}^{(v)}=\tilde{\lambda}_{w(v)+2}^{(v)}, \cdots, n-\lambda_{t(v)-i+1}^{(v)}=\tilde{\lambda}_{w(v)+i}^{(v)}, \cdots, n-\lambda_{w(v)+1}^{(v)}=\tilde{\lambda}_{t(v)}^{(v)}$, and
(d) $n-\theta_m=\tilde{\theta}_1, n-\theta_{m-1}=\tilde{\theta}_2, \cdots, n-\theta_{m-i+1}=\tilde{\theta}_i, \cdots, n-\theta_1=\tilde{\theta}_m$.

(II) An element $(1, 0, 0, 0, \cdots, 0, 0, \delta, \eta, \lambda^{(1)}, \lambda^{(2)}, \cdots, \lambda^{(r)}, \theta)$ of $Z_n^+(g; \mathscr{X})$ with $g\geqq 1$ is *η-equivalent* to an element $(1, 0, 0, 0, \cdots, 0, 0, \tilde{\delta}, \tilde{\eta}, \tilde{\lambda}^{(1)}, \tilde{\lambda}^{(2)}, \cdots, \tilde{\lambda}^{(r)}, \tilde{\theta})$ of $Z_n^+(g; \mathscr{X})$, iff

$$(\delta, \eta, \lambda^{(1)}, \lambda^{(2)}, \cdots, \lambda^{(r)}, \theta) \underset{\eta}{\sim} (\tilde{\delta}, \tilde{\eta}, \tilde{\lambda}^{(1)}, \tilde{\lambda}^{(2)}, \cdots, \tilde{\lambda}^{(r)}, \tilde{\theta}) .$$

In the same way as in [9] and [10], that is, by using the generating set of the homeotopy group of a compact surface, we will give the complete sets of equivalence classes of $Z_{n/2}^+(g; \mathscr{X})$, $Z_n^+(g; \mathscr{X})$, $Z_{n/2}^-(g; \mathscr{X})$ and $Z_n^-(g; \mathscr{X})$ respectively, in the following Theorems 3.1–3.5.

THEOREM 3.1. *The complete set of equivalence classes of $Z_n^+(g; \mathscr{X})$ is represented by the following set $\mathfrak{X}_n^+(g; \mathscr{X})$:*

(1) *For $g\geqq 1$,*

$$\mathfrak{X}_n^+(g; \mathscr{X})=\{(1, 0, 0, 0, \cdots, 0, 0, \delta, \eta, \lambda^{(1)}, \lambda^{(2)}, \cdots, \lambda^{(r)}, \theta) \text{ ; the conditions (A) and (B)}\}/\eta ,$$

where

(A) $$0\leqq\delta_1\leqq\delta_2\leqq\cdots\leqq\delta_l<n , \quad 0\leqq\eta_1\leqq\eta_2\leqq\cdots\leqq\eta_q<n ,$$
$$0\leqq\lambda_1^{(v)}\leqq\lambda_2^{(v)}\leqq\cdots\leqq\lambda_{t(v)}^{(v)}<n \ (v=1, 2, \cdots, r) , \quad 1\leqq\theta_1\leqq\theta_2\leqq\cdots\leqq\theta_m<n ,$$

(B) $$\delta_1+\delta_2+\cdots+\delta_l+\eta_1+\eta_2+\cdots+\eta_q+\lambda_1^{(1)}+\lambda_2^{(1)}+\cdots+\lambda_{t(1)}^{(1)}+\lambda_1^{(2)}+\lambda_2^{(2)}+$$
$$\cdots+\lambda_{t(2)}^{(2)}+\cdots+\lambda_1^{(r)}+\lambda_2^{(r)}+\cdots+\lambda_{r(t)}^{(r)}+\theta_1+\theta_2+\cdots+\theta_m\equiv 0 \pmod n) .$$

(2) $$\mathfrak{X}_n^+(0; \mathscr{X})=\{(\delta, \eta, \lambda^{(1)}, \lambda^{(2)}, \cdots, \lambda^{(r)}, \theta) \text{ ; the conditions (A), (B) and (C)}\}/\eta ,$$

where

(C) $$\text{g.c.d.}\{\delta_1, \delta_2, \cdots, \delta_l, \eta_1, \eta_2, \cdots, \eta_q, \lambda_1^{(1)}, \lambda_2^{(1)}, \cdots, \lambda_{t(1)}^{(1)},$$
$$\lambda_1^{(2)}, \lambda_2^{(2)}, \cdots, \lambda_{t(2)}^{(2)}, \cdots, \lambda_1^{(r)}, \lambda_2^{(r)}, \cdots, \lambda_{t(r)}^{(r)}, \theta_1, \theta_2, \cdots, \theta_m, n\}=1 .$$

OUTLINE OF THE PROOF. We can check that any element of $Z_n^+(g, \mathscr{X})$ is equivalent to an element of $\mathfrak{X}_n^+(g; \mathscr{X})$ in a similar way to Lemmas 2, 4 and 5 in [9], and that two elements of $\mathfrak{X}_n^+(g; \mathscr{X})$ are not equivalent in a similar way to Lemmas 1 and 3 in [9]. So, the proof is the same as that of Theorem 1 in [9].

THEOREM 3.2. *The complete set of equivalence classes of* $Z_n^-(2g+1, \mathscr{X})$, $g \geqq 1$, *is represented by the set* $\mathscr{X}_n^-(2g+1; \mathscr{X})$ *of disjoint union of the following four sets* $\mathscr{X}_n^-(2g+1; \mathscr{X})_1^\circ$, $\mathscr{X}_n^-(2g+1; \mathscr{X})_1^*$, $\mathscr{X}_n^-(2g+1; \mathscr{X})_2^\circ$ *and* $\mathscr{X}_n^-(2g+1; \mathscr{X})_2^*$;

$$\mathscr{X}_n^-(2g+1; \mathscr{X})_1^\circ = \{(1, 0, 0, 0, \cdots, 0, 0, \gamma, \boldsymbol{\delta}, \boldsymbol{\eta}, \boldsymbol{\lambda}^{(1)}, \boldsymbol{\lambda}^{(2)}, \cdots, \boldsymbol{\lambda}^{(r)}, \boldsymbol{\theta}) ;\ \textit{the conditions } (\mathrm{A})' \textit{ and } (\mathrm{B})'\} ,$$

where

(A)′ $0 \leqq \delta_1 \leqq \delta_2 \leqq \cdots \leqq \delta_l < n/2$, $0 \leqq \eta_1 \leqq \eta_2 \leqq \cdots \leqq \eta_q < n/2$,

$0 \leqq \lambda_1^{(v)} \leqq \lambda_2^{(v)} \leqq \cdots \leqq \lambda_{t(v)}^{(v)} < n/2$ $(v = 1, 2, \cdots, r)$, $1 \leqq \theta_1 \leqq \theta_2 \leqq \cdots \leqq \theta_m < n/2$,

(B)′ $2\gamma + \delta_1 + \delta_2 + \cdots + \delta_l + \eta_1 + \eta_2 + \cdots + \eta_q + \lambda_1^{(1)} + \lambda_2^{(1)} + \cdots + \lambda_{t(1)}^{(1)} + \lambda_1^{(2)} + \lambda_2^{(2)} + \cdots + \lambda_{t(2)}^{(2)} + \cdots + \lambda_1^{(r)} + \lambda_2^{(r)} + \cdots + \lambda_{t(r)}^{(r)} + \theta_1 + \theta_2 + \cdots + \theta_m \equiv 0 \pmod{n}$.

$$\mathscr{X}_n^-(2g+1; \mathscr{X})_1^* = \{(1, 0, 0, 0, \cdots, 0, 0, \gamma, \boldsymbol{\delta}, \boldsymbol{\eta}, \boldsymbol{\lambda}^{(1)}, \boldsymbol{\lambda}^{(2)}, \cdots, \boldsymbol{\lambda}^{(r)}, \boldsymbol{\theta}) ;\ \textit{the conditions } (*), (\mathrm{A})'_* \textit{ and } (\mathrm{B})'\} ,$$

where

$(*)$ $\delta_l = n/2$, $\eta_q = n/2$, $\lambda_{t(v)}^{(v)} = n/2$ $(1 \leqq \exists v \leqq r)$ *or* $\theta_m = n/2$,

$(\mathrm{A})'_*$ $0 \leqq \delta_1 \leqq \delta_2 \leqq \cdots \leqq \delta_l \leqq n/2$, $0 \leqq \eta_1 \leqq \eta_2 \leqq \cdots \leqq \eta_q \leqq n/2$, $0 \leqq \gamma < n/2$,

$0 \leqq \lambda_1^{(v)} \leqq \lambda_2^{(v)} \leqq \cdots \leqq \lambda_{t(v)}^{(v)} \leqq n/2$ $(v = 1, 2, \cdots, r)$, $1 \leqq \theta_1 \leqq \theta_2 \leqq \cdots \leqq \theta_m \leqq n/2$,

$$\mathscr{X}_n^-(2g+1; \mathscr{X})_2^\circ = \{(2, 0, 0, 0, \cdots, 0, 0, \gamma, \boldsymbol{\delta}, \boldsymbol{\eta}, \boldsymbol{\lambda}^{(1)}, \boldsymbol{\lambda}^{(2)}, \cdots, \boldsymbol{\lambda}^{(r)}, \boldsymbol{\theta}) ;\ \textit{the conditions } (\mathrm{OE}), (\mathrm{A})' \textit{ and } (\mathrm{B})'\} ,$$

where

(OE) γ *is odd*, δ_j *is even*, η_q *is even*, $\lambda_w^{(v)}$ *is even*, θ_m *is even*,

$$\mathscr{X}_n^-(2g+1; \mathscr{X})_2^* = \{(2, 0, 0, 0, \cdots, 0, 0, \gamma, \boldsymbol{\delta}, \boldsymbol{\eta}, \boldsymbol{\lambda}^{(1)}, \boldsymbol{\lambda}^{(2)}, \cdots, \boldsymbol{\lambda}^{(r)}, \boldsymbol{\theta}) ;\ \textit{the conditions } (*), (\mathrm{OE}), (\mathrm{A})' \textit{ and } (\mathrm{B})'\} .$$

THEOREM 3.3. *The complete set of equivalence classes of* $Z_n^-(1, \mathscr{X})$ *is represented by the disjoint union* $\mathscr{X}_n^-(1; \mathscr{X})$ *of the following two sets* $\mathscr{X}_n^-(1; \mathscr{X})^\circ$ *and* $\mathscr{X}_n^-(1; \mathscr{X})^*$;

$$\mathscr{X}_n^-(1; \mathscr{X})^\circ = \{(\gamma, \boldsymbol{\delta}, \boldsymbol{\eta}, \boldsymbol{\lambda}^{(1)}, \boldsymbol{\lambda}^{(2)}, \cdots, \boldsymbol{\lambda}^{(r)}, \boldsymbol{\theta}) ;\ \textit{the conditions } (\mathrm{A})', (\mathrm{B})' \textit{ and } (\mathrm{C})'\} ,$$

where

(C)′ $\text{g.c.d.}\{\gamma, \delta_1, \delta_2, \cdots, \delta_l, \eta_1, \eta_2, \cdots, \eta_q, \lambda_1^{(1)}, \lambda_2^{(1)}, \cdots, \lambda_{t(1)}^{(1)}, \lambda_1^{(2)}, \lambda_2^{(2)}, \cdots, \lambda_{t(2)}^{(2)}, \cdots, \lambda_1^{(r)}, \lambda_2^{(r)}, \cdots, \lambda_{t(r)}^{(r)}, \theta_1, \theta_2, \cdots, \theta_m, n\} = 1$.

$$\mathscr{Z}_n^-(1;\mathscr{L})^* = \{(\gamma, \boldsymbol{\delta}, \boldsymbol{\eta}, \boldsymbol{\lambda}^{(1)}, \boldsymbol{\lambda}^{(2)}, \cdots, \boldsymbol{\lambda}^{(r)}, \boldsymbol{\theta})\ ;\ \textit{the conditions } (*), (\mathrm{A})'_*, (\mathrm{B})' \textit{ and } (\mathrm{C})'\} .$$

OUTLINE OF PROOFS OF THEOREMS 3.2 AND 3.3. We can check that any element of $Z_n^-(2g+1, \mathscr{L})$ is equivalent to an element of $\mathscr{Z}_n^-(2g+1; \mathscr{L})$ in a similar way to Lemmas 2.2, 2.3 and 2.5 in [10], and that two elements of $\mathscr{Z}_n^-(2g+1; \mathscr{L})$ are not equivalent in a similar way to Lemmas 2.4, 2.6 and 2.7 in [10]. So, the proofs are the same as those of Theorems 2.1 and 2.2 in [10].

THEOREM 3.4. *The complete set of equivalence classes of* $Z_n^-(2g+2; \mathscr{L})$, $g \geqq 1$, *is represented by the set* $\mathscr{Z}_n^-(2g+2; \mathscr{L})$ *of disjoint union of the following four sets* $\mathscr{Z}_n^-(2g+2; \mathscr{L})_1^\circ$, $\mathscr{Z}_n^-(2g+2; \mathscr{L})_1^*$, $\mathscr{Z}_n^-(2g+2; \mathscr{L})_2^\circ$ *and* $\mathscr{Z}_n^-(2g+2; \mathscr{L})_2^*$;

$$\mathscr{Z}_n^-(2g+2;\mathscr{L})_1^\circ = \{(1,0,0,0,\cdots,0,0,\gamma_1,\gamma_2,\boldsymbol{\delta},\boldsymbol{\eta},\boldsymbol{\lambda}^{(1)},\boldsymbol{\lambda}^{(2)},\cdots,\boldsymbol{\lambda}^{(r)},\boldsymbol{\theta})\ ;\ \gamma_1=0, \textit{ and the conditions } (\mathrm{A})', (\mathrm{B})''\} ,$$

where

$$(\mathrm{B})'' \qquad 2\gamma_2+\delta_1+\delta_2+\cdots+\delta_l+\eta_1+\eta_2+\cdots+\eta_q+\lambda_1^{(1)}+\lambda_2^{(1)}+\cdots+\lambda_{t(1)}^{(1)}+\lambda_1^{(2)}+ \lambda_2^{(2)}+\cdots+\lambda_{t(2)}^{(2)}+\cdots+\lambda_1^{(r)}+\lambda_2^{(r)}+\cdots+\lambda_{t(r)}^{(r)}+\theta_1+\theta_2+\cdots+\theta_m \equiv 0 \pmod n ,$$

$$\mathscr{Z}_n^-(2g+2;\mathscr{L})_1^* = (1,0,0,0,\cdots,0,0,\gamma_1,\gamma_2,\boldsymbol{\delta},\boldsymbol{\eta},\boldsymbol{\lambda}^{(1)},\boldsymbol{\lambda}^{(2)},\cdots,\boldsymbol{\lambda}^{(r)},\boldsymbol{\theta})\ ;\ \textit{the conditions } (*), (\mathrm{A})'_* \textit{ and } (\mathrm{B})''\} ,$$

$$\mathscr{Z}_n^-(2g+2;\mathscr{L})_2^\circ = \{(2,0,0,0,\cdots,0,0,\gamma_1,\gamma_2,\boldsymbol{\delta},\boldsymbol{\eta},\boldsymbol{\lambda}^{(1)},\boldsymbol{\lambda}^{(2)},\cdots,\boldsymbol{\lambda}^{(r)},\boldsymbol{\theta})\ ;\ \gamma_1=1, \textit{ and the conditions } (\mathrm{OE})', (\mathrm{A})' \textit{ and } (\tilde{\mathrm{B}})''\} ,$$

where

$$(\mathrm{OE})' \qquad \gamma_2 \textit{ is odd}, \ \delta_j \textit{ is even}, \ \eta_q \textit{ is even}, \ \lambda_w^{(v)} \textit{ is even}, \ \theta_m \textit{ is even} ,$$

$$(\tilde{\mathrm{B}})'' \quad 2+2\gamma_2+\delta_1+\delta_2+\cdots+\delta_l+\eta_1+\eta_2+\cdots+\eta_q+\lambda_1^{(1)}+\lambda_2^{(1)}+\cdots+\lambda_{t(1)}^{(1)}+\lambda_1^{(2)} +\lambda_2^{(2)}+\cdots+\lambda_{t(2)}^{(2)}+\cdots+\lambda_1^{(r)}+\lambda_2^{(r)}+\cdots+\lambda_{t(r)}^{(r)}+\theta_1+\theta_2+\cdots+\theta_m \equiv 0 \pmod n ,$$

$$\mathscr{Z}_n^-(2g+2;\mathscr{L})_2^* = \{(2,0,0,0,\cdots,0,0,\gamma_1,\gamma_2,\boldsymbol{\delta},\boldsymbol{\eta},\boldsymbol{\lambda}^{(1)},\boldsymbol{\lambda}^{(2)},\cdots,\boldsymbol{\lambda}^{(r)},\boldsymbol{\theta})\ ;\ \gamma_1=1, \textit{ the conditions } (*), (\mathrm{OE})', (\mathrm{A})' \textit{ and } (\mathrm{B})'\} .$$

OUTILINE OF THE PROOF. We can check that any element of $Z_n^-(2g+2, \mathscr{L})$ is equivalent to an element of $\mathscr{Z}_n^-(2g+2, \mathscr{L})$ in a similar way to Lemmas 3.2, 3.3 and 3.5 in [10], and that two elements of $\mathscr{Z}_n^-(2g+2, \mathscr{L})$ are not equivalent in a similar way to Lemmas 3.4, 3.6 and 3.7 in [10]. So, the proof is the same as those of Theorems 3.1 and 3.2 in [10].

THEOREM 3.5. *The complete set of equivalence classes of* $Z_n^-(2; \mathscr{L})$ *is represented by the set* $\mathscr{Z}_n^-(2; \mathscr{L})$ *of disjoint union of the following two sets* $\mathscr{Z}_n^-(2; \mathscr{L})^\circ$ *and* $\mathscr{Z}_n^-(2; \mathscr{L})^*$;

$$\mathscr{Z}_n^-(2;\mathscr{L})^\circ = \{(\gamma_1,\gamma_2,\boldsymbol{\delta},\boldsymbol{\eta},\boldsymbol{\lambda}^{(1)},\boldsymbol{\lambda}^{(2)},\cdots,\boldsymbol{\lambda}^{(r)},\boldsymbol{\theta})\ ;\ \textit{the conditions } (\tilde{\mathrm{A}})', (\mathrm{D}), (\tilde{\tilde{\mathrm{B}}}) \textit{ and } (\mathrm{C})''\} ,$$

where

(D) $\quad 0 \leqq \gamma_1 \leqq \gamma_2 < n\,,\quad 0 \leqq \gamma_1 \leqq \lceil \alpha/2 \rceil,\quad \gamma_1 + \gamma_2 \leqq n\,,$

$(\tilde{\tilde{\mathrm{B}}})$ $\quad 2\gamma_1 + 2\gamma_2 + \delta_1 + \delta_2 + \cdots + \delta_l + \eta_1 + \eta_2 + \cdots + \eta_q + \lambda_1^{(1)} + \lambda_2^{(1)} + \cdots + \lambda_{t(1)}^{(1)} + \lambda_1^{(2)} + \lambda_2^{(2)} + \cdots + \lambda_{t(2)}^{(2)} + \cdots + \lambda_1^{(r)} + \lambda_2^{(r)} + \cdots + \lambda_{t(r)}^{(r)} + \theta_1 + \theta_2 + \cdots + \theta_m \equiv 0 \pmod{n}\,,$

(C)″ $\quad$ g.c.d.$\{\gamma_1, \gamma_2, \delta_1, \delta_2, \cdots, \delta_l, \eta_1, \eta_2, \cdots, \eta_q, \lambda_1^{(1)}, \lambda_2^{(1)}, \cdots, \lambda_{t(1)}^{(1)}, \lambda_1^{(2)}, \lambda_2^{(2)}, \cdots, \lambda_{t(2)}^{(2)}, \cdots, \lambda_1^{(r)}, \lambda_2^{(r)}, \cdots, \lambda_{t(r)}^{(r)}, \theta_1, \theta_2, \cdots, \theta_m, n\} = 1\,,$

$\mathscr{Z}_n^-(2; \mathscr{X})^* = \{(\gamma_1, \gamma_2, \boldsymbol{\delta}, \boldsymbol{\eta}, \boldsymbol{\lambda}^{(1)}, \boldsymbol{\lambda}^{(2)}, \cdots, \boldsymbol{\lambda}^{(r)}, \boldsymbol{\theta})$; *the conditions* $(*)$, (A)′, (D), $(\tilde{\tilde{\mathrm{B}}})$ *and* (C)″$\}$,

where

$$\alpha = \text{g.c.d.}\{\gamma_1 + \gamma_2, \delta_1, \delta_2, \cdots, \delta_l, \eta_1, \eta_2, \cdots, \eta_q, \lambda_1^{(1)}, \lambda_2^{(1)}, \cdots, \lambda_{t(1)}^{(1)}, \lambda_1^{(2)}, \lambda_2^{(2)}, \cdots, \lambda_{t(2)}^{(2)}, \cdots, \lambda_1^{(r)}, \lambda_2^{(r)}, \cdots, \lambda_{t(r)}^{(r)}, \theta_1, \theta_2, \cdots, \theta_m, n\}\,.$$

OUTLINE OF THE PROOF. We can check that any element of $Z_n^-(2, \mathscr{X})$ is equivalent to an element of $\mathscr{Z}_n^-(2; \mathscr{X})$ in a similar way to Lemmas 4.1, 4.2, 4.3, 4.4 and 4.7 in [10], and that two elements of $\mathscr{Z}_n^-(2; \mathscr{X})$ are not equivalent in a similar way to Lemmas 4.5, 4.6 and 4.8 in [10]. So, the proof is the same as that of Theorem 4.2 in [10].

By topological and geometrical consideration, we have the following (cf. Proposition 2.4 in [10]).

PROPOSITION 3.2.

(1) *An element* (f, M) *of* $\ddot{P}_n$ *corresponding to an element of* $\mathscr{Z}_{n/2}^+(g; \mathscr{X})$ *belongs to* $\ddot{P}_n^+$.

(2) *An element* (f, M) *of* $\ddot{P}_n$ *corresponding to an element of* $\mathscr{Z}_{n/2}^-(g; \mathscr{X})$ *belongs to* $\ddot{P}_n^{\circ}$.

(3) *An element* (f, M) *of* $\hat{P}_n$ *corresponding to an element of* $\mathscr{Z}_n^+(g; \mathscr{X})$ *belongs to* $\hat{P}_n^{+-}$.

(4) *An element* (f, M) *of* $\hat{P}_n$ *corresponding to an element of* $\mathscr{Z}_n^-(g; \mathscr{X})_1^{\circ}$ *or* $Z_n^-(g; l, q, t(1), t(2), \cdots, t(r), m)_1^*$ *belongs to* $\hat{P}_n^{\circ\circ}$ *if* $g \geqq 3$.

(5) *An element* (f, M) *of* $\hat{P}_n$ *corresponding to an element of* $\mathscr{Z}_n^-(g; \mathscr{X})_2^{\circ}$ *or* $Z_n^-(g; \mathscr{X})_2^*$ *belongs to* $\hat{P}_n^{-+}$ *if* $n/2$ *is odd and* $g \geqq 3$.

(6) *An element* (f, M) *of* $\hat{P}_n$ *corresponding to an element of* $\mathscr{Z}_n^-(g; \mathscr{X})_2^{\circ}$ *or* $Z_n^-(g; \mathscr{X})_2^*$ *belongs to* $\hat{P}_n^{--}$ *if* $n/2$ *is even and* $g \geqq 3$.

(7) *An element* (f, M) *of* $\hat{P}_n$ *corresponding to an element of* $\mathscr{Z}_n^-(1; \mathscr{X})$ *belongs to* $\hat{P}_n^{\circ\circ}$ *if* δ *is odd.*

(8) *An element* (f, M) *of* $\hat{P}_n$ *corresponding to an element of* $\mathscr{Z}_n^-(1; \mathscr{X})$ *belongs to* $\hat{P}_n^{-+}$ *if* $n/2$ *is odd and* δ *is even.*

(9) *An element* (f, M) *of* $\hat{P}_n$ *corresponding to an element of* $\mathscr{Z}_n^-(1; \mathscr{X})$ *belongs to* $\hat{P}_n^{--}$ *if* $n/2$ *is even and* δ *is even.*

(10) *An element* (f, M) *of* $\hat{P}_n$ *corresponding to an element of* $\mathscr{Z}_n^-(2; \mathscr{X})$ *belongs to* $\hat{P}_n^{\circ\circ}$ *if* δ *is odd, or* δ *is even and* $\gamma_1 + \gamma_2$ *is odd.*

(11) *An element* (f, M) *of* $\hat{P}_n$ *corresponding to an element of* $\mathscr{Z}_n^-(2; \mathscr{Z})$ *belongs to* $\hat{P}_n^{-+}$ *if* $n/2$ *is odd,* δ *is even and* $\gamma_1+\gamma_2$ *is even.*

(12) *An element* (f, M) *of* $\hat{P}_n$ *corresponding to an element of* $\mathscr{Z}_n^-(2; \mathscr{Z})$ *belongs to* $\hat{P}_n^{--}$ *if* $n/2$ *is even,* δ *is even and* $\gamma_1+\gamma_2$ *is even.*

Here

$$\delta=\text{g.c.d.}\{\delta_1, \delta_2, \cdots, \delta_l, \eta_1, \eta_2, \cdots, \eta_q, \lambda_1^{(1)}, \lambda_2^{(1)}, \cdots, \lambda_{t(1)}^{(1)}, \lambda_1^{(2)}, \lambda_2^{(2)}, \cdots, \lambda_{t(2)}^{(2)}, \cdots, \lambda_1^{(r)}, \lambda_2^{(r)}, \cdots, \lambda_{t(r)}^{(r)}, \theta_1, \theta_2, \cdots, \theta_m, n\}.$$

§4. Examples.

In the next section, we will give the proofs of the main theorems using the combinatorial theory. In this section, we discuss periodic maps on the Klein bottle Kb and the torus T, which will be helpful to understand its complicated arguments.

Example 1. Periodic maps on the Klein bottle Kb.

First we consider the case that $\mathscr{S}^1(f)\neq\varnothing$. Let (f, Kb) be an element of P_n such that $\mathscr{S}^1(f)\neq\varnothing$. Then $(f, Kb)\in\hat{P}^o\cup\hat{P}^{+-}\cup\hat{P}^{--}\cup P^{oo}$.

Case (1) $(f, Kb)\in\hat{P}^o$. Since $\tilde{l}=0$, $\tilde{r}=0$ and $\hat{g}=\frac{1}{2}(2-2\hat{q}+2)$, where $\hat{g}$ is the genus of $\hat{Kb}$. $\hat{g}>0$ implies that $2>\hat{q}$, so $\hat{q}=1$. Since $q_a=\hat{q}_a/a=\tilde{q}_a/a=1/a=1$, $a=1$. Hence $q_a=0$ for every a with $a|n$ unless $q_1=1$ $(=\tilde{q}_1=\hat{q}_1)$. $g=\frac{1}{n}\{\sum_{a|n}(a-n)m_{2/a}+n\}$ is the genus of $X=Kb/f$. Hence $m_{2/a}=0$ for every a with $a|n$, and $g=1$, and so, X is a Möbius strip. By Theorem 3.3, $\{(\gamma, \eta)\,;\,2\gamma+\eta\equiv0\ (\text{mod}\ n/2)\}$ is representative. Hence, in this case, there exist $\lfloor\varphi(n/2)/2\rfloor$ non-equivalent periodic maps on Kb with period n.

Case (2) $(f, Kb)\in\hat{P}^{+-}$. Since $\hat{g}=\frac{1}{2}(2-2\tilde{q}^+-\tilde{q}^-)$, where $\hat{g}$ is the genus of $\hat{M}$, is a non-negative integer, $1\geqq\tilde{q}^++\frac{1}{2}\tilde{q}^-$. Hence (a) $\tilde{q}^+=1$ and $\tilde{q}^-=0$ or (b) $\tilde{q}^+=0$ and $\tilde{q}^-=2$. Hence $\hat{Kb}$ is an annulus.

Subcase (a) Since $\hat{q}_{2a}^+=2\tilde{q}_a^+=2$ and $q_{2a}^+=\hat{q}_{2a}^+/(2a)=\tilde{q}_a^+/(2a)=2/(2a)$, $a=1$. Hence $q_a^+=0$ for every a with $a|n$ unless $q_2^+=1$ ($\hat{q}_2^+=1$ and $\tilde{q}_1^+=2$). $g=\frac{1}{2n}\{\sum_{a|n}(a-n)m_a+n\}$ is the genus of $X=Kb/f$. Hence $\sum_{a|n}(a-n)m_a=-n$. This equation has a unique solution $m_a=n/a=2$, since $a\neq n$ and $a|n$. Hence, $n=2$ and $g=0$, and so, X is a disk with two singular points. In this case, there exists a unique involution on Kb.

Subcase (b) It is easily checked that exactly one a with $a|n$ satisfies $\tilde{q}_a^-\neq0$. Since $\hat{q}_a^-=\tilde{q}_a^-=2$ and $q_a^-=\hat{q}_a^-/a=2/a$, $a=1$ or $a=2$. If $a=2$, $q_2^-=1$. $q=1$, so g is not an integer, which is a contradiction. Hence $a=1$, and so, $\hat{q}_1^-=\tilde{q}_1^-=q_1^-=2$. $g=\frac{1}{2n}\{\sum_{a|n}(a-n)m_a\}$ is the genus of $X=Kb/f$. Hence $\sum_{a|n}(a-n)m_a=0$ and $g=0$. X is an annulus. By Theorem 3.1 (2), $\{(\eta_1, \eta_2)\,;\,\eta_1+\eta_2\equiv0\ (\text{mod}\ n)\ \text{g.c.d.}(\eta_1, \eta_2)=1\}$ is representative. Hence, in this case, there exist $\lfloor\varphi(n)/2\rfloor$ non-equivalent periodic maps on Kb with period n.

Case (3) $(f, Kb)\in\hat{P}^{oo}$. Since $\hat{g}=2-2\tilde{q}^+-\tilde{q}^-$, where $\hat{g}$ is the genus of $\hat{Kb}$, is a positive integer, $2\geqq2\tilde{q}^++\tilde{q}^-$, so $\tilde{q}^+=0$ and $\tilde{q}^-=1$. Hence $\hat{Kb}$ is a Möbius strip. Since

$q_a^- = \hat{q}_a^-/a = \tilde{q}_a^-/a = 1/a$, $a=1$. But, this is impossible, because $q_a^- = 0$ for every odd integer a with $a|n$.

Case (4) $(f, Kb) \in \hat{P}^{--}$. Since $\hat{g} = \frac{1}{2}(2 - 2\tilde{q}^+ - \tilde{q}^-)$, where $\hat{g}$ is the genus of $\hat{M}$, is a non negative integer, $1 \geqq \tilde{q}^+ + \frac{1}{2}\tilde{q}^-$. Hence (a) $\tilde{q}^+ = 1$ and $\tilde{q}^- = 0$ or (b) $\tilde{q}^+ = 0$ and $\tilde{q}^- = 2$. Hence $\hat{Kb}$ is an annulus.

Subcase (a) Since $\hat{q}_{2a}^+ = 2\tilde{q}_a^+ = 2$ and $q_{2a}^+ = \hat{q}_{2a}^+/(2a) = \tilde{q}_a^+/(2a) = 2/(2a)$, $a=1$. Hence $q_a^+ = 0$ for every a with $a|n$ unless $q_2^+ = 1$ ($\hat{q}_2^+ = 2$ and $\hat{q}_1^+ = 1$). But, this is impossible, because $q_a^+ = 0$ for every integer a with $a|(n/2)$ and $n/2$ is even.

Subcase (b) It is easy to check that exactly one a with $a|n$ satisfies $\tilde{q}_a^- \neq 0$. Since $\hat{q}_a^- = \tilde{q}_a^- = 2$ and $q_a^- = \hat{q}_a^-/a = 2/a$, $a=1$ or $a=2$. If $a=1$, $q_1^- = \hat{q}_1^- = 2$, so $g = \frac{1}{n}\{\sum_{a|n}(a-n)m_a\}$ is a positive integer, which is impossible. Hence $a=2$, and so, $q_2^- = 1$ and $\tilde{q}_2^- = \hat{q}_2^- = 2$. $g = \frac{1}{n}\{\sum_{a|n}(a-n)m_a + n\}$ is the genus of $X = Kb/f$. Hence $m_a = 0$ for every a with $a|n$ and $g=1$. X is Möbius strip. By Theorem 3.3, $\{(\gamma, \eta); 2\gamma + \eta \equiv 0 \pmod{n}\}$ is representative. Hence, in this case, there exist $\lfloor \varphi(n/2) \rfloor$ non-equivalent periodic maps on Kb with period n.

Next we consider the case that $\mathscr{S}^1(f) = \varnothing$. Let (f, Kb) be an element of P_n such that $\mathscr{S}^1(f) = \varnothing$. Then $(f, Kb) \in P_n^o$, where P_n^o is the set of elements $(f, Kb) \in P_n$ such that $\mathscr{S}^1(f) = \varnothing$.

Case (5) $(f, Kb) \in P_n^o$. Since $g = \frac{1}{n}\{\sum_{a|n}(a-n)m_a\}$ is a positive integer, $\sum_{a|n}(a-n)m_a = 0$ (if $\mathscr{S}^0(f) = \varnothing$) or $\sum_{a|n}(a-n)m_a = -n$ (if $\mathscr{S}^0(f) \neq \varnothing$).

Subcase (a) $\mathscr{S}^0(f) = \varnothing$. In this case, $X = Kb/f$ is a Klein bottle. By Theorem 4.2 in [10], $\{\{(\gamma_1, \gamma_2); 2\gamma_1 + 2\gamma_2 \equiv 0 \pmod{n}$, g.c.d.$\{\gamma_1, \gamma_2, n\} = 1\}$ is representative. Hence, in this case, there exist $\lfloor \varphi(n/2) \rfloor$ non-equivalent periodic maps on Kb with period n.

Subcase (b) $\mathscr{S}^0(f) \neq \varnothing$. In this case, $m_a = 0$ for every a with $a|n$ unless $m_1 = 2$. So we have $n=2$ and $g=1$, and so, $X = Kb/f$ is a projective plane. By Theorem 2.2 (2) in [10], $\{(\gamma, m_1, m_2); m_1 + m_2 \equiv 0 \pmod{2}\}$ is representative. Hence, in this case, there exists a unique involution on Kb represented by (0, 1, 1). That is, let $Kb = X \cup X$ and g be an involution (Lemma 3.2) on the boundary circle $\partial X = S^1$. Then f is an extension of g on Kb such that one X maps onto the other X by f.

Example 2. Periodic maps on the torus T.

First we consider the case that $\mathscr{S}^1(f) \neq \varnothing$. Let (f, T) be an element of P_n such that $\mathscr{S}^1(f) \neq \varnothing$. Then $(f, T) \in \ddot{P}^+ \cup \hat{P}^{-+}$.

Case (1) $(f, T) \in \ddot{P}^+$. Since $\hat{g} = \frac{1}{2}(2 - \tilde{q})$, where $\hat{g}$ is the genus of $\hat{T}$, is a non-negative integer, $2 \geqq \tilde{q}$. Hence $\tilde{q} = 2$, and so, $\hat{T}$ is an annulus. Since $\hat{q}_a = \tilde{q}_a = 2$ and $q_a = \hat{q}_a/a = \tilde{q}_a/a = 2/a$, $a=1$ or $a=2$. Hence (a) $q_a = 0$ for every a with $a|n$ unless $q_1 = 2$ ($\hat{q}_1 = 2$ and $\tilde{q}_1 = 2$) or (b) $q_a = 0$ for every a with $a|n$ unless $q_2 = 1$ ($\hat{q}_2 = 2$ and $\tilde{q}_2 = 2$).

Subcase (a) $g = \frac{1}{n}\{\sum_{a|n}((a-n)/2)m_{a/2} + n/2\}$ is the genus of $X = T/f$. Hence $\sum_{a|n}((a-n)/2)m_{a/2} = -n/2$. This equation has a unique solution $m_{a/2} = 2$, since $a \neq n$ and $a|n$. Hence, $n=4$, $a=2$, $m_1 = 2$ and $g=0$, and so, X is a disk with two singular points. In this case, there exists a unique periodic map with period 4 (represented by

$\{(\eta, \theta_1, \theta_2); \eta=0, \theta_1=\theta_2=1\})$.

Subcase (b) $g=\frac{1}{n}\{\sum_{a|n}((a-n)/2)m_{a/2}\}$ is the genus of $X=T/f$. Hence $\sum_{a|n}((a-n)/2)m_{a/2}=0$ and $g=0$. X is an annulus. By Theorem 3.1 (2), $\{(\eta_1, \eta_2); 1 \leqq \eta_1, \eta_2 < n/2, \eta_1+\eta_2 \equiv 0 \pmod{n/2}$, g.c.d.$\{\eta_1, n/2\}=1\}$ is representative. Hence, in this case, there exist $\lfloor \varphi(n/2)/2 \rfloor$ non-equivalent periodic maps on T with period n.

Case (2) $(f, T) \in \hat{P}^{-+}$. Since $\tilde{l}=0$ and $\tilde{r}=0$, $\hat{g}=1-\tilde{q}$, where $\hat{g}$ is the genus of $\hat{T}$. $\hat{g} \geqq 0$ implies that $1 \geqq \tilde{q}$, and so, $\tilde{q}=1$, Since $q_{2a}=\hat{q}_{2a}/(2a)=2\tilde{q}_a/(2a)=2/(2a)$, $a=1$. Hence $q_a=0$ for every a with $a|n$ unless $q_2=1$ ($\tilde{q}_1=1, \hat{q}_2=2$). $g=\frac{1}{n}\{\sum_{a|n}(a-n)m_a+n\}$ is the genus of $X=T/f$. Hence $m_a=0$ for every a with $a|n$, and $g=1$, and so, X is a Möbius strip. By Theorem 3.3, $\{(\gamma, \eta); 2\gamma+\eta \equiv 0 \pmod{n}\}$ is representative. Hence, in this case, there exist $\lfloor \varphi(n/2)/2 \rfloor$ non-equivalent periodic maps on T with period n.

Next we consider the case that $\mathscr{S}^1(f)=\varnothing$. Let (f, T) be an element of P_n such that $\mathscr{S}^1(f)=\varnothing$. Then $(f, T) \in P_n^+ \cup P_n^-$. Here P_n^+ is the set of elements $(f, M) \in P_n$ such that $\mathscr{S}^1(f)=\varnothing$ where M is an orientable surface and f is an orientation preserving periodic map on M, and P_n^- is the set of elements $(f, M) \in P_n$ such that $\mathscr{S}^1(f)=\varnothing$ where M is an orientable surface and f is an orientation reversing periodic map on M.

Case (3) $(f, M) \in P^+$. Since $g=\frac{1}{2n}\{\sum_{a|n}(a-n)m_a+2n\}$ is a non-negative integer, $\sum_{a|n}(a-n)m_a=0$ (if $\mathscr{S}^0(f)=\varnothing$) or $\sum_{a|n}(a-n)m_a=-2n$ (if $\mathscr{S}^0(f) \neq \varnothing$).

Subcase (a) $\mathscr{S}^0(f) \neq \varnothing$. In this case, $X=T/f$ is a sphere. By Theorem 1 (2) in [9], there are the following four representative elements; (1) $\{(1, 1, 1, 1)\}$ if $n=2$, (2) $\{(1, 1, 1)\}$ if $n=3$, (3) $\{(1, 1, 2)\}$ if $n=4$ or (4) $\{(1, 2, 3)\}$ if $n=6$. In every case, there exists a unique periodic map on T with period n.

Subcase (b) $\mathscr{S}^0(f)=\varnothing$. In this case, $X=T/f$ is a torus. By Theorem 1 (1) in [9], $\{(0, 1)\}$ is representative. Hence, in this case, there exists a unique periodic map on T.

Case (4) $(f, T) \in P^-$. Since $g=\frac{1}{n}\{\sum_{a|n}(a-n)m_a+2n\}$ is a positive integer, $\sum_{a|n}(a-n)m_a=0$ (if $\mathscr{S}^0(f)=\varnothing$) or $\sum_{a|n}(a-n)m_a=-n$ (if $\mathscr{S}^0(f) \neq \varnothing$).

Subcase (a) $\mathscr{S}^0(f) \neq \varnothing$. In this case, $X=T/f$ is a torus. By Theorem B in [10], there are no periodic maps on T.

Subcase (b) $\mathscr{S}^0(f)=\varnothing$. In this case, $X=T/f$ is a Klein bottle. By Theorem 2.2 (2) in [10], since n is even, $\{(\gamma_1, \gamma_2); 2\gamma_1+2\gamma_2 \equiv 0 \pmod{n}$, g.c.d.$\{\gamma_1, \gamma_2\}=1, \gamma=\gamma_1+\gamma_2$ is even$\}$ is representative. Hence, in this case, there exist $\lfloor \varphi(n)/2 \rfloor$ periodic maps on T if $n/2$ is odd, or there exist $\lfloor \varphi(n)+\varphi(n/2)/2 \rfloor$ periodic maps on T if $n/2$ is even.

§5. Proof of the theorems.

In this section, we give the proofs of our main theorems, that is, Proposition A, Theorems A1, A2, A3, A4, Proposition B, Theorems B1 and B2. In fact, we completely classify the sets $\hat{P}_n$ and $\check{P}_n$, up to the equivalence, but not many details of the proofs are given as they may be found in [9] and [10].

PROOF OF PROPOSITION A. Let (f, M) be an element of $\hat{P}_n^{\varepsilon\varepsilon'}(\mathscr{D})$, where $(\varepsilon, \varepsilon')=(+, -), (-, +), (-, -)$ or (o, o). Clearly we have (1) and (2).

We put $RD(f, M)=(\hat{f}, \hat{M}, S_*)$. Then $(\hat{f}, \hat{M}, S_*)$ is an element of $\hat{\boldsymbol{P}}_n^{\varepsilon\varepsilon'}(\mathscr{D})$, and so, satisfies all the equations in the statement of Proposition 2.5.

We put

$$l_a=\frac{\hat{l}_a}{a}\left(=\frac{\tilde{l}_a}{a}\right), \quad m_a=\frac{\hat{m}_a}{a}\left(=\frac{\tilde{m}_a}{a}\right), \quad q_a^+=\frac{\hat{q}_a^+}{a}\left(=\frac{2\cdot\tilde{q}_{a/2}^+}{a}\right), \quad q_a^-=\frac{\hat{q}_a^-}{a}\left(=\frac{\tilde{q}_a^-}{a}\right),$$

$$t_a^+(v)=\frac{\hat{t}_a^+(nv/a)}{a}\left(=\frac{\hat{t}_a^+(\hat{v})}{a}=\frac{2\cdot\tilde{t}_{a/2}^+(\tilde{v})}{a}\right),$$

$$t_a^-(v)=\frac{\hat{t}_a^-(nv/a)}{a}\left(=\frac{\hat{t}_a^-(\hat{v})}{a}=\frac{\tilde{t}_a^-(\hat{v}/2)}{a}=\frac{\tilde{t}_a^-(\tilde{v})}{a}\right),$$

$$q^+=\sum_{a|n}q_a^+, \quad q^-=\sum_{a|n}q_a^-, \quad t^+(v)=\sum_{a|n}t_a^+(v), \quad t^-(v)=\sum_{a|n}t_a^-(v),$$

$$t^+=\sum_v t^+(v), \quad t^-=\sum_v t^-(v), \quad q=q^++q^- \quad \text{and} \quad t=t^++t^-.$$

Using the orbit space $X=\hat{M}/\hat{f}$ and the branched covering $\hat{p}\colon \hat{M}\to X$, we easily have $(4)_+$, $(4)_-$, $(5)_+$, $(5)_-$, $(6)_{+-}$, $(6)_{-+}$, $(6)_{--}$ and $(6)_{oo}$.

If $\hat{P}_n^{\varepsilon\varepsilon'}(\mathscr{D})\neq\varnothing$, then by Proposition 2.2, n is even; and in the case of $(\varepsilon, \varepsilon')=(-, +)$, by Lemma 2.1, $n/2$ is odd. Thus we have (0) and $(0)_*$. Clearly we have $\tilde{q}_a^-=\tilde{t}_a^-=0$ for each divisor a of n. Therefore we have Proposition A.

PROOF OF THEOREM A.1. Under the conditions (0), (1), (2), $(4)_+$, $(4)_-$, $(5)_+$, $(5)_-$ and $(6)_{+-}$ in Proposition A, there are a compact surface X and a subset $S=S^0\cup S^1$ in X satisfying the following conditions;

(i) $(\hat{f}, \hat{M}, S_*)\in\hat{P}_n(X, S)$.

(ii) X is a compact orientable surface of genus g_{+-} with $l+q+t$ boundary components, where g_{+-} is as in Proposition A.

(iii) S^0 consists of m points in $\mathring{X}$.

(iv) S^1 consists of q loops and $\sum_v v\cdot t(v)$ arcs in ∂X.

We put q_a and $t_a(v)$ as follows;

$$q_a=\begin{cases} q_a^+ & \text{if } a \text{ is even and } a \text{ is not a divisor of } n/2, \\ q_a^- & \text{if } a \text{ is a divisor of } n/2, \end{cases}$$

$$t_a(v)=\begin{cases} t_a^+(v) & \text{if } a \text{ is even and } a \text{ is not a divisor of } n/2, \\ t_a^-(v) & \text{if } a \text{ is a divisor of } n/2, \end{cases}$$

and take $\boldsymbol{l}=(l_a)_{a|n}$, $\boldsymbol{m}=(m_a)_{\substack{a|n\\ a\neq n}}$, $\boldsymbol{q}=(q_a)_{a|n}$, $\boldsymbol{t}=(t_a(v))_{\substack{a|n\\ v}}$, where l_a, m_a, q_a^+, q_a^-, $t_a^+(v)$ and

$t_a^-(v)$ are non-negative integers given in the proof of Proposition A.

For $n, \boldsymbol{l}, \boldsymbol{m}, \boldsymbol{q}$ and $\boldsymbol{t}$, we take the set $D^+(n; \boldsymbol{l}, \boldsymbol{m}, \boldsymbol{q}, \boldsymbol{t})$ of systems of integers

$$(\delta_1, \delta_2, \cdots, \delta_l, \eta_1, \eta_2, \cdots, \eta_q, \lambda_1^{(1)}, \lambda_2^{(1)}, \cdots, \lambda_{t(1)}^{(1)},$$
$$\lambda_1^{(2)}, \lambda_2^{(2)}, \cdots, \lambda_{t(2)}^{(2)}, \cdots, \lambda_1^{(r)}, \lambda_2^{(r)}, \cdots, \lambda_{t(r)}^{(r)}, \theta_1, \theta_2, \cdots, \theta_m),$$

(we abbreviate it as $(\boldsymbol{\delta}, \boldsymbol{\eta}, \boldsymbol{\lambda}^{(1)}, \boldsymbol{\lambda}^{(2)}, \cdots, \boldsymbol{\lambda}^{(r)}, \boldsymbol{\theta})$), satisfying

(1) $$0 \leqq \delta_1 \leqq \delta_2 \leqq \cdots \leqq \delta_l < n, \quad 0 \leqq \eta_1 \leqq \eta_2 \leqq \cdots \leqq \eta_q < n,$$
$$0 \leqq \lambda_1^{(v)} \leqq \lambda_2^{(v)} \leqq \cdots \leqq \lambda_{t(v)}^{(v)} < n \ (v=1, 2, \cdots, r), \quad 1 \leqq \theta_1 \leqq \theta_2 \leqq \cdots \leqq \theta_m < n,$$

(2) $$\delta_1+\delta_2+\cdots+\delta_l+\eta_1+\eta_2+\cdots+\eta_q+\lambda_1^{(1)}+\lambda_2^{(1)}+\cdots+\lambda_{t(1)}^{(1)}+\lambda_1^{(2)}+\lambda_2^{(2)}+$$
$$\cdots+\lambda_{t(2)}^{(2)}+\cdots+\lambda_1^{(r)}+\lambda_2^{(r)}+\cdots+\lambda_{t(r)}^{(r)}+\theta_1+\theta_2+\cdots+\theta_m \equiv 0 \pmod n,$$

(3) $$l_a = \#\{\delta_j\, ;\ \text{g.c.d.}\{\delta_j, n\}=a\}, \qquad m_a = \#\{\theta_k\, ;\ \text{g.c.d.}\{\theta_k, n\}=a\},$$
$$q_a = \#\{\eta_u\, ;\ \text{g.c.d.}\{\eta_u, n\}=a\} \quad \text{and} \quad t_a(v) = \#\{\lambda_w^{(v)}\, ;\ \text{g.c.d.}\{\lambda_w^{(v)}, n\}=a\}.$$

Then, in a similar way to [7], the number $C^+(n; \boldsymbol{l}, \boldsymbol{m}, \boldsymbol{q}, \boldsymbol{t})$ of elements of $D^+(n; \boldsymbol{l}, \boldsymbol{m}, \boldsymbol{q}, \boldsymbol{t})$ is given as follows;

Let $n=p_1^{e_1} \cdot p_2^{e_2} \cdot \cdots \cdot p_s^{e_s}$ be the prime decomposition of n, where p_i is a prime number and e_i is a positive integer, and put $a=p_1^{f_1} \cdot p_2^{f_2} \cdot \cdots \cdot p_s^{f_s}$, where $0 \leqq f_i \leqq e_i$. Without loss of generality, we may assume that $0 \leqq f_1 < e_1$, $0 \leqq f_2 < e_2$, $\cdots$, $0 \leqq f_{v_a} < e_{v_a}$, $f_{v_a+1}=e_{v_a+1}$, $f_{v_a+2}=e_{v_a+2}$, $\cdots$, $f_s=e_s$ for some v_a $(0 \leqq v_a \leqq s)$. Let

$$g_a(x, y, z, u, w(1), w(2), \cdots, w(r))$$
$$= \prod_{j=1}^{n/a-1} (1+yx^{ja}+y^2x^{2ja}+\cdots)(1+zx^{ja}+z^2x^{2ja}+\cdots)(1+ux^{ja}+u^2x^{2ja}+\cdots)$$
$$\cdot \left(\prod_{v=1}^{r} (1+w(v)x^{ja}+w(v)^2x^{2ja}+\cdots) \right)$$

be a formal power series, and

$$f_a(x, y, z, u, w(1), w(2), \cdots, w(r))$$
$$= g_a(x, y, z, u, w(1), w(2), \cdots, w(r)) \cdot \left(\prod_{i=1}^{v_a} g_{p_i a}^{-1}(x, y, z, u, w(1), w(2), \cdots, w(r)) \right)$$
$$\cdot \left(\prod_{1 \leqq i < j \leqq v_a} g_{p_i p_j a}(x, y, z, u, w(1), w(2), \cdots, w(r)) \right) \cdots$$
$$\cdot \left(\prod_{1 \leqq j_1 < j_2 < \cdots < j_t \leqq v_a} g_{p_{j_1} \cdot p_{j_2} \cdots p_{j_t} \cdot a}^{(-1)^t}(x, y, z, u, w(1), w(2), \cdots, w(r)) \right)$$
$$\cdots g_{p_1 p_2 \cdots p_{v_a} \cdot a}^{(-1)^{v_a}}(x, y, z, u, w(1), w(2), \cdots, w(r)).$$

Let $y=(y_a)_{\substack{a|n\\a\neq n}}$, $z=(z_a)_{\substack{a|n\\a\neq n}}$, $u=(u_a)_{\substack{a|n\\a\neq n}}$, $w(1)=(w_a(1))_{\substack{a|n\\a\neq n}}$, $w(2)=(w_a(2))_{\substack{a|n\\a\neq n}}$, $\cdots$, $w(r)=(w_a(r))_{\substack{a|n\\a\neq n}}$ be vectors of variables y_a, z_a, u_a, $w_a(1)$, $w_a(2)$, $\cdots$, $w_a(r)$, respectively, and put

$$F(x, y, z, u, w(1), w(2), \cdots, w(r))=\prod_{\substack{a|n\\a\neq n}} f_a(x, y_a, z_a, u_a, w_a(1), w_a(2), \cdots, w_a(r)) .$$

Then we have $F(x, y, z, u, w(1), w(2), \cdots, w(r))$ as a generating function.

Let d be a divisor of n, and ω be a primitive d-th root of unity. Then,

$$C^+(n; l, m, q, t)=\frac{1}{n}\sum_{d|n} \varphi(d)C_d(n; l, m, q, t) ,$$

where $C_d(n; l, m, q, t)$ is the coefficient of the term

$$\prod_{\substack{a|n\\a\neq n}} y_a^{l_a}z_a^{m_a}u_a^{q_a}w_a(1)^{t_a(1)}w_a(2)^{t_a(2)}\cdots w_a(r)^{t_a(r)}$$

of $F(\omega; y, z, u, w(1), w(2), \cdots, w(r))$.

Let $Q(n; l, m, q, t)$ be the number of elements of $D^+(n; l, m, q, t)$ whose η-equivalence class consists of itself alone. If (1) l_a is even, (2) m_a is even, (3) q_a is even and (4) $t_a(v)$ is even for each divisor a of n with $0<a<n/2$ and any positive integer v $(1\leqq v\leqq r)$, then $Q(n; l, m, q, t)$ is equal to

$$\prod_{\substack{a|n\\0<a<n/2}}\left[\binom{\frac{\varphi(n/a)}{2}+\frac{l_a}{2}-1}{\frac{l_a}{2}}\binom{\frac{\varphi(n/a)}{2}+\frac{m_a}{2}-1}{\frac{m_a}{2}}\binom{\frac{\varphi(n/a)}{2}+\frac{q_a}{2}-1}{\frac{q_a}{2}}\cdot\prod_{v=1}^{r}\binom{\frac{\varphi(n/a)}{2}+\frac{t_a(v)}{2}-1}{\frac{t_a(v)}{2}}\right].$$

Otherwise, $Q(n; l, m, q, t)$ is equal to 0.

Under the conditions (0), (1), (2), $(4)_+$, $(4)_-$, $(5)_+$, $(5)_-$ and $(6)_{+-}$ in Proposition A, if $g=g_{+-}$ is a positive integer, the function $\hat{P}_n^{+-}(\mathscr{D})\to\hat{\boldsymbol{P}}_n^{+-}(\mathscr{D})\to\hat{P}_n(X, S)\to Z_n(g; l, q, t(1), t(2), \cdots, t(r), m)\to D^+(n; l, m, q, t)/\eta$ is a bijection. Hence we have (I).

We take an integer $d=p_{i_1}p_{i_2}\cdots p_{i_t}$ with $1\leqq i_1<i_2<\cdots<i_t\leqq s$. Then we consider the subset $D^+(n, d; l, m, q, t)$ of $D^+(n; l, m, q, t)$ consisting of elements $(\delta, \eta, \lambda^{(1)}, \lambda^{(2)}, \cdots, \lambda^{(r)}, \theta)$ satisfying that d is a divisor of g.c.d.$\{\delta_1, \delta_2, \cdots, \delta_l, \eta_1, \eta_2, \cdots, \eta_q, \lambda_1^{(1)}, \lambda_2^{(1)}, \cdots, \lambda_{t(1)}^{(1)}, \lambda_1^{(2)}, \lambda_2^{(2)}, \cdots, \lambda_{t(2)}^{(2)}, \cdots, \lambda_1^{(r)}, \lambda_2^{(r)}, \cdots, \lambda_{t(r)}^{(r)}, \theta_1, \theta_2, \cdots, \theta_m, n\}$. If a divisor a (of n) is not a multiple of d, then we have $l_a=0$, $m_a=0$, $q_a=0$ and $t_a(v)=0$ $(1\leqq\forall v\leqq r)$ in $D^+(n, d; l, m, q, t)$. Suppose that a is a multiple of d. We take $l^{(d)}$, $m^{(d)}$, $q^{(d)}$, $t^{(d)}$ as in the statement of the theorem. Then we define a function T: $D^+(n, d; l, m, q, t)\to D^+(n/d; l^{(d)}, m^{(d)}, q^{(d)}, t^{(d)})$ as follows: for an element of $D^+(n, d; l, m, q, t)$, divide each component by d, then we get the corresponding T-image of the element. Then the function

T is bijective. Hence the number of elements of $D^+(n, d; \boldsymbol{l}, \boldsymbol{m}, \boldsymbol{q}, \boldsymbol{t})$ is equal to $C^+(n/d; \boldsymbol{l}^{(d)}, \boldsymbol{m}^{(d)}, \boldsymbol{q}^{(d)}, \boldsymbol{t}^{(d)})$. This completes the proof of (II).

PROOF OF THEOREM A.2. Under the conditions (0), (1), (2), $(4)_+$, $(4)_-$, $(5)_+$, $(5)_-$ and $(6)_{oo}$ in Proposition A, there are a compact surface X and a subset $S=S^0\cup S^1$ in ∂X satisfying the following conditions (i), (ii), (iii) and (iv).

(i) $(\hat{f}, \hat{M}, S_*)\in \hat{P}_n(X, S)$.

(ii) X is a compact non-orientable surface of genus g with $l+q+t$ boundary components, where $g=g_{oo}$ is as in Proposition A.

(iii) S^0 consists of m points in $\mathring{X}$.

(iv) S^1 consists of q loops and $\sum_v v\cdot t(v)$ arcs in ∂X.

For $n, \boldsymbol{l}, \boldsymbol{m}, \boldsymbol{q}$ and $\boldsymbol{t}$ satisfying $l_{n/2}=m_{n/2}=q_{n/2}=t_{n/2}(v)=0$ for any v $(1\leqq v\leqq t)$, we take the set $D(n; \boldsymbol{l}, \boldsymbol{m}, \boldsymbol{q}, \boldsymbol{t})^{\circ}$ of systems of integers $(\boldsymbol{\delta}, \boldsymbol{\eta}, \boldsymbol{\lambda}^{(1)}, \boldsymbol{\lambda}^{(2)}, \cdots, \boldsymbol{\lambda}^{(r)}, \boldsymbol{\theta})$ satisfying

$$(1)' \qquad 0\leqq\delta_1\leqq\delta_2\leqq\cdots\leqq\delta_l<n/2\,, \qquad 0\leqq\eta_1\leqq\eta_2\leqq\cdots\leqq\eta_q<n/2,$$
$$0\leqq\lambda_1^{(v)}\leqq\lambda_2^{(v)}\leqq\cdots\leqq\lambda_{t(v)}^{(v)}<n/2\ \ (v=1, 2, \cdots, r)\,,$$
$$1\leqq\theta_1\leqq\theta_2\leqq\cdots\leqq\theta_m<n/2\,,$$

$$(3) \qquad l_a=\#\{\delta_j\ ;\ \text{g.c.d.}\{\delta_j, n\}=a\}\,, \qquad m_a=\#\{\theta_k\ ;\ \text{g.c.d.}\{\theta_k, n\}=a\}\,,$$
$$q_a=\#\{\eta_u\ ;\ \text{g.c.d.}\{\eta_u, n\}=a\} \quad \text{and} \quad t_a(v)=\#\{\lambda_w^{(v)}\ ;\ \text{g.c.d.}\{\lambda_w^{(v)}, n\}=a\}\,.$$

For $n, \boldsymbol{l}, \boldsymbol{m}, \boldsymbol{q}$ and $\boldsymbol{t}$ satisfying $l_{n/2}+m_{n/2}+q_{n/2}+q_{n/2}+\sum_v t_{n/2}(v)\neq 0$, we take the set $D(n; \boldsymbol{l}, \boldsymbol{m}, \boldsymbol{q}, \boldsymbol{t})^*$ of systems of integers $(\boldsymbol{\delta}, \boldsymbol{\eta}, \boldsymbol{\lambda}^{(1)}, \boldsymbol{\lambda}^{(2)}, \cdots, \boldsymbol{\lambda}^{(r)}, \boldsymbol{\theta})$ satisfying the conditions (1)′ and (3) above and

$$(*) \qquad \delta_l=n/2\,, \quad \eta_q=n/2\,, \quad \lambda_{t(v)}^{(v)}=n/2\ \ (1\leqq\exists v\leqq r)\,, \quad \theta_m=n/2\,.$$

Then we have clearly that;

LEMMA 5.1. *For an element of the set $D(n; \boldsymbol{l}, \boldsymbol{m}, \boldsymbol{q}, \boldsymbol{t})^{\circ}$ (resp. $D(n; \boldsymbol{l}, \boldsymbol{m}, \boldsymbol{q}, \boldsymbol{t})^*$), the condition $(7)_e$ in Theorem A.2 is a necessary and sufficient condition that $\delta_1+\delta_2+\cdots+\delta_l+\eta_1+\eta_2+\cdots+\eta_q+\lambda_1^{(1)}+\lambda_2^{(1)}+\cdots+\lambda_{t(1)}^{(1)}+\lambda_1^{(2)}+\lambda_2^{(2)}+\cdots+\lambda_{t(2)}^{(2)}+\cdots+\lambda_1^{(r)}+\lambda_2^{(r)}+\cdots+\lambda_{t(r)}^{(r)}+\ \theta_1+\theta_2+\cdots+\theta_m$ is even.*

Moreover $\# D(n; \boldsymbol{l}, \boldsymbol{m}, \boldsymbol{q}, \boldsymbol{t})^{\circ}=\# D(n; \boldsymbol{l}, \boldsymbol{m}, \boldsymbol{q}, \boldsymbol{t})^*\equiv C(n; \boldsymbol{l}, \boldsymbol{m}, \boldsymbol{q}, \boldsymbol{t})$ equals

$$\prod_{\substack{a|n\\ a\neq n\\ a\neq n/2}}\left[\binom{\frac{\varphi(n/a)}{2}+l_a-1}{l_a}\binom{\frac{\varphi(n/a)}{2}+m_a-1}{m_a}\binom{\frac{\varphi(n/a)}{2}+q_a-1}{q_a}\cdot\prod_{v=1}^{r}\binom{\frac{\varphi(n/a)}{2}+t_a(v)-1}{t_a(v)}\right].$$

We denote by $D_e(n; \boldsymbol{l}, \boldsymbol{m}, \boldsymbol{q}, \boldsymbol{t})^{\circ}$ (resp. $D_e(n; \boldsymbol{l}, \boldsymbol{m}, \boldsymbol{q}, \boldsymbol{t})^*$) the set $D(n; \boldsymbol{l}, \boldsymbol{m}, \boldsymbol{q}, \boldsymbol{t})^{\circ}$ (resp. $D(n; \boldsymbol{l}, \boldsymbol{m}, \boldsymbol{q}, \boldsymbol{t})^*$) when $n, \boldsymbol{l}, \boldsymbol{m}, \boldsymbol{q}$ and $\boldsymbol{t}$ satisfy the condition $(7)_e$, and by $D_o(n; \boldsymbol{l}, \boldsymbol{m}, \boldsymbol{q}, \boldsymbol{t})^{\circ}$ (resp. $D_o(n; \boldsymbol{l}, \boldsymbol{m}, \boldsymbol{q}, \boldsymbol{t})^*$) the set $D(n; \boldsymbol{l}, \boldsymbol{m}, \boldsymbol{q}, \boldsymbol{t})^{\circ}$ (resp. $D(n; \boldsymbol{l}, \boldsymbol{m}, \boldsymbol{q}, \boldsymbol{t})^*$) when $n, \boldsymbol{l}, \boldsymbol{m}, \boldsymbol{q}$ and $\boldsymbol{t}$ satisfy the condition $(7)_o$.

Note that n is even. We can give the proof of Theorem A.2 in respective cases in the same way as that of Theorem A in [10]. For example, in case that g is odd and

$g \geqq 3$, if $l_{n/2}+m_{n/2}+q_{n/2}+\sum_v t_{n/2}(v)\neq 0$, then the function $\hat{P}_n^{oo}(\mathscr{D})\to\hat{\boldsymbol{P}}_n^{oo}(\mathscr{D})\to\hat{P}_n(X,S)\to Z_n^-(g; l, q, t(1), t(2), \cdots, t(r), m)_1^o \to 2\cdot D_e(n; \boldsymbol{l}, \boldsymbol{m}, \boldsymbol{q}, \boldsymbol{t})^o$ is a bijection. The other cases follow similarly, completing the proof.

We can prove Theorems A.3 and A.4, in a similar way as above. Since many details of the proofs are found in Theorem B in [10], we omit the proof.

Proof of Proposition B. Let (f, M) be an element of $\ddot{P}_n^\varepsilon(\mathscr{D})$, where $\varepsilon=+$ or o. Clearly we have (1) and (2).

We put $RD(f, M)=(\ddot{f}, \ddot{M}, S_*)\in\ddot{\boldsymbol{P}}_n^\varepsilon(\mathscr{D})$. Then, from Proposition 2.6, we have (3).

Using the orbit space $X=\ddot{M}/\ddot{f}$ and the branched covering $p: \ddot{M}\to X$, we have (4), (5), $(6)_{2+}$ and $(6)_{2o}$. Here we put

$$l_a=\frac{\hat{l}_a}{a}\left(=\frac{\tilde{l}_a}{a}\right),\quad m_a=\frac{\hat{m}_a}{a}\left(=\frac{\tilde{m}_a}{a}\right),\quad q_a=\frac{\hat{q}_a}{a}\left(=\frac{\tilde{q}_a}{a}\right),$$

$$t_a(v)=\frac{\hat{t}_a(nv/a)}{a}\left(=\frac{\tilde{t}_a(nv/(2a))}{a}=\frac{\tilde{t}_a(\tilde{v})}{a}\right),\quad q=\sum_{a\mid n} q_a,\quad t=\sum_v\sum_{a\mid n} t_a(v).$$

If $\ddot{P}_n^\varepsilon(\mathscr{D})\neq\varnothing$, then by Proposition 2.2 and Lemma 2.1, we have $(0)_*$. Thus we have Proposition B.

Proof of Theorem B.1. Under the conditions $(0)_*$, (1), (2), (3), (4), (5) and $(6)_{2+}$ in Proposition B, there is a compact surface X and a subset $S=S^0\cup S^1$ in X satisfying the following conditions (i)–(v):

(i) $(\ddot{f}, \ddot{M}, S_*)\in\ddot{P}_n(X, S)$.

(ii) X is a compact orientable surface of genus g_{2+} with $l+q+t$ boundary components, where g_{2+} is as in Proposition B.

(iii) S^0 consists of m points in $\mathring{X}$.

(iv) S^1 consists of q loops and $\sum_v v\cdot t(v)$ arcs in ∂X.

(v) l is the number of boundary components which does not intersect with S^1, q is the number of boundary components which is contained in S^1, $t(v)$ is the number of boundary components which intersects v arcs with S^1.

With l_a, m_a, q_a, $t_a(v)$ given in the proof of Prop. B, we have vectors $\boldsymbol{l}=(l_a)_{a\mid n}$, $\boldsymbol{m}=(m_a)_{a\mid n}$, $\boldsymbol{q}=(q_a)_{a\mid n}$ and $\boldsymbol{t}=(t_a(v))_{a\mid n, v}$. Then we take a set $D^{2+}(n; \boldsymbol{l}, \boldsymbol{m}, \boldsymbol{q}, \boldsymbol{t})$ of systems of integers $(\boldsymbol{\delta}, \boldsymbol{\eta}, \boldsymbol{\lambda}^{(1)}, \boldsymbol{\lambda}^{(2)}, \cdots, \boldsymbol{\lambda}^{(r)}, \boldsymbol{\theta})$ such that

(1) $$0\leqq\delta_1\leqq\delta_2\leqq\cdots\leqq\delta_l<n/2,\qquad 0\leqq\eta_1\leqq\eta_2\leqq\cdots\leqq\eta_q<n/2,$$
$$0\leqq\lambda_1^{(v)}\leqq\lambda_2^{(v)}\leqq\cdots\leqq\lambda_{t(v)}^{(v)}<n/2\quad(v=1,2,\cdots,r),$$
$$1\leqq\theta_1\leqq\theta_2\leqq\cdots\leqq\theta_m<n/2,$$

(2)′ $$\delta_1+\delta_2+\cdots+\delta_l+\eta_1+\eta_2+\cdots+\eta_q+\lambda_1^{(1)}+\lambda_2^{(1)}+\cdots+\lambda_{t(1)}^{(1)}+\lambda_1^{(2)}+\lambda_2^{(2)}+\cdots+\lambda_{t(2)}^{(2)}+\cdots+\lambda_1^{(r)}+\lambda_2^{(r)}+\cdots+\lambda_{t(r)}^{(r)}+\theta_1+\theta_2+\cdots+\theta_m\equiv 0 \pmod{n/2},$$

(3)′ $l_a = \#\{\delta_j\ ;\ \text{g.c.d.}\{\delta_j, n/2\} = a\}$, $m_a = \#\{\theta_k\ ;\ \text{g.c.d.}\{\theta_k, n/2\} = a\}$,
$q_a = \#\{\eta_u\ ;\ \text{g.c.d.}\{\eta_u, n/2\} = a\}$ and $t_a(v) = \#\{\lambda_w^{(v)}\ ;\ \text{g.c.d.}\{\lambda_w^{(v)}, n/2\} = a\}$.

Clearly, it holds that $D^{2+}(n; \boldsymbol{l}, \boldsymbol{m}, \boldsymbol{q}, \boldsymbol{t}) = D^{+}(n/2; \boldsymbol{l}, \boldsymbol{m}, \boldsymbol{q}, \boldsymbol{t})$. Hence we have Theorem B.1.

PROOF OF THEOREM B.2. Under the conditions $(0)_*$, (1), (2), (3), (4), (5) and $(6)_{2o}$ in Proposition B, there are a compact surface X and a subset $S = S^0 \cup S^1$ in ∂X satisfying the same conditions as (i)–(v) in the proof of Theorem B.1, except that we should replace the condition (ii) with

(ii)′ X is a compact non-orientable surface of genus g with $l+q+t$ boundary components, where $g = g_{2o}$ is as in Proposition B.

Then we take $\boldsymbol{l}$, $\boldsymbol{m}$, $\boldsymbol{q}$ and $\boldsymbol{t}$ as before. For n, $\boldsymbol{l}$, $\boldsymbol{m}$, $\boldsymbol{q}$ and $\boldsymbol{t}$, we take the set $D^{2o}(n; \boldsymbol{l}, \boldsymbol{m}, \boldsymbol{q}, \boldsymbol{t})$ of systems of integers $(\boldsymbol{\delta}, \boldsymbol{\eta}, \boldsymbol{\lambda}^{(1)}, \boldsymbol{\lambda}^{(2)}, \cdots, \boldsymbol{\lambda}^{(r)}, \boldsymbol{\theta})$ satisfying

(1)″ $0 \leqq \delta_1 \leqq \delta_2 \leqq \cdots \leqq \delta_l < n/4$, $0 \leqq \eta_1 \leqq \eta_2 \leqq \cdots \leqq \eta_q < n/4$,
$0 \leqq \lambda_1^{(v)} \leqq \lambda_2^{(v)} \leqq \cdots \leqq \lambda_{t(v)}^{(v)} < n/4\ (v = 1, 2, \cdots, r)$,
$1 \leqq \theta_1 \leqq \theta_2 \leqq \cdots \leqq \theta_m < n/4$,

and (3)′ in the proof of Theorem B.1.

Then $\# D^{2o}(n; \boldsymbol{l}, \boldsymbol{m}, \boldsymbol{q}, \boldsymbol{t}) \equiv C^2(n; \boldsymbol{l}, \boldsymbol{m}, \boldsymbol{q}, \boldsymbol{t})$ is equal to

$$\prod_{\substack{a \mid n/2 \\ a \neq n/2}} \binom{\dfrac{\varphi(n/(2a))}{2} + l_a - 1}{l_a} \binom{\dfrac{\varphi(n/(2a))}{2} + m_a - 2}{m_a} \binom{\dfrac{\varphi(n/(2a))}{2} + q_a - 1}{q_a} \cdot \prod_{v=1}^{r} \binom{\dfrac{\varphi(n/(2a))}{2} + t_a(v) - 1}{t_a(v)}.$$

Then, in a similar way to Theorem A in [10], we have the proof of Theorem B.2.

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